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Finite Geometry — *Sets of type* $(3,h)$ *in* PG$(3,q)$, by Vito Napolitano and Domenico Olanda, communicated on 22 June 2012.

Abstract. — In this paper, a complete classification of subsets of points of P$G(3,q)$ of type $(3,q+3)$ with respect to planes is given.

Key words: Two-character sets, strongly regular graphs.

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1. Introduction

A subset K of PG$(3,q)$ is a *two character* (m,h)-set, m, h integers such that $0 \leq m < h$, *with respect to planes* if it intersects every plane either in m or h points. If there exist both an m-secant plane and an h-secant plane, then K is of *type* (m,h).

Subsets in P$G(3,q)$ with two intersection numbers with respect to planes have been studied by many Authors but they have not yet been completely classified.

The interest in their study is also motivated by the well known equivalence between two-character sets, two-weight codes and some strongly regular graphs; see [3]. In our setting, let K be a set of n points in $PG(3,q)$ with characters (m,h) with respect to the planes, and consider the code $\mathscr{C}$ whose generator matrix contains as columns the coordinates of the points of K. Then, $\mathscr{C}$ is an $[n,4]_q$ linear code whose non-zero words have either weight $w_2 = n-m$ or $w_1 = n-h$. Denote now by $d(x,y)$ the usual Hamming distance and fix $i \in \{1,2\}$. We say that two codewords $x, y \in \mathscr{C}$ are *adjacent* if, and only if $d(x,y) = w_i$. The graph Γ_i with the elements of $\mathscr{C}$ as vertices and adjacency defined as above for turns out to be strongly regular, see [6, 12] and to sport several further properties. Some of the papers in the references, see [6, 3, 12], contain further details on this topic.

The subsets of PG$(3,q)$ of type $(1,h)$ and $(2,h)$ with respect to planes have been completely determined in [11] and [8], respectively.

In this paper, we consider the case $(3,h)$, that is we consider sets in P$G(3,q)$ which are intersected by every plane either in 3 or in h-points, $h > 3$.

All two character $(3,h)$-sets for $q=2$ are described in Section 3. If $q>2$, then we prove that $h-3$ divides q so $h \leq q+3$.

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By running a computer programme for admissible values of $h < q + 3$ results were obtained which led the Authors to think that sets of type $(3, h)$ corresponding to such values of h, if they do exist, they have to be considered sporadic. Thus, the Authors study sets in $PG(3, q)$ of type $(3, q + 3)$ with respect to planes.

The following two theorems are proved.

THEOREM I. *Let K be a subset of $PG(3, 2)$ intersected by every plane either in 3 or in h points, then either $h = 7$ or $h = 5$ and K is either a plane, or the whole space $PG(3, 2)$ or the set of points on three pairwise skew lines.*

THEOREM II. *Let K be a subset of $PG(3, q)$ $(q > 2)$ of type $(3, q + 3)$ with respect to planes. Then,*

(i) *If $q = 3$, then K is the set of points on three pairwise skew lines, or one of the three sets described in Examples 4.1, 4.2 and 4.3.*
(ii) *If $q = 4$, then K is either the set of points on three pairwise skew lines or $PG(3, 2)$ embedded in $PG(3, 4)$.*
(iii) *If $q > 4$, then K is the set of points on three pairwise skew lines.*

2. FIRST PROPERTIES AND EXAMPLES

Throughout the paper K will denote a subset of $PG(3, q)$ of size k intersecting every plane of $PG(3, q)$ either in 3 or in h points.

Let ℓ be a line of $PG(3, q)$, ℓ is *external* if it intersects K in the empty set, ℓ is *tangent* if it intersects K in exactly one point and ℓ is *s-secant* if it intersects K in exactly s-points, $s \geq 2$.

A plane of $PG(3, q)$ is a 3-*secant plane* if it intersects K in exactly 3 points, and a plane is a *h-secant plane* if it meets K in exactly h points.

An example of k-set in $PG(3, q)$ of type $(3, q + 3)$ with respect to the planes is the following.

EXAMPLE I. Let ℓ, m, n be three pairwise skew lines of $PG(3, q)$. The set $K = \ell \cup m \cup n$ is intersected by every plane either in 3 or in $q + 3$ points.

Let us recall some known properties on sets in $PG(3, q)$ with two intersection numbers with respect to planes [10].

Let t_3 and t_h denote the number of 3-secant and h-secant planes, respectively. The following equations hold.

