Rend. Lincei Mat. Appl. 23 (2012), 405–428
DOI 10.4171/RLM/636

Mechanics — *Onset of convection for ternary fluid mixtures saturating horizontal porous layers with large pores*, by Florinda Capone and Roberta De Luca, communicated on 22 June 2012.

Abstract. — Ternary fluid mixtures saturating horizontal porous layers with large pores, uniformly rotating around the vertical axis, are investigated. The layers are heated from below, salted from above and from below by two salts. The stabilizing effects of both the rotation and Brinkman terms on the conduction solution are analyzed.

Sommario. — Vengono studiate miscele fluide ternarie saturanti uno strato poroso orizzontale uniformemente rotante attorno all'asse verticale, nell'ipotesi che i pori siano sufficientemente grandi da tener conto della viscosità di Brinkman. Si ammette inoltre che lo strato sia riscaldato dal basso e salato dal basso e dall'alto da due diversi sali. Si studia la stabilità non lineare (globale) della soluzione di conduzione, mettendo in evidenza gli effetti stabilizzanti della rotazione e del termine di Brinkman, al variare dei numeri di Prandtl.

Key words: Porous media, global stability, Brinkman law, rotation.

2010 Mathematics Subject Classification: 76R10, 76S05, 35B35.

1. Introduction

Because of their relevance in the real world phenomena (geophysical applications, artificial porous materials used for insulating purposes and in heat transfer devices ...), multicomponent fluid mixtures in porous layers have attracted, in the past as nowadays, the interest of many authors {cfr. [1]–[32]}. Recently, in [27], the case of porous rotating layers, heated from below and salted from above by two salts (the most destabilizing case), has been analyzed under the assumption of the validity of the Brinkman model (large pores) [2]. Through suitable scalings, the linear operator of the model has been symmetrized and a necessary and sufficient condition ensuring the global, nonlinear, asymptotic L^2-stability of the conduction solution, has been found for any values of the Prandtl numbers. In the present paper we consider porous rotating layers with large pores, heated from below, salted from below by one salt and from above by another salt, and apply the methodology recently introduced by Rionero [32]. Although we generalize the Rionero procedure to the case of rotating porous layers with large pores, uniformly heated from below and salted from below and from above by two salts, our main scope is to analyze and evaluate—in finite forms—the stabilizing effects of rotation and Brinkman viscosity either when act separately or when they act together. Precisely our aim is to show that the global nonlinear stability of the conduction solution is guaranteed by

$$R^2 < R_1^2 - R_2^2 + A^*, \quad \text{for } P_1 \le 1,\ P_2 \ge 1, \tag{1.1}$$

$$R^2 < \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + A^*, \quad \text{for } P_1 \ge 1,\ P_2 \le 1, \tag{1.2}$$

$$R^2 < R_1^2 - \frac{R_2^2}{P_2} + A^*, \quad \text{for } P_1 \le 1,\ P_2 \le 1, \tag{1.3}$$

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + A^*, \quad \text{for } P_1 \ge 1,\ P_2 \ge 1, \tag{1.4}$$

where R^2, R_i^2 $(i = 1,2)$, are the thermal Rayleigh number and the solutal Rayleigh numbers respectively, P_i $(i = 1,2)$, are the Prandtl numbers and $A^* = A^*(D_a, \mathcal{T}) > 0$, with $D_a(> 0)$ and $\mathcal{T}(> 0)$ the Darcy and the Taylor-Darcy numbers (cfr. Sect. 2) linked to the Brinkman viscosity and uniform rotation respectively. It is to remark that

1) $A^*(0,0) = 4\pi^2$ and (1.1)–(1.4), in absence of rotation and Brinkman terms, reduce to the conditions found in [32];
2) $A^*(D_a, \mathcal{T}) > 0$, is a measure of the stabilizing effects of rotation and Brinkman terms {cfr. Sect. 6}.

The plan of the paper is as follows. Section 2 is devoted to the introduction of the mathematical model. In Section 3 the main boundary value problem at stake is resolved and the independent fields involved in the model are reduced only to three. In the subsequent Section 4, new fields and positive scalings are introduced, in order to write the system in a suitable way. Sections 5 regards the global nonlinear stability of the conduction solution. It is shown that, for $\{P_1 \le 1, P_2 \ge 1\}$, the linear operator of the model can be symmetrized and (1.1), which is a necessary and sufficient condition for the nonlinear stability, is obtained. For the other values of the Prandtl numbers (the partial skew-symmetric cases) the conditions (1.2)–(1.4) are proved to be only sufficient to guarantee the global nonlinear stability of the conduction solution. Some numerical estimation of the stability thresholds are furnished in Section 6, on showing the stabilizing effects of both Brinkman and rotation terms. The paper ends with an Appendix (Section 7) in which an uniqueness theorem is proved.

2. Mathematical model

We consider fluid mixtures saturating an horizontal porous layer, uniformly rotating around the vertical axis. We denote by d the depth of the layer, by S_i $(i = 1,2)$ two chemical species (or salts) dissolved in the fluid and by $Oxyz$ an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). The equations governing the motion of the fluid, in the Darcy-Oberbeck-Boussinesq scheme, according to the Brinkman law, are:

$$(2.1)\qquad \begin{cases} \nabla P = -\dfrac{\mu_1}{K}\mathbf{v} + \mu_2\Delta\mathbf{v} - 2\rho_0\omega\mathbf{k}\times\mathbf{v} - \mathbf{g}\rho_f, \\ \nabla\cdot\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = K_T\Delta T, \\ \dfrac{\partial C_1}{\partial t} + \mathbf{v}\cdot\nabla C_1 = K_1\Delta C_1, \\ \dfrac{\partial C_2}{\partial t} + \mathbf{v}\cdot\nabla C_2 = K_2\Delta C_2, \end{cases}$$

where

$$(2.2)\qquad \begin{aligned} &P = p - \frac{\rho_0}{2}|\underline{\omega}\times\mathbf{x}|^2, \\ &\rho_f = \rho_0[1 - \alpha(T - T_0) + \beta_1(C_1 - C_1^0) + \beta_2(C_2 - C_2^0)], \end{aligned}$$

with

p = pressure field, $\underline{\omega} = \omega\mathbf{k}$ = angular velocity, $\mathbf{x} = (x, y, z)$,
ρ_f = fluid mixture density, ρ_0 = reference density,
α = thermal expansion coefficient, β_i = solutal expansion coefficients, $(i = 1, 2)$,
T = temperature field, T_0 = reference temperature,
C_i = solutal concentrations, C_i^0 = reference solutal concentrations, $(i = 1, 2)$,
$\mathbf{v}$ = seepage velocity, μ_i = viscosity coefficients, $(i = 1, 2)$,
K = permeability, K_T = thermal diffusivity,
K_i = solute diffusivity, $(i = 1, 2)$.

To (2.1) we append the boundary conditions

$$(2.3)\qquad \begin{cases} T(x, y, 0, t) = T_l, \quad T(x, y, d, t) = T_u, \\ C_i(x, y, 0, t) = C_{il}, \quad C_i(x, y, d, t) = C_{iu}, \quad i = 1, 2, \\ \mathbf{v}\cdot\mathbf{k} = 0, \quad \text{on } z = 0, d, \end{cases}$$

where T_l, T_u, C_{il}, C_{iu} $(i = 1, 2)$, are positive constants such that $T_l > T_u$, $C_{1l} > C_{1u}$, and $C_{2l} < C_{2u}$.

