



**Functional Analysis** — *Bisobolev mappings and homeomorphisms with finite distortion*, by ANTONIA PASSARELLI DI NAPOLI, communicated on 20 April 2012.

ABSTRACT. — Let  $\Omega$  and  $\Omega'$  be bounded open sets in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $\text{Hom}(\Omega; \Omega')$  the class of homeomorphisms  $f : \Omega \rightarrow \Omega'$ . We illustrated some properties of bisobolev homeomorphisms and their connections with homeomorphisms with finite distortion<sup>1</sup>.

KEY WORDS: Bisobolev mappings, homeomorphisms with finite distortion, area formula.

AMS CLASSIFICATIONS: 46E30, 46E99.

## 1. INTRODUCTION

Let  $\Omega$  and  $\Omega'$  be bounded domains in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $\text{Hom}(\Omega; \Omega')$  the class of homeomorphisms  $f : \Omega \rightarrow \Omega'$ .

In this paper we will be concerned with the subclasses of  $\text{Hom}(\Omega; \Omega')$  consisting of the bisobolev homeomorphisms and the homeomorphisms with finite distortion.

In the last few years, homeomorphisms with finite distortion have attracted a great interest thanks to their connection with relevant topics such as elliptic partial differential equations, differential geometry and calculus of variations (see [18] and the references therein).

In [17], the class of bisobolev maps has been introduced as the class of homeomorphisms  $f : \Omega \rightarrow \Omega'$  such that

$$f \in W_{\text{loc}}^{1,1}(\Omega; \Omega') \quad \text{and} \quad f^{-1} \in W_{\text{loc}}^{1,1}(\Omega'; \Omega).$$

Similarly, for  $p > 1$ , a homeomorphism  $f : \Omega \rightarrow \Omega'$  is said to be a  $W^{1,p}$ -bisobolev mapping if

$$f \in W_{\text{loc}}^{1,p}(\Omega; \Omega') \quad \text{and} \quad f^{-1} \in W_{\text{loc}}^{1,p}(\Omega'; \Omega).$$

Recall that a homeomorphism  $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$  has finite outer distortion if its Jacobian  $J_f$  is strictly positive a.e. on the set where  $|Df| \neq 0$ . In case  $J_f(x) \geq 0$  a.e., we define its outer distortion function as

$$(1.1) \quad K_{O,f}(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases}$$

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<sup>1</sup>The results of this paper are related to the lecture that the Author gave at the Conference “Geometric Function Theory”, which took place at the Accademia dei Lincei on November 3rd 2011.

Similarly, we say that a homeomorphism  $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbb{R}^n)$  has finite inner distortion if its Jacobian is strictly positive a.e. on the set where the adjugate  $\text{adj } Df$  of the differential matrix does not vanish. In case  $J_f(x) \geq 0$  a.e., we define its inner distortion function as

$$K_{I,f}(x) = \begin{cases} \frac{|\text{adj } Df(x)|^n}{J_f(x)^{n-1}} & \text{for } J_f(x) > 0 \\ 1 & \text{otherwise.} \end{cases}$$

Obviously these two notions coincide in the planar case, while for  $n > 2$  Hadamard's inequality yields that they are related by the inequality

$$K_{I,f}(x) \leq K_{O,f}^{n-1}(x).$$

In order to illustrate the connection between bisobolev homeomorphisms and homeomorphisms with finite distortion, let us begin recalling that in the planar case, i.e. for  $n = 2$ , in [15, 3], the authors prove that each bisobolev map has finite distortion.

Such a conclusion is not valid in higher dimension. In fact, there exists a bisobolev map in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that its Jacobian determinant is zero a.e. and the modulus of its differential matrix is strictly positive on a set of positive measure [17].

Hence, if  $f$  is a planar bisobolev map the zero set of its Jacobian determinant coincides a.e. with the zero set of the differential matrix, while for bisobolev mappings in dimension greater than 2 the norm of the differential matrix need not vanish on the zero set of the Jacobian. Actually, the example constructed in [17] shows that none of the matrix of the  $l \times l$  minors of the differential matrix need to vanish on the zero set of the Jacobian determinant for  $1 \leq l \leq n - 2$ . Actually, in [17], it is proven that bisobolev mappings in dimension greater than 2 have finite inner distortion (see Theorem 2.3 in Subsection 2.3).

Obviously the question can be reversed, wondering what are the conditions on a homeomorphism  $f$  in the Sobolev class  $W^{1,1}$  that guarantee that it is a bisobolev map.

Recall that, in general, the inverse of a homeomorphism  $f \in W_{\text{loc}}^{1,n-1}(\Omega, \Omega')$  belongs to  $BV_{\text{loc}}$  only [3]. For planar homeomorphisms, the explicit expression of the total variation of the components of the inverse is given in [4] (see also [5]).

On the other hand, in [3], it has been proved that if  $f \in W^{1,n-1}(\Omega, \Omega')$  is a homeomorphism with finite outer distortion, then the inverse map  $f^{-1}$  belongs to  $W^{1,1}(\Omega', \Omega)$  and has finite distortion too.