(i) $$t_3 + t_h = q^3 + q^2 + q + 1$$

(ii) $$3t_3 + ht_h = k(q^2 + q + 1)$$

(iii) $$6t_3 + h(h-1)t_h = k(k-1)(q+1).$$

From (i), (ii) and (iii) it follows that:

$$k^2(q+1) - k[(h+3)(q^2+q+1) - q^2] + 3h(q^3+q^2+q+1) = 0 \tag{2.1}$$

$$t_3 = \frac{h(q^3+q^2+q+1) - k(q^2+q+1)}{h-3} \tag{2.2}$$

$$t_h = \frac{k(q^2+q+1) - 3(q^3+q^2+q+1)}{h-3}. \tag{2.3}$$

Let p be a point of K, denote with $v_3(p)$ and with $v_h(p)$ the numbers of 3-secant and h-secant planes on p, respectively.

Then,

$$v_3(p) = \frac{h(q^2+q+1) - k(q+1)}{h-3} - \frac{q^2}{h-3} \tag{2.4}$$

$$v_h(p) = \frac{k(q+1) - 3(q^2+q+1)}{h-3} + \frac{q^2}{h-3}. \tag{2.5}$$

Let p be a point not in K, denote with $u_3(p)$ and with $u_h(p)$ the numbers of 3-secant and h-secant planes on p, respectively. We have that

$$u_3(p) = \frac{h(q^2+q+1) - k(q+1)}{h-3} \tag{2.6}$$

$$u_h(p) = \frac{k(q+1) - 3(q^2+q+1)}{h-3}. \tag{2.7}$$

Thus, the integers $v_3(p)$, $v_h(p)$, $u_3(p)$, $u_h(p)$ are independent from p and comparing Equation (2.4) and Equation (2.6) gives

$$(h-3) \mid q^2. \tag{2.8}$$

3. Two character $(3,h)$-sets in $\mathrm{PG}(3,2)$

Let K be a subset of points of $\mathrm{PG}(3,2)$ intersected by every plane either in 3 or in h points. By Equation (2.8) it follows that $h \in \{4,5,7\}$.

For $h=4$, $q=2$ Equation (2.1) has no solution. When $h=5$ or $h=7$ the corresponding values of k are 7, 15, 9. It follows that at least one plane is h-secant, otherwise $k \leq 5$.

For $h=7$, $q=2$ Equation (2.1) gives $k=7$ or $k=15$.

If $k=7$, the set K is a plane. If $k=15$ the set K is $\mathrm{PG}(3,2)$.

Next, assume that $h=5$. From Equation (2.1) it follows that $k=9$. In such a case, we are going to prove that K is the union of three pairwise skew lines. The proof will proceed by steps.

Step 1. *Every point of K belongs to at least one line contained in K.*
This is clear, since $k = 9$.

Step 2. *Through every point of K there are at most two lines contained in K.*
Assume to the contrary that it there exists a point p of K on three lines, say ℓ_1, ℓ_2, ℓ_3 contained in K. Namely such lines are not coplanar, and they contain seven points of K. Let p' and p'' be the other two remaining points of K, and let π_1, π_2, π_3 be the three distinct planes obtained by connecting each of the lines ℓ_1, ℓ_2, ℓ_3 with the point p'. Each of these planes has to contain a fifth point which is p'', necessarily.

Thus, p'' belongs to the line pp' and so there are four lines containing p and contained in K. A plane π not containing p does contain none of those four lines, and so it has exactly four points of K, a contradiction.

Step 3. *K is the union of three pairwise skew lines.*
Let ℓ be a line contained in K and let π_1, π_2, π_3 be the three planes passing through ℓ. Each of such planes is 5-secant, and so besides ℓ it contains another line. Let ℓ_1, ℓ_2, ℓ_3 be the lines different from ℓ and contained in π_1, π_2, π_3, respectively.

By Step 2 the lines ℓ_1, ℓ_2 and ℓ_3 meet ℓ in distinct points. Thus, they are three pairwise skew lines and they contain all the points of K. □

4. k-sets in $PG(3,q)$, $q > 2$, of type $(3,h)$

From now on, by Section 2, we may assume that $q > 2$.

Equation (2.8) implies that $h \leq q^2 + 3$ and so K cannot contain planes.