The conduction solution is

$$(2.4)\qquad \begin{cases} \tilde{\mathbf{v}} = 0,\ \tilde{T} = T_l - \dfrac{\delta T}{d}z,\ \tilde{C}_1 = C_{1l} - \dfrac{\delta C_1}{d}z,\ \tilde{C}_2 = C_{2l} + \dfrac{\delta C_2}{d}z, \\ \delta T = T_l - T_u,\ \delta C_1 = C_{1l} - C_{1u},\ \delta C_2 = C_{2u} - C_{2l}, \\ \tilde{p} = p_0 + \rho_0 g z^2\left[-\dfrac{\alpha\delta T}{2d} + \beta_1\dfrac{\delta C_1}{2d} + \beta_2\dfrac{\delta C_2}{2d}\right] \\ \qquad - \rho_0 g z[1 - \alpha(T_l - T_0) + \beta_1(C_{1l} - C_1^0) + \beta_2(C_{2l} - C_2^0)], \end{cases}$$

with $\tilde{p}(z=0) = p_0 = \text{const}$. Setting

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}, \quad p = \tilde{p} + \pi, \quad T = \tilde{T} + \theta, \quad C_i = \tilde{C}_i + \gamma_i \quad (i=1,2), \tag{2.5}$$

with $\mathbf{u} = (u, v, w)$, introducing the dimensionless variables:

$$\begin{cases} t^* = \dfrac{K_T}{d^2} t,\ \mathbf{x}^* = \dfrac{\mathbf{x}}{d},\ \mathbf{u}^* = \dfrac{d}{K_T}\mathbf{u},\ \pi^* = \dfrac{K}{\mu_1 K_T}\pi,\ \gamma_i^* = \varphi_i \gamma_i, \\ \theta^* = T^\sharp \theta,\ T^\sharp = \Big(\dfrac{\alpha\rho_0 g K d}{\mu_1 K_T \delta T}\Big)^{1/2},\ \varphi_i = \Big(\dfrac{\beta_i \rho_0 g K d}{\mu_1 K_T P_i \delta C_i}\Big)^{1/2} \quad (i=1,2), \end{cases} \tag{2.6}$$

with $P_i = \dfrac{K_T}{K_i}$ Prandtl number for the solute S_i $(i=1,2)$ and omitting all the stars, (2.1) becomes:

$$\begin{cases} \nabla\pi = -\mathbf{u} + D_a \Delta\mathbf{u} + \mathscr{T}\mathbf{u}\times\mathbf{k} + (R\theta - R_1\gamma_1 - R_2\gamma_2)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = Rw + \Delta\theta, \\ P_1\Big(\dfrac{\partial\gamma_1}{\partial t} + \mathbf{u}\cdot\nabla\gamma_1\Big) = R_1 w + \Delta\gamma_1, \\ P_2\Big(\dfrac{\partial\gamma_2}{\partial t} + \mathbf{u}\cdot\nabla\gamma_2\Big) = -R_2 w + \Delta\gamma_2, \end{cases} \tag{2.7}$$

where

$$D_a = \frac{\mu_2 K}{\mu_1 d^2} \text{ (Darcy number)}, \quad \mathscr{T} = \frac{2\rho_0\omega K}{\mu_1} \text{ (Taylor-Darcy number)},$$

$$R = \Big(\frac{\alpha\rho_0 g K d \delta T}{\mu_1 K_T}\Big)^{1/2} \text{ (thermal Rayleigh number)},$$

$$R_i = \Big(\frac{\beta_i \rho_0 g K d P_i \delta C_i}{\mu_1 K_T}\Big)^{1/2} \text{ (solute Rayleigh numbers)}, \quad (i=1,2).$$

To (2.7), in view of (2.3), the boundary conditions:

$$u_z = v_z = w = \theta = \gamma_1 = \gamma_2 = 0, \quad \text{on } z = 0, 1, \tag{2.8}$$

are appended.

In the sequel we shall assume that:

i) the solutions of (2.7)–(2.8) are periodic in the x and y directions of period $\dfrac{2\pi}{a_x}$, $\dfrac{2\pi}{a_y}$ respectively;

ii) $\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1]$, is the periodicity cell;

iii) $\mathbf{u}, \theta, \gamma_1, \gamma_2 \in L^*(\Omega)$, where $L^*(\Omega)$ is the set of the functions belonging to $W^{2,2}(\Omega)$, verifying i), such that all their first derivatives and second spatial derivatives can be expanded in Fourier series absolutely, uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$.

3. The boundary value problem

This Section is devoted to solve the boundary value problem

$$\begin{cases} \nabla\pi = -\mathbf{u} + D_a\Delta\mathbf{u} + \mathscr{T}\mathbf{u}\times\mathbf{k} + (R\theta - R_1\gamma_1 - R_2\gamma_2)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ w = \theta = \gamma_1 = \gamma_2 = 0, \quad \text{on } z = 0, 1. \end{cases} \tag{3.1}$$

Since the set $\{\sin n\pi z\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2(0,1)$, then

$$\Gamma = \sum_{n=1}^{\infty}\Gamma_n = \sum_{n=1}^{\infty}\tilde{\Gamma}_n(x,y,t)\sin(n\pi z), \quad \forall\Gamma \in \{w, \theta, \gamma_1, \gamma_2\}. \tag{3.2}$$

On the other hand, by virtue of the periodicity in the x and y directions, one obtains:

$$\tilde{\Gamma}_n(x,y,t) = \Gamma_n^*(t)e^{i(a_x x + a_y y)}, \tag{3.3}$$

and hence

$$\Delta_1\Gamma_n = -a^2\Gamma_n, \quad \Delta\Gamma_n = -\xi_n\Gamma_n, \tag{3.4}$$

with

$$a^2 = a_x^2 + a_y^2, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta = \Delta_1 + \frac{\partial^2}{\partial z^2}, \quad \xi_n = a^2 + n^2\pi^2. \tag{3.5}$$

Lemma 3.1. *Let* $(\mathbf{u}, \theta, \gamma_1, \gamma_2) \in [L^*(\Omega)]^6$ *be a solution of* (3.1). *Then* $(\mathbf{u}, \theta, \gamma_1, \gamma_2)$ *is a solution of the boundary value problem*

$$\begin{cases} (D_a\Delta - 1)^2\Delta w + \mathscr{T}^2 w_{zz} + (D_a\Delta - 1)\Delta_1(R\theta - R_1\gamma_1 - R_2\gamma_2) = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ w = \theta = \gamma_1 = \gamma_2 = 0, \quad \textit{on } z = 0, 1. \end{cases} \tag{3.6}$$

Proof. On setting

$$\zeta = (\nabla\times\mathbf{u})\cdot\mathbf{k}, \tag{3.7}$$

from $(3.1)_2$ one obtains

$$\begin{cases} \Delta_1 u = -\dfrac{\partial^2 w}{\partial x \partial z} - \dfrac{\partial \zeta}{\partial y}, \\ \Delta_1 v = -\dfrac{\partial^2 w}{\partial y \partial z} + \dfrac{\partial \zeta}{\partial x}. \end{cases} \tag{3.8}$$

On the other hand, the third component of the curl of $(3.1)_1$ and the third component of the double curl of $(3.1)_1$, are respectively:

$$D_a \Delta \zeta - \zeta + \mathscr{T} w_z = 0, \tag{3.9}$$

$$-\Delta w + D_a \Delta \Delta w - \mathscr{T} \zeta_z + (R \Delta_1 \theta - R_1 \Delta_1 \gamma_1 - R_2 \Delta_1 \gamma_2) = 0. \tag{3.10}$$

From (3.10), on applying the operator $(D_a \Delta - 1)$, one obtains

$$(D_a \Delta - 1)^2 \Delta w - (D_a \Delta - 1) \mathscr{T} \zeta_z + (D_a \Delta - 1) \Delta_1 (R\theta - R_1 \gamma_1 - R_2 \gamma_2) = 0, \tag{3.11}$$

moreover, since (3.9), it follows that

$$(D_a \Delta - 1) \mathscr{T} \zeta_z = -\mathscr{T}^2 w_{zz}. \tag{3.12}$$

Hence, substituting (3.12) in (3.11), one has that a solution of (3.1) is also a solution of (3.6).