Previous result has been extended to the wider class of homeomorphisms with finite inner distortion in [7], where it is shown that homeomorphisms in the Sobolev class  $W_{\text{loc}}^{1,n-1}(\Omega, \Omega')$  with finite inner distortion are bisobolev maps, i.e.  $f^{-1}$  belongs to  $W^{1,1}(\Omega', \Omega)$ . The sharpness of previous Theorem has been shown in [14].

Hence, in case  $n \geq 3$ , if  $f \in \text{Hom}(\Omega, \Omega') \cap W^{1,n-1}$  has finite inner distortion then it is a bisobolev homeomorphism. Note that  $f \in W^{1,n-1}$  while  $f^{-1} \in W^{1,1}(\Omega', \Omega)$ . So this result still leave open some questions about the regularity

of the inverse. In the planar case, the  $L^1$ -integrability of the distortion  $K_f$  of a homeomorphism of the Sobolev class  $W^{1,1}$  is a sufficient condition to the  $L^2$ -integrability of the differential matrix of the inverse mapping [15].

For  $n > 2$  a similar result has been established in [25], under the stronger assumption  $f \in W^{1,p}(\Omega, \Omega')$ , for some  $p > n - 1$ . This stronger assumption has been removed in [23], where it is shown that the analogous conclusion of the quoted result of [15] holds true if  $f \in W^{1,n-1}(\Omega, \Omega')$  (see Theorem 2.5 in subsection 2.3).

Hence the regularity of the distortion function influences the regularity of the inverse mapping and this happens also in the scale of Orlicz-Zygmund spaces (see [15, 10, 23]).

The aim of this paper is to identify suitable Orlicz spaces, more general than the one treated in the above mentioned papers, in which the same phenomenon holds true. We will confine ourselves to the case in which the distortion is assumed to belong to a Orlicz class of functions not too far from  $L^1$ .

More precisely, we will deal with functions  $\mathcal{P} : [1, +\infty) \rightarrow [0, +\infty)$  which are smooth, non decreasing and onto such that

$$(1.2) \quad \int_1^{+\infty} \frac{\mathcal{P}(t)}{t^2} dt = +\infty.$$

Condition (1.2) is called divergence condition and it is a critical assumption for the regularity of mappings with finite distortion  $K$  with  $e^{\mathcal{P}(K)} \in L^1$  (see for example [8, 9, 13, 2]). We will stay not too far from the borderline case of (1.2), i.e. we will represent  $\mathcal{P}(t)$  as

$$\mathcal{P}(t) = \frac{t}{\mathcal{L}(t)}$$

where  $\mathcal{L} : [1, +\infty) \rightarrow [0, +\infty)$  is a smooth, non decreasing function, growing at  $\infty$  slower than any power. More precisely, we will assume that

$$(1.3) \quad \lim_{t \rightarrow +\infty} \mathcal{L}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\log \mathcal{L}(t)}{\log t} = 0.$$

We extend  $\mathcal{L}(t)$  and  $\mathcal{P}(t)$  for  $0 \leq t \leq 1$  letting  $\mathcal{L}(t) = 1$  and  $\mathcal{P}(t) = 0$ . For  $t \geq 1$ , let us introduce the function

$$(1.4) \quad \mathcal{A}(t) = 1 + \int_1^t \frac{\mathcal{P}(t)}{t^2} dt.$$

We are going to establish the following

**THEOREM 1.1.** *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism with finite inner distortion such that*

$$(1.5) \quad \int_{\Omega} K_{I,f} \mathcal{A}^{\alpha}(K_{I,f}) dx < \infty$$

for some  $\alpha \geq 0$ . Then

$$(1.6) \quad \int_{E'} |Df^{-1}|^n \mathcal{A}^\alpha(\log(e + |Df^{-1}|)) \, dx < \infty$$

for every  $E' \Subset \Omega'$  compact.

When the inner distortion function belongs to an Orlicz class slightly larger than  $L^1$ , we have the following

**THEOREM 1.2.** *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism with finite inner distortion such that  $|\text{adj } Df| \in L^p(\Omega)$  for some  $p > 1$ . If*

$$\frac{K_{I,f}}{\mathcal{A}^\alpha(|K_{I,f}|)} \in L^1(\Omega)$$

for some  $\alpha \geq 0$ , then

$$\frac{|Df^{-1}|^n}{\mathcal{A}^\alpha(|Df^{-1}|)} \in L^1(\Omega').$$

Notice that

$$\mathcal{P}(t) = t \quad \Rightarrow \quad \mathcal{A}(t) \simeq \log(e + t)$$

Hence, in this case, our theorems give back previous results contained in [15, 10, 23].

Our proofs are based on the validity of the area formula for homeomorphisms [6] and on a chain rule formula proved in [7] for Sobolev functions.

## 2. PRELIMINARIES

### 2.1. The area formula

Let  $\Omega$  and  $\Omega'$  be bounded domains in  $\mathbb{R}^n$ . We shall denote by  $\text{Hom}(\Omega; \Omega')$  the set of all Sobolev homeomorphisms  $f : \Omega \rightarrow \Omega' = f(\Omega)$ , by  $|Df|$  the operator norm of the differential matrix and by  $\text{adj } Df$  the adjugate of  $Df$  which is defined by the formula

$$(2.1) \quad Df \cdot \text{adj } Df = \mathbf{I} \cdot J_f,$$

where, as usual,  $J_f = \det Df$  and  $\mathbf{I}$  is the identity matrix.