Proposition 4.1. *There are both* 3*-secant planes and h-secant planes.*

Proof. Assume by way of contradiction that all the planes are h-secant. Let ℓ be a s-secant line. Clearly, $s < h$, otherwise K should be the set of s collinear points, and so not of type $(3,h)$. Computing the size of K via the planes through ℓ gives

$$|K| = (q+1)(h-s) + s. \tag{4.1}$$

Counting in double way the incident point-plane pairs (x, π), with $x \in K$, gives

$$|K|(q^2+q+1) = h(q+1)(q^2+1). \tag{4.2}$$

Comparing equations (4.1) and (4.2) it follows that

$$(q+1)(h-s) + s = \frac{h(q^2+1)(q+1)}{q^2+q+1}$$

and so

$$\begin{aligned} h - s + \frac{s}{q+1} &= \frac{h(q^2+1)}{q^2+q+1} \\ h - \frac{h(q^2+1)}{q^2+q+1} &= s - \frac{s}{q+1} \\ \frac{h}{q^2+q+1} &= \frac{s}{q+1} \\ h(q+1) - s(q+1) &= sq^2 \\ (h-s)(q+1) &= s(q^2-1)+s \end{aligned}$$

from which it follows that $q+1 \mid s$ and so $s = q+1$. Since K is of type $(3,h)$ it cannot be a line, thus there is at least one point p of K outside the line ℓ. The plane connecting p and ℓ is contained in K since all the lines px, with $x \in \ell$, are secant lines and so they are contained in K. Since K does not contain planes, we have a contradiction.

Finally, we prove that there is at least one h-secant plane.

Assume to the contrary that all the planes are 3-secant. Let x and y be two distinct points of K, and let ℓ be the line containing both x and y. Such a line is a 2-secant line, otherwise K should be the set of three collinear points. It follows that, $|K| = q+3$. Being $q > 2$ it there exists an external line, let E be such a line. Computing $|K|$ via the planes on E gives $|K| = 3q+3$, a contradiction. □

PROPOSITION 4.2. *$h-3$ divides q and so $h \leq q+3$.*

PROOF. Since there are 3-secant planes and $q > 2$, it follows that there are both external lines and tangent lines. Let E be an external line, computing the size k of K via the planes on E gives

$$k = 3\alpha + (q+1-\alpha)h = (q+1)h - h(\alpha - 3),$$

where α denotes the number of 3-secant planes through E.

Now, let t be a tangent line, computing k via the planes on t gives

$$k = 1 + 2\mu + (q+1-\mu)(h-1) = (q+1)h - q - \mu(h-3)$$

where μ denotes the number of 3-secant planes on t. Comparing the two above values of k one gets that $(h-3) \mid q$. □

Consider the discriminant

$$\Delta = [q^2 - (h+3)(q^2+q+1)]^2 - 12h(q+1)(q^3+q^2+q+1)$$

of Equation (2.1).

Fixed $q = p^t$, p prime, and chosen $h = 3 + d$, with $1 \leq d < q$ and $d \mid q$, running a computer programme the Authors have checked the possible values of Δ allowing p and t to assume the following values:

$$p \text{ prime} \quad 1 \leq p \leq 30011 \; 1 \leq t \leq 100$$
$$p \text{ prime} \quad 1 \leq p \leq 1979 \; 1 \leq t \leq 200$$
$$p \text{ prime} \quad 1 \leq p \leq 1699 \; 1 \leq t \leq 500.$$

The only values of q and h that make Δ a perfect square are $q = 8$ and $h = 7$.

Hence, it seems that sets of type $(3, h)$ with respect to the planes with $h < q + 3$ and such that $h - 3 \mid q$, if they do exist they are very rare.

Therefore, from now on we will assume $h = q + 3$.

PROPOSITION 4.3. *If K contains no line then either $q = 3$ or $q = 4$. If $q = 4$, then K is $PG(3, 2)$. If $q = 3$ then K is determined.*

PROOF. When $h = q + 3$, Equation 2.1 gives $k = 3q + 3$ or $q = 3$ and $k = 12 = 3q + 3$ or $k = 15$.

Assume $k = 3q + 3$. Since there are 3-secant planes, then at least one external line exists. Let E be an external line, and let α be the number of $(q + 3)$-secant planes passing through E. Then,

$$k = \alpha(q + 3) + (q + 1 - \alpha)3 = 3q + 3$$

from which it follows that $\alpha \cdot q = 0$ and so $\alpha = 0$. Let π be a $(q + 3)$-secant plane and let $B = \pi \cap K$. By the previous argumentation, the set B has no external line and since by assumptions it contains no line, it is a blocking set of π.