Theorem 3.1. *Let* $(u_n, v_n, w_n, \theta_n, \gamma_{1n}, \gamma_{2n}) \in [L^*(\Omega)]^6$ *verifying* $(3.6)_3$. *Then, the solutions of the boundary value problem* (3.6) *are given by*

$$\mathbf{u} = \sum_{n=1}^{\infty} [u_n \mathbf{i} + v_n \mathbf{j} + w_n \mathbf{k}], \tag{3.13}$$

where

$$\begin{cases} u_n = \dfrac{1}{a^2} \dfrac{\partial^2 w_n}{\partial x \partial z} + \dfrac{\mathscr{T}}{a^2(1 + D_a \xi_n)} \dfrac{\partial^2 w_n}{\partial y \partial z}, \\ v_n = \dfrac{1}{a^2} \dfrac{\partial^2 w_n}{\partial y \partial z} - \dfrac{\mathscr{T}}{a^2(1 + D_a \xi_n)} \dfrac{\partial^2 w_n}{\partial x \partial z}, \\ w_n = \eta_n (R\theta_n - R_1 \gamma_{1n} - R_2 \gamma_{2n}), \\ \eta_n = \dfrac{a^2(1 + D_a \xi_n)}{\xi_n (1 + D_a \xi_n)^2 + n^2 \pi^2 \mathscr{T}^2}. \end{cases} \tag{3.14}$$

Proof. In view of (3.8) one obtains

$$\begin{cases} (D_a \Delta - 1) \Delta_1 u = -(D_a \Delta - 1) \dfrac{\partial^2 w}{\partial x \partial z} - (D_a \Delta - 1) \dfrac{\partial \zeta}{\partial y}, \\ (D_a \Delta - 1) \Delta_1 v = -(D_a \Delta - 1) \dfrac{\partial^2 w}{\partial y \partial z} + (D_a \Delta - 1) \dfrac{\partial \zeta}{\partial x}, \end{cases} \tag{3.15}$$

moreover, by virtue of (3.9), since $(D_a\Delta - 1)\zeta_y = -\mathscr{T} w_{yz}$ and $(D_a\Delta - 1)\zeta_x = -\mathscr{T} w_{xz}$, system (3.15) becomes

$$
\begin{cases} (D_a\Delta - 1)\Delta_1 u = -(D_a\Delta - 1)\dfrac{\partial^2 w}{\partial x \partial z} + \mathscr{T}\dfrac{\partial^2 w}{\partial y \partial z}, \\ (D_a\Delta - 1)\Delta_1 v = -(D_a\Delta - 1)\dfrac{\partial^2 w}{\partial y \partial z} - \mathscr{T}\dfrac{\partial^2 w}{\partial x \partial z}. \end{cases} \tag{3.16}
$$

In view of (3.2)–(3.4), one has that the first two components of **u**, are given by

$$
\begin{aligned} u &= \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \tilde{u}_n(x, y, t)\sin(n\pi z), \\ v &= \sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \tilde{v}_n(x, y, t)\sin(n\pi z), \end{aligned} \tag{3.17}
$$

where

$$
\begin{cases} u_n = \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial x \partial z} + \dfrac{\mathscr{T}}{a^2(1 + D_a\xi_n)}\dfrac{\partial^2 w_n}{\partial y \partial z}, \\ v_n = \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial y \partial z} - \dfrac{\mathscr{T}}{a^2(1 + D_a\xi_n)}\dfrac{\partial^2 w_n}{\partial x \partial z}. \end{cases} \tag{3.18}
$$

Hence the solution of $(3.6)_1$, (3.16) is

$$
\mathbf{u} = \sum_{n=1}^{\infty}[u_n\mathbf{i} + v_n\mathbf{j} + w_n\mathbf{k}], \tag{3.19}
$$

with

$$
\begin{cases} w_n = \eta_n(R\theta_n - R_1\gamma_{1n} - R_2\gamma_{2n}), \\ \eta_n = \dfrac{a^2(1 + D_a\xi_n)}{\xi_n(1 + D_a\xi_n)^2 + n^2\pi^2\mathscr{T}^2}. \end{cases} \tag{3.20}
$$

One can easily verify that **u**, given in (3.19), satisfies also $(3.6)_2$ and hence the thesis is hold.

Remark 3.1. *In view of Theorem* 3.1, *it follows that the independent fields of* (2.7) *are reduced to the three fields* θ, γ_1, γ_2.

4. Introduction of new fields and scaling

By virtue of Remark 3.1, in order to study the fluid motion, we can confine ourselves to consider the last three equations in (2.7), i.e.

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial\theta}{\partial t} = \displaystyle\sum_{n=1}^{\infty}[(R^2\eta_n - \xi_n)\theta_n - RR_1\eta_n\gamma_{1n} - RR_2\eta_n\gamma_{2n}] - \mathbf{u}\cdot\nabla\theta, \\ \dfrac{\partial\gamma_1}{\partial t} = \displaystyle\sum_{n=1}^{\infty}\left[\frac{RR_1\eta_n}{P_1}\theta_n - \frac{R_1^2\eta_n + \xi_n}{P_1}\gamma_{1n} - \frac{R_1R_2\eta_n}{P_1}\gamma_{2n}\right] - \mathbf{u}\cdot\nabla\gamma_1, \\ \dfrac{\partial\gamma_2}{\partial t} = \displaystyle\sum_{n=1}^{\infty}\left[-\frac{RR_2\eta_n}{P_2}\theta_n + \frac{R_1R_2\eta_n}{P_2}\gamma_{1n} + \frac{R_2^2\eta_n - \xi_n}{P_2}\gamma_{2n}\right] - \mathbf{u}\cdot\nabla\gamma_2, \end{cases}$$

which on setting

$$\text{(4.2)} \qquad \mathrm{L}_n = \begin{pmatrix} R^2\eta_n - \xi_n & -RR_1\eta_n & -RR_2\eta_n \\ \dfrac{RR_1\eta_n}{P_1} & -\dfrac{R_1^2\eta_n + \xi_n}{P_1} & -\dfrac{R_1R_2\eta_n}{P_1} \\ -\dfrac{RR_2\eta_n}{P_2} & \dfrac{R_1R_2\eta_n}{P_2} & \dfrac{R_2^2\eta_n - \xi_n}{P_2} \end{pmatrix}$$

become

$$\text{(4.3)} \qquad \frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \gamma_1\\ \gamma_2\end{pmatrix} = \sum_{n=1}^{\infty}\mathrm{L}_n\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta\\ \mathbf{u}\cdot\nabla\gamma_1\\ \mathbf{u}\cdot\nabla\gamma_2\end{pmatrix},$$

i.e.

$$\text{(4.4)} \qquad \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} = \sum_{n=1}^{\infty}\mathrm{L}_n\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \sum_{n=1}^{\infty}\begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\gamma_{1n}\\ \mathbf{u}\cdot\nabla\gamma_{2n}\end{pmatrix},$$

under the initial-boundary conditions

$$\text{(4.5)} \qquad \begin{cases}(\theta_n)_{t=0} = \theta_{0n}, \quad (\gamma_{in})_{t=0} = \gamma_{0in}, \quad i = 1,2,\\ \theta_n = w_n = \gamma_{in} = 0, \quad (i = 1,2), \quad \text{on } z = 0,1,\end{cases}$$

$\theta_0 = \sum_{n=1}^{\infty}\theta_{0n}$; $\gamma_{i0} = \sum_{n=1}^{\infty}\gamma_{i0n}$, $(i = 1,2)$, being the initial values of θ and γ_i, $(i = 1,2)$.

REMARK 4.1. *Let us consider the "nonlinear evolution system of the n-th Fourier component* $(\theta_n, \gamma_{1n}, \gamma_{2n})$ *of the perturbation* $(\theta, \gamma_1, \gamma_2)$*"* [32], *i.e.*

$$\text{(4.6)} \qquad \frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} = L_n\begin{pmatrix}\theta_n\\ \gamma_{1n}\\ \gamma_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\gamma_{1n}\\ \mathbf{u}\cdot\nabla\gamma_{2n}\end{pmatrix}.$$

Since the uniqueness theorem for (4.4)–(4.5) (*cfr. Appendix*) *implies, as particular case, the uniqueness theorem for* (4.6)–(4.5) (*cfr.* [26] *for details*)*—by inspection of* (4.4) *and* (4.6)*—it follows that the global nonlinear stability of the null solution of*

(4.3) *is guaranteed if exist conditions—independent of n—guaranteeing the global nonlinear stability of* (4.6).

On introducing the two new fields:

$$\Phi_1 = R_1\theta - P_1 R\gamma_1, \quad \Phi_2 = R_2\theta + P_2 R\gamma_2, \tag{4.7}$$

system (4.4) becomes

$$\sum_{n=1}^{\infty} \frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} = \sum_{n=1}^{\infty} \mathscr{L}_n \begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} - \sum_{n=1}^{\infty}\begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\Phi_{1n}\\ \mathbf{u}\cdot\nabla\Phi_{2n}\end{pmatrix}, \tag{4.8}$$

with

$$\mathscr{L}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \dfrac{R_1}{P_1}\eta_n & -\dfrac{R_2}{P_2}\eta_n \\ -\dfrac{R_1}{P_1}(P_1-1)\xi_n & -\dfrac{\xi_n}{P_1} & 0 \\ -\dfrac{R_2}{P_2}(P_2-1)\xi_n & 0 & -\dfrac{\xi_n}{P_2} \end{pmatrix}, \tag{4.9}$$

and

$$R^* = R^2 - \frac{R_1^2}{P_1} + \frac{R_2^2}{P_2}. \tag{4.10}$$

To (4.8)–(4.9) we append the initial-boundary conditions

$$\begin{cases} (\theta_n)_{t=0} = \theta_{0n}, \quad (\Phi_{in})_{t=0} = \Phi_{0in}, \quad i = 1,2, \\ \theta_n = w_n = \Phi_{in} = 0, \quad (i=1,2), \quad \text{on } z = 0,1. \end{cases} \tag{4.11}$$

Setting

$$\theta_n^* = \theta_n, \quad \Phi_i^* = \frac{1}{\mu_{in}}\Phi_i, \quad (i=1,2), \tag{4.12}$$

with

$$\mu_{1n} = \sqrt{|1-P_1|\frac{\xi_n}{\eta_n}}, \quad \mu_{2n} = \sqrt{|P_2-1|\frac{\xi_n}{\eta_n}}, \tag{4.13}$$

and omitting the stars, the nonlinear evolution system of the n-th Fourier component of (4.8) is given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} = \tilde{\mathscr{L}}_n \begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta_n\\ \mathbf{u}\cdot\nabla\Phi_{1n}\\ \mathbf{u}\cdot\nabla\Phi_{2n}\end{pmatrix}, \tag{4.14}$$

under the i.b.c. (4.11) and $\tilde{\mathscr{L}}_n$ given by

$$
\tilde{\mathscr{L}}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \dfrac{R_1}{P_1}\sqrt{|1-P_1|\xi_n\eta_n} & -\dfrac{R_2}{P_2}\sqrt{|P_2-1|\xi_n\eta_n} \\ \dfrac{R_1(1-P_1)\sqrt{\xi_n\eta_n}}{P_1\sqrt{|1-P_1|}} & -\dfrac{\xi_n}{P_1} & 0 \\ -\dfrac{R_2(P_2-1)\sqrt{\xi_n\eta_n}}{P_2\sqrt{|P_2-1|}} & 0 & -\dfrac{\xi_n}{P_2} \end{pmatrix}. \tag{4.15}
$$

5. Global nonlinear stability of the conduction solution

Setting

$$
A^* = \inf_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \frac{\xi_n}{\eta_n}, \quad E = \sum_{n=1}^{\infty} E_n, \tag{5.1}
$$

with

$$
E_n = \int_\Omega (\theta_n^2 + \Phi_{1n}^2 + \Phi_{2n}^2)\, d\Omega, \tag{5.2}
$$

in view of

$$
\int_\Omega \mathbf{u}\cdot\nabla\left(\frac{\varphi_n^2}{2}\right) d\Omega = 0, \quad \varphi_n \in \{\theta_n, \Phi_{1n}, \Phi_{2n}\}, \tag{5.3}
$$

it follows that, along the solutions of (4.14), $\dfrac{dE_n}{dt}$ is given by

$$
\frac{dE_n}{dt} = \int_\Omega \eta_n Q_n \, d\Omega, \tag{5.4}
$$

where

1) in the case $\{P_1 \le 1, P_2 \ge 1\}$, Q_n reduces to

$$
\begin{aligned} Q_n = \left(R^* - \frac{\xi_n}{\eta_n}\right)\theta_n^2 - \sum_{i=1}^{2} \frac{\xi_n}{\eta_n P_i}\Phi_{in}^2 + \frac{2R_1}{P_1\eta_n}\sqrt{(1-P_1)\xi_n\eta_n}\,\theta_n\Phi_{1n} \\ - 2\frac{R_2}{P_2\eta_n}\sqrt{(P_2-1)\xi_n\eta_n}\,\theta_n\Phi_{2n}; \end{aligned} \tag{5.5}
$$

2) in the case $\{P_1 \ge 1, P_2 \le 1\}$, Q_n reduces to

$$
Q_n = \left(R^* - \frac{\xi_n}{\eta_n}\right)\theta_n^2 - \frac{1}{P_1}\frac{\xi_n}{\eta_n}\Phi_{1n}^2 - \frac{1}{P_2}\frac{\xi_n}{\eta_n}\Phi_{2n}^2; \tag{5.6}
$$

3) in the case $\{P_1 \le 1, P_2 \le 1\}$, Q_n reduces to

$$Q_n = \Big(R^* - \frac{\xi_n}{\eta_n}\Big)\theta_n^2 - \frac{1}{P_1}\frac{\xi_n}{\eta_n}\Phi_{1n}^2 - \frac{1}{P_2}\frac{\xi_n}{\eta_n}\Phi_{2n}^2 + \frac{2R_1}{P_1\eta_n}\sqrt{(1-P_1)\xi_n\eta_n}\,\theta_n\Phi_{1n}; \tag{5.7}$$

4) in the case $\{P_1 \ge 1, P_2 \ge 1\}$, Q_n reduces to

$$Q_n = \Big(R^* - \frac{\xi_n}{\eta_n}\Big)\theta_n^2 - \frac{1}{P_1}\frac{\xi_n}{\eta_n}\Phi_{1n}^2 - \frac{1}{P_2}\frac{\xi_n}{\eta_n}\Phi_{2n}^2 - 2\frac{R_2}{P_2\eta_n}\sqrt{(P_2-1)\xi_n\eta_n}\,\theta_n\Phi_{2n}. \tag{5.8}$$