From the results contained in [2] it follows that

$$|B| = q + 3 \geq q + \sqrt{q} + 1$$

from which it follows that either $q = 3$ or $q = 4$. If $q = 4$, $k = 3q + 3 = 15$ and B is a Baer subplane. Let r be a line containing two points, say p and p' of K. At least one plane, through the line r is $(q + 3)$-secant otherwise $k \leq q + 3$. Let π be such a plane. The set $K \cap \pi$ is a Baer subplane and so r has three points in common with K. Thus, on every 3-secant plane π, the three points of K in π are collinear. If any plane intersects K either in three collinear points or in a Baer subplane then $K = PG(3, 2)$, (cf e.g. [13]).

If $q = 3$, as already remarked, the set K has size either 12 or 15. The sets K of $PG(3, 3)$ not containing lines and intersecting every plane in either 3 or 6 points have been determined with the help of a computer. Using a package in MAGMA similar to that contained in [9] an **exhaustive** search provided an unique 12-set and two 15-sets in $PG(3, 3)$, not projectively equivalent and with the required properties. Below, such sets are described via the homogeneous coordinates (x, y, z, t).

EXAMPLE 4.1. $k = 12$

$$A(1,0,0,0), B(0,1,0,0), C(0,1,1,1), D(0,0,1,0), E(0,1,0,1), F(0,0,0,1)$$
$$G(1,0,0,1), H(1,1,0,1), I(1,0,2,0), L(1,2,2,0), M(1,0,2,1), N(0,1,1,0).$$

EXAMPLE 4.2. $k = 15$

$$A(1,1,2,1), B(1,0,0,0), C(0,1,0,0), D(0,0,1,0), E(0,0,0,1),$$
$$F(0,0,1,2), G(1,1,1,1), H(1,1,1,2), I(1,0,2,0), L(1,2,2,0),$$
$$M(0,1,2,2), N(0,1,1,0), O(1,0,2,2), P(1,2,1,1), Q(1,2,1,2).$$

EXAMPLE 4.3. $k = 15$

$$A(1,0,0,0), B(0,1,1,0), C(0,1,0,0), D(0,0,1,0), E(0,0,0,1),$$
$$F(1,1,2,1), G(1,1,1,1), H(1,0,1,2), I(1,1,1,2), L(1,2,2,0),$$
$$M(0,1,2,2), N(1,1,2,2), O(0,1,2,1), P(1,0,1,1), Q(1,0,2,0).$$

5. The characterization theorem

In view of the previous Sections we may assume that $q > 2$, $h = q + 3$ and K contains at least one line.

PROPOSITION 5.1. *If $q > 2$ and K contains at least one line, then K is the point-set of the union of three pairwise skew lines.*

PROOF. Let L_1 denote a line contained in K. Being $q > 2$ every plane passing through L_1 is a $(q + 3)$-secant plane, and so counting $k = |K|$ via the planes on L_1 gives:

$$k = q + 1 + 2(q + 1) = 3q + 3.$$

Put $K' = K - L_1$ and let π be a plane. If π contains L_1, then it is a $(q + 3)$-secant plane and hence it intersects K' in exactly two points. If π does not contain L_1, then it meets K' in exactly two points if it is a plane meeting K in three points and it meets K' in $q + 2$ points if it is a plane meeting K in $q + 3$ points. Hence every plane meets K' either in two or in $q + 2$ points. From results contained in [5] it follows that K' is the union of two pairwise skew lines L_2 and L_3. Hence $K = L_1 \cup L_2 \cup L_3$ and the assertion is proved. □

6. Final remarks

The aim of this last section is to analyse some properties of the linear codes related to the subsets of $PG(3,3)$ described in examples 4.1, 4.2, 4.3 of Section 4.

The set of P$G(3,3)$ described in Example 4.1 gives rise to a linear $[12,4,6]_3$-code with second weight 9. This code is the subcode generated by the first 4 rows of

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Observe that G generates a $[12,6,6]_3$ code, equivalent to the usual ternary extended Golay code.

The strongly regular graphs arising from the sets in examples 4.2 and 4.3 have different automorphism groups. More in detail, the automorphism group of the graph from Example 4.2 has order 5832, while that from Example 4.3 has order 116640. In view of this latter property, we can compare the structure here obtained with that described in [4] and the code with that of [12]. In particular, the two linear $[15,4,9]_3$-codes with second weight 12 associated to the sets of examples 4.2 and 4.3 are different.

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