REMARK 5.1. *On accounting for* $(5.1)_2$–(5.3)*, it follows that, along the solutions of*

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \Phi_1\\ \Phi_2\end{pmatrix} = \sum_{n=1}^{\infty}\tilde{\mathscr{L}}_n\begin{pmatrix}\theta_n\\ \Phi_{1n}\\ \Phi_{2n}\end{pmatrix} - \begin{pmatrix}\mathbf{u}\cdot\nabla\theta\\ \mathbf{u}\cdot\nabla\Phi_1\\ \mathbf{u}\cdot\nabla\Phi_2\end{pmatrix}, \tag{5.9}$$

the temporal derivative of E, is given by

$$\frac{dE}{dt} = \sum_{n=1}^{\infty}\frac{dE_n}{dt}. \tag{5.10}$$

Hence, if $\frac{dE_n}{dt}$ *is negative definite along the solutions of* (4.14)*,* $\forall n \in \mathbb{N}$*, then* $\frac{dE}{dt}$ *is negative definite along the solutions of* (5.9).

We begin by analyzing the stability in the case $\{P_1 \le 1, P_2 \ge 1\}$, i.e. when the matrix $\tilde{\mathscr{L}}_n$ is symmetric. The following theorem holds.

THEOREM 5.1. *The conduction solution is linearly stable and nonlinearly, globally, asymptotically stable, if and only if* (1.1) *holds.*

PROOF. In view of (4.15), it follows that if and only if $\{P_1 \le 1, P_2 \ge 1\}$, $\tilde{\mathscr{L}}_n$ is symmetric and it is given by

$$\tilde{\mathscr{L}}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} & -\frac{R_2}{P_2}\sqrt{(P_2-1)\xi_n\eta_n}\\ \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} & -\frac{\xi_n}{P_1} & 0\\ -\frac{R_2}{P_2}\sqrt{(P_2-1)\xi_n\eta_n} & 0 & -\frac{\xi_n}{P_2}\end{pmatrix}. \tag{5.11}$$

Introducing the $L^2(\Omega)$-norm E_n given in (5.2), it follows that, along the solution of (4.14), $\frac{dE_n}{dt}$ is given by (5.4) with Q_n given by (5.5). Hence Q_n is negative definite $\forall n \in \mathbb{N}$ if and only if

$$R^* < \frac{\xi_n}{\eta_n}, \quad \det \tilde{A}_{33} > 0, \quad \det \tilde{\mathscr{L}}_n < 0, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \tag{5.12}$$

with

$$\tilde{A}_{33} = \begin{vmatrix} R^*\eta_n - \xi_n & \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} \\ \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} & -\frac{\xi_n}{P_1} \end{vmatrix}. \tag{5.13}$$

In view of Remark 5.1 and

$$\begin{cases} \det \tilde{A}_{33} = -\frac{\xi_n\eta_n}{P_1}\left(R^* + \frac{R_1^2}{P_1} - R_1^2 - \frac{\xi_n}{\eta_n}\right), \\ \det \tilde{\mathscr{L}}_n = \frac{\xi_n^2\eta_n}{P_1P_2}\left(R^* + R_2^2 - R_1^2 - \frac{R_2^2}{P_1} + \frac{R_1^2}{P_1} - \frac{\xi_n}{\eta_n}\right), \end{cases} \tag{5.14}$$

one has that (5.12) are implied by $(1.1)_1$. It remains to show that (1.1) is also necessary. This follows immediately since $\det \tilde{\mathscr{L}}_n < 0$ is one of the Routh-Hurwitz conditions [33] requested for all eigenvalues of $\tilde{\mathscr{L}}_n$ have negative real part.

Now we prove that the conditions (1.2)–(1.4) are sufficient to ensure the global, nonlinear, asymptotic stability of the conduction solution in the partial skew-symmetric cases.

Theorem 5.2. *The global nonlinear stability of the conduction solution is guaranteed by* (1.2)–(1.4).

Proof. In the case $\{P_1 \geq 1, P_2 \leq 1\}$, $\tilde{\mathscr{L}}_n$ becomes

$$\tilde{\mathscr{L}}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \frac{R_1}{P_1}\sqrt{(P_1-1)\xi_n\eta_n} & -\frac{R_2}{P_2}\sqrt{(1-P_2)\xi_n\eta_n} \\ -\frac{R_1}{P_1}\sqrt{(P_1-1)\xi_n\eta_n} & -\frac{\xi_n}{P_1} & 0 \\ \frac{R_2}{P_2}\sqrt{(1-P_2)\xi_n\eta_n} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix} \tag{5.15}$$

Hence, the temporal derivative of E_n, along the solution of (4.14) is given by (5.4) with Q_n given by (5.6). It is easy to prove that Q_n is negative definite, $\forall n \in \mathbb{N}$, when $(1.2)_1$ holds.

In the case $\{P_1 \le 1, P_2 \le 1\}$, $\tilde{\mathscr{L}}_n$ reduces to

$$(5.16)\quad \tilde{\mathscr{L}}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} & -\frac{R_2}{P_2}\sqrt{(1-P_2)\xi_n\eta_n} \\ \frac{R_1}{P_1}\sqrt{(1-P_1)\xi_n\eta_n} & -\frac{\xi_n}{P_1} & 0 \\ \frac{R_2}{P_2}\sqrt{(1-P_2)\xi_n\eta_n} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}$$

and hence the temporal derivative of E_n along the solutions of system (4.14) is given by (5.4) with Q_n as in (5.7). Then, the condition $(1.3)_1$ guarantees that Q_n is negative definite $\forall n \in \mathbb{N}$.

In the last case $\{P_1 \ge 1, P_2 \ge 1\}$, one has that

$$(5.17)\quad \tilde{\mathscr{L}}_n = \begin{pmatrix} R^*\eta_n - \xi_n & \frac{R_1}{P_1}\sqrt{(P_1-1)\xi_n\eta_n} & -\frac{R_2}{P_2}\sqrt{(P_2-1)\xi_n\eta_n} \\ -\frac{R_1}{P_1}\sqrt{(P_1-1)\xi_n\eta_n} & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2}{P_2}\sqrt{(P_2-1)\xi_n\eta_n} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}$$

and hence the temporal derivative of E_n along the solutions of system (4.14) is given by (5.4) with Q_n given by (5.8). Therefore $(1.4)_1$ assures that Q_n is negative definite $\forall n \in \mathbb{N}$.

Remark 5.2. *For any values of the Prandtl numbers, it follows that*

$$(5.18)\qquad \det \tilde{\mathscr{L}}_n < 0, \quad \forall n \in \mathbb{N} \Leftrightarrow R^2 - R_1^2 + R_2^2 < A^*.$$

Since (5.18) *is one of the Routh-Hurwitz conditions, necessary for all eigenvalues of* $\tilde{\mathscr{L}}_n$ *have negative real part, then it follows that*

$$(5.19)\qquad R^2 < R_1^2 - R_2^2 + A^*,$$

is necessary for the linear stability of the conduction solution, for any values of the Prandtl numbers.

6. Estimates of the stability thresholds

Setting

$$(6.1)\qquad \mathscr{A}(n^2, a^2, D_a, \mathscr{T}) = \frac{\xi_n^2(1+D_a\xi_n)}{a^2} + \frac{n^2\pi^2\mathscr{T}^2\xi_n}{a^2(1+D_a\xi_n)},$$

by virtue of $(3.5)_4$, $(3.20)_2$, it follows that A^* in (5.1) is given by

$$(6.2)\qquad A^* = \inf_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \mathscr{A}(n^2, a^2, D_a, \mathscr{T}).$$

REMARK 6.1. *We remark that:*

i) *since*

$$\mathscr{A}(n^2, a^2, D_a, \mathscr{T}) > \frac{\xi_n^2(1 + D_a\xi_n)}{a^2} = \frac{\xi_n^2}{a^2} + \frac{D_a\xi_n^3}{a^2} > \frac{\xi_n^2}{a^2}, \tag{6.3}$$

i.e.

$$A^* > \min \mathscr{A}(n^2, a^2, 0, 0) = 4\pi^2, \tag{6.4}$$

either rotation or Brinkman terms have stabilizing effect on the conduction solution;

ii) *in view of*

$$\frac{\partial \mathscr{A}}{\partial n^2} > 0, \tag{6.5}$$

one has that

$$A^* = \min \mathscr{A}_1, \tag{6.6}$$

with

$$\mathscr{A}_1 = \mathscr{A}(1, a^2, D_a, \mathscr{T}) = \frac{(a^2 + \pi^2)^2[1 + D_a(a^2 + \pi^2)]}{a^2} + \frac{(a^2 + \pi^2)\pi^2\mathscr{T}^2}{a^2[1 + D_a(a^2 + \pi^2)]} \tag{6.7}$$

iii) *since*

$$\frac{\partial \mathscr{A}_1}{\partial \mathscr{T}} = \frac{2(a^2 + \pi^2)\pi^2\mathscr{T}}{a^2[1 + D_a(a^2 + \pi^2)]} > 0, \tag{6.8}$$

$\mathscr{A}_1$ *is an increasing function of* $\mathscr{T}$.

REMARK 6.2. *We remark that*

i) *in view of*

$$\frac{\partial \mathscr{A}_1}{\partial D_a} = \frac{(a^2 + \pi^2)^2}{a^2}\left(a^2 + \pi^2 - \frac{\pi^2\mathscr{T}^2}{[1 + D_a(a^2 + \pi^2)]^2}\right), \tag{6.9}$$

it follows that $\mathscr{A}_1$ *for*

$$D_a > D_a^* = \frac{\mathscr{T} - 1}{\pi^2}, \tag{6.10}$$

is an increasing function of D_a.

ii) *In the absence of Brinkman term, i.e.* $D_a = 0$, $\mathscr{A}_1$ *reduces to*

$$\mathscr{A}_2 = \mathscr{A}_1(a^2, 0, \mathscr{T}) = \frac{(a^2+\pi^2)\pi^2\mathscr{T}^2 + (a^2+\pi^2)^2}{a^2}; \tag{6.11}$$

and the minimum is reached for

$$a^2 = (a^2)_* = \pi^2(1+\mathscr{T}^2)^{1/2}, \tag{6.12}$$

and it is given by

$$A^*_{\mathscr{T}} = \pi^2(1+\sqrt{1+\mathscr{T}^2})^2. \tag{6.13}$$

Hence, the global stability of the conduction solution is guaranteed by (1.1)–(1.4) *with* (6.13) *at the place of* A^*.

iii) *In the absence of rotation, i.e.* $\mathscr{T} = 0$, $\mathscr{A}_1$ *is given by*

$$\mathscr{A}_3 = \mathscr{A}_1(a^2, D_a, 0) = \frac{D_a(a^2+\pi^2)^3 + (a^2+\pi^2)^2}{a^2}. \tag{6.14}$$

Since

$$X^* = \frac{3D_a\pi^2 - 1 + \sqrt{(3D_a\pi^2-1)^2 + 16\pi^2 D_a}}{4D_a}, \tag{6.15}$$

minimizes $\mathscr{A}_3$, *the global stability of the conduction solution is guaranteed by* (1.1)–(1.4) *with*

$$A^*_{D_a} = \frac{(X^*)^2(1+D_aX^*)}{X^*-\pi^2}, \tag{6.16}$$

at the place of A^*.

iv) *Evaluating, together and in closed form, both the stabilizing effects of rotation and Brinkman terms, in the case* $\mathscr{T} \leq 1$, *the global stability of the conduction solution is guaranteed by* (1.1)–(1.4) *with*

$$\frac{A^*_{\mathscr{T}} + A^*_{D_a}}{2}, \tag{6.17}$$

at the place of A^* (*for the proof see* [27]).

In Tables 1–3 some numerical values of $A^*_{\mathscr{T}}$ (when $D_a = 0$), $A^*_{D_a}$ (when $\mathscr{T} = 0$) and of $\frac{A^*_{\mathscr{T}} + A^*_{D_a}}{2}$ (when $\mathscr{T} \leq 1$) are, respectively, listed. In Figures 1 and 2, the graphics of $A^*_{\mathscr{T}}$ and $A^*_{D_a}$ are showed.

$\mathscr{T}$	D_a	$A^*_{\mathscr{T}}$
0	0	$4\pi^2$
0.1	0	39.6756
0.2	0	40.2641
0.5	0	44.2757
1.2	0	64.7851
1.5	0	77.5312
2	0	103.356

Table 1. Numerical values of $A^*_{\mathscr{T}}$.

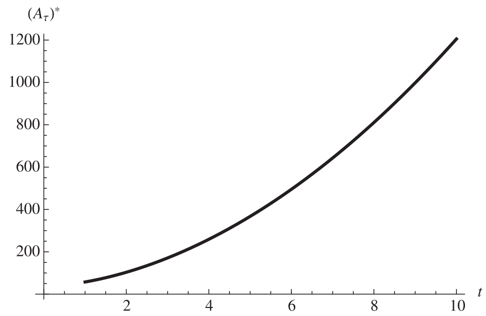

Figure 1. Graphic of $A^*_{\mathscr{T}}$ as function of $\mathscr{T}$

$\mathscr{T}$	D_a	$A^*_{D_a}$
0	0	$4\pi^2$
0	0.1	108.573
0	0.5	372.722
0	1	701.689
0	1.5	1030.52
0	2	1359.31
0	2.5	1688.09
0	3	2016.87

Table 2. Numerical values of $A^*_{D_a}$.

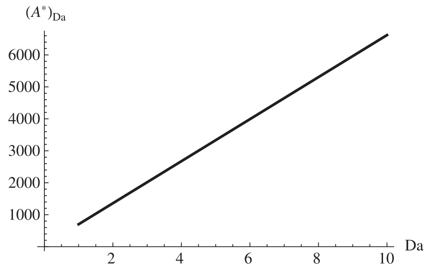

Figure 2. Graphic of $A^*_{D_a}$ as function of D_a

$\mathscr{T}$	D_a	$\frac{A^*_{\mathscr{T}} + A^*_{D_a}}{2}$
0	0	$4\pi^2$
0.1	0.1	74.1243
0.2	0.5	206.493
0.5	0.5	208.499
0.3	1	371.462
0.7	1.5	539.595
0.9	1.5	542.405
0.9	2	706.8

Table 3. Numerical values of $\frac{A^*_{\mathscr{T}} + A^*_{D_a}}{2}$ when $\mathscr{T} \leq 1$.

We furnish here some new estimates of A^* not present in [27].

LEMMA 6.1. *Let*

$$1 < \mathcal{T} < 1 + D_a\pi^2. \tag{6.18}$$

Setting

$$A^*_{D^*_a} = \frac{(Y^*)^2[1 + D^*_a Y^*]}{Y^* - \pi^2}, \tag{6.19}$$

with D^*_a *given by* (6.10) *and* Y^* *given by* (6.15) (*with* $D_a = D^*_a$)*, then the global stability of the conduction solution is guaranteed by* (1.1)–(1.4) *with*

$$\max\{A^*_{\mathcal{T}}; A^*_{D^*_a}\}, \tag{6.20}$$

at the place of A^*.

PROOF. In view of (6.9) it easily follows that when $\mathcal{T} < 1$, $\mathscr{A}_1$ is an increasing function of $D_a(> 0)$ and takes the minimum at D^*_a given by (6.10). Hence by virtue of (6.14), one obtains

$$\begin{aligned} \mathscr{A}_1(a^2, D_a, \mathcal{T}) &> \mathscr{A}_1(a^2, D^*_a, \mathcal{T}) > \mathscr{A}_1(a^2, D^*_a, 0) \\ &= \mathscr{A}_3(a^2, D^*_a, 0) = \frac{D^*_a(a^2+\pi^2)^3 + (a^2+\pi^2)^2}{a^2}. \end{aligned} \tag{6.21}$$

In view of

$$a^2\frac{\partial\mathscr{A}_3}{\partial a^2} = \frac{Y}{Y-\pi^2}[2D^*_a Y^2 + (1 - 3D^*_a\pi^2)Y - 2\pi^2], \tag{6.22}$$

with $Y = a^2 + \pi^2$, it follows that the minimum of $\mathscr{A}_3(a^2, D^*_a, 0)$ is reached for

$$Y^* = \frac{3D^*_a\pi^2 - 1 + \sqrt{(3D^*_a\pi^2 - 1)^2 + 16\pi^2 D^*_a}}{4D^*_a}, \tag{6.23}$$

and it is given by

$$A^*_{D^*_a} = \frac{(Y^*)^2[1 + D^*_a Y^*]}{Y^* - \pi^2}. \tag{6.24}$$

In view of $\mathcal{T} < 1 + D_a\pi^2$ and ii) of Remark 6.2, it follows that

$$\mathscr{A}_1(a^2, D_a, \mathcal{T}) > \mathscr{A}_1(a^2, 0, \mathcal{T}) > \mathscr{A}_2 > A^*_{\mathcal{T}} \tag{6.25}$$

Collecting (6.21) and (6.25), (6.20) immediately follows.

The graphics of $A^*_{\mathscr{T}}$ and $A^*_{D_a^*}$, as functions of $\mathscr{T}$, are showed in Fig. 3.
Denoting with

$$T_1 \approx 1.60013, \quad T_2 \approx 3.38337, \tag{6.26}$$

it follows that

$$\begin{cases} \mathscr{T} < T_1 \quad \Rightarrow \quad A^*_{\mathscr{T}} > A^*_{D_a^*}, \\ T_1 < \mathscr{T} < T_2 \quad \Rightarrow \quad A^*_{D_a^*} > A^*_{\mathscr{T}}, \\ \mathscr{T} > T_2 \quad \Rightarrow \quad A^*_{\mathscr{T}} > A^*_{D_a^*}. \end{cases} \tag{6.27}$$

Hence, in view of Lemma 6.1 and (6.27), the following theorems hold true.

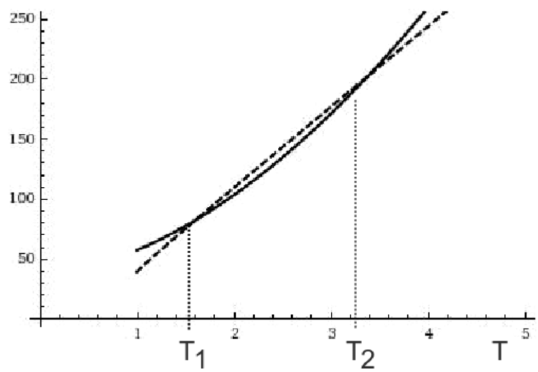

Figure 3. Graphics of $A^*_{\mathscr{T}}$ (continue line) and $A^*_{D_a^*}$ (dashed line).

Theorem 6.1. *Let either*

$$1 < \mathscr{T} < T_1 < 1 + D_a \pi^2, \tag{6.28}$$

or

$$T_2 < \mathscr{T} < 1 + D_a \pi^2. \tag{6.29}$$

Then the global stability of the conduction solution is guaranteed by (1.1)–(1.4) *with* $A^*_{\mathscr{T}}$ *at the place of* A^*.

In Table 4 some numerical values of $A^*_{\mathscr{T}}$, either in the case (6.28) or in the case (6.29), are listed.

$\mathscr{T}$	D_a	D_a^*	$A^*_{\mathscr{T}}$
1.1	0.5	0.0101321	61.0259
1.2	1	0.0202642	64.7851
1.4	2.5	0.0405285	73.0443
1.5	3.5	0.0506606	77.5312
3.4	4	0.243171	203.788
3.5	5.5	0.253303	212.494
4	8	0.303964	259.04
4.5	10	0.354624	310.592

Table 4. Numerical values of $A^*_{\mathscr{T}}$ either in the case (6.28) or in the case (6.29).

THEOREM 6.2. *Let*

$$\mathscr{T}_1 < \mathscr{T} < \mathscr{T}_2 < 1 + D_a\pi^2. \tag{6.30}$$

Then the global stability of the conduction solution is guaranteed by (1.1)–(1.4) *with* $A^*_{D_a^*}$ *at the place of* A^*.

$\mathscr{T}$	D_a	D_a^*	$A^*_{D_a^*}$
1.61	0.5	0.0618059	82.9319
1.8	1.5	0.0810569	95.8946
2	1.7	0.101321	109.456
2.5	2	0.151982	143.152
2.7	3	0.172246	156.581
3	3.5	0.202642	176.695
3.1	4	0.212774	183.393
3.3	5	0.233039	196.781

Table 5. Numerical values of $A^*_{D_a^*}$ in the case (6.30).

In Table 5 some numerical values of $A^*_{D_a^*}$, in the case (6.30), are listed.

7. APPENDIX

In this Section we prove an uniqueness theorem for system (2.7) with the initial-boundary conditions (7.2).

THEOREM 7.1 ***(Uniqueness Theorem)***. *The problem*

$$
(7.1)\qquad \begin{cases} \nabla\pi = -\mathbf{u} + D_a\Delta\mathbf{u} + \mathscr{T}\mathbf{u}\times\mathbf{k} + (R\theta - R_1\gamma_1 - R_2\gamma_2)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = Rw + \Delta\theta, \\ P_1\Big(\dfrac{\partial\gamma_1}{\partial t} + \mathbf{u}\cdot\nabla\gamma_1\Big) = R_1 w + \Delta\gamma_1, \\ P_2\Big(\dfrac{\partial\gamma_2}{\partial t} + \mathbf{u}\cdot\nabla\gamma_2\Big) = -R_2 w + \Delta\gamma_2, \end{cases}
$$

with the initial-boundary conditions

$$
(7.2)\qquad \begin{cases} \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \pi(\mathbf{x},0) = \pi_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), \\ \gamma_i(\mathbf{x},0) = \gamma_{i0}(\mathbf{x}), \quad i = 1,2, \\ u_z = v_z = w = \theta = \gamma_1 = \gamma_2 = 0, \quad on\ z = 0,1, \end{cases}
$$

can admit an unique solution $(\mathbf{u},\theta,\gamma_1,\gamma_2,\pi) \in [L^*(\Omega)]^7$.

PROOF. Even if the proof of the theorem can be obtained following, step by step, the procedure used in [23], for the sake of completeness, we give here a sketch of the proof. Let $(\mathbf{u},\theta,\gamma_1,\gamma_2,\pi)$ and $(\tilde{\mathbf{u}},\tilde{\theta},\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\pi})$ be two solutions of (7.1)–(7.2). Setting

$$
(7.3)\qquad \Psi = \tilde{\theta} - \theta, \quad \mathbf{U} = \tilde{\mathbf{u}} - \mathbf{u}, \quad \Pi^* = \tilde{\Pi} - \Pi, \quad \Psi_i = \tilde{\gamma}_i - \gamma_i, \quad (i = 1,2),
$$

it follows that

$$
(7.4)\qquad \begin{cases} \nabla\Pi^* = -\mathbf{U} + D_a\Delta\mathbf{U} + \mathscr{T}\mathbf{U}\times\mathbf{k} + (R\Psi - R_1\Psi_1 - R_2\Psi_2)\mathbf{k}, \\ \nabla\cdot\mathbf{U} = 0, \\ \Psi_t + \tilde{\mathbf{u}}\cdot\nabla\Psi + \mathbf{U}\cdot\nabla\theta = \mathbf{U}\cdot\mathbf{k} + \Delta\Psi, \\ P_1(\Psi_{1t} + \tilde{\mathbf{u}}\cdot\nabla\Psi_1 + \mathbf{U}\cdot\nabla\gamma_1) = R_1\mathbf{U}\cdot\mathbf{k} + \Delta\Psi_1, \\ P_2(\Psi_{2t} + \tilde{\mathbf{u}}\cdot\nabla\Psi_2 + \mathbf{U}\cdot\nabla\gamma_2) = -R_2\mathbf{U}\cdot\mathbf{k} + \Delta\Psi_2, \end{cases}
$$

with $\mathbf{U} = (U,V,W)$ and

$$
(7.5)\qquad U_z = V_z = W = \Psi = \Psi_1 = \Psi_2 = 0, \quad z = 0,1.
$$

From (7.4), one obtains that

$$
(7.6)\qquad \begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|\Psi\|^2 \le \|W\|\cdot\|\Psi\| + \langle\mathbf{U}\cdot\nabla\Psi,\theta\rangle - \|\nabla\Psi\|^2, \\ \dfrac{1}{2}P_1\dfrac{d}{dt}\|\Psi_1\|^2 \le \|W\|\cdot\|\Psi_1\| + \langle\mathbf{U}\cdot\nabla\Psi_1,\gamma_1\rangle - \|\nabla\Psi_1\|^2, \\ \dfrac{1}{2}P_2\dfrac{d}{dt}\|\Psi_2\|^2 \le \|W\|\cdot\|\Psi_2\| + \langle\mathbf{U}\cdot\nabla\Psi_2,\gamma_2\rangle - \|\nabla\Psi_2\|^2, \\ \|U\| \le R\|\Psi\| + R_1\|\Psi_1\| + R_2\|\Psi_2\|. \end{cases}
$$

Setting

$$E = \frac{1}{2}(\|\Psi\|^2 + P_1\|\Psi_1\|^2 + P_2\|\Psi_2\|^2), \tag{7.7}$$

one has

$$\frac{dE}{dt} \leq \|W\| \cdot (\|\Psi\| + \|\Psi_1\| + \|\Psi_2\|) + \langle \mathbf{U}\cdot\nabla\Psi, \theta\rangle + \langle \mathbf{U}\cdot\nabla\Psi_1, \gamma_1\rangle + \langle \mathbf{U}\cdot\nabla\Psi_2, \gamma_2\rangle - (\|\nabla\Psi\|^2 + \|\nabla\Psi_1\|^2 + \|\nabla\Psi_2\|^2). \tag{7.8}$$

Since

$$\begin{cases} \|W\|\cdot\|\Psi\| \leq \|U\|\cdot\|\Psi\| \leq R\|\Psi\|^2 + (R_1\|\Psi_1\| + R_2\|\Psi_2\|)\|\Psi\| \\ \qquad \leq \left[R + \frac{1}{2}(R_1+R_2)\right]\|\Psi\|^2 + \frac{1}{2}(R_1\|\Psi_1\|^2 + R_2\|\Psi_2\|^2), \\ \|W\|\cdot\|\Psi_1\| \leq \|U\|\,\|\Psi_1\| \leq \left[R_1 + \frac{1}{2}(R+R_2)\right]\|\Psi_1\|^2 \\ \qquad + \frac{1}{2}(R\|\Psi\|^2 + R_2\|\Psi_2\|^2), \\ \|W\|\cdot\|\Psi_2\| \leq \left[R_2 + \frac{1}{2}(R+R_1)\right]\|\Psi_2\|^2 + \frac{1}{2}(R\|\Psi\|^2 + R_1\|\Psi_1\|^2), \end{cases} \tag{7.9}$$

then

$$2\frac{dE}{dt} \leq (4R + R_1 + R_2)\|\Psi\|^2 + (R + 4R_1 + R_2)\|\Psi_1\|^2 + (R + R_1 + 4R_2)\|\Psi_2\|^2 + |\langle \mathbf{U}\cdot\nabla\Psi, \theta\rangle| + |\langle \mathbf{U}\cdot\nabla\Psi_1, \Phi_1\rangle| + |\langle \mathbf{U}\cdot\nabla\Psi_2, \Phi_2\rangle| - 2(\|\nabla\Psi\|^2 + \|\nabla\Psi_1\|^2 + \|\nabla\Psi_2\|^2). \tag{7.10}$$

But in view of the boundedness of the solutions of (7.1)–(7.2) (see [27] for details), there exists a positive constant m_1 such that

$$\sup_{\Omega\times\mathbb{R}^+} (|\theta|, |\Phi_1|, |\Phi_2|) \leq m_1, \tag{7.11}$$

and hence one has that

$$\begin{cases} |\langle \mathbf{U}\cdot\nabla\Psi, \theta\rangle| \leq m_1\langle |U|, |\nabla\Psi|\rangle \leq \frac{1}{2}m_1\left(\frac{\|\mathbf{U}\|^2}{\varepsilon} + \varepsilon\|\nabla\Psi\|^2\right), \\ |\langle \mathbf{U}\cdot\nabla\Psi_1, \Phi_1\rangle| \leq \frac{1}{2}m_1\left(\frac{\|\mathbf{U}\|^2}{\varepsilon_1} + \varepsilon_1\|\nabla\Psi_1\|^2\right), \\ |\langle \mathbf{U}\cdot\nabla\Psi_2, \Phi_2\rangle| \leq \frac{1}{2}m_1\left(\frac{\|\mathbf{U}\|^2}{\varepsilon_2} + \varepsilon_2\|\nabla\Psi_2\|^2\right). \end{cases} \tag{7.12}$$

Choosing

$$\varepsilon = \varepsilon_1 = \varepsilon_2, \tag{7.13}$$

from (7.10), by virtue of (7.12), it turns out that

$$\frac{dE}{dt} \leq qE, \tag{7.14}$$

and hence

$$E \leq E_0 e^{qt}, \quad q = \text{const.} > 0. \tag{7.15}$$

From (7.15), it follows that

$$E_0 = 0 \Rightarrow E(t) = 0, \quad \forall t \in \mathbb{R}^+, \tag{7.16}$$

and uniqueness is proved.

ACKNOWLEDGMENTS. This paper has been performed under the auspices of the G.N.F.M. of I.N.d.A.M. and Programma F.A.R.O. (Finanziamenti per l' Avvio di Ricerche Originali, III tornata) "Controllo e stabilitá di processi diffusivi nell'ambiente", Polo delle Scienze e Tecnologie, Universitá degli Studi di Napoli Federico II

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Received 04 April 2012,
and in revised form 31 May 2012.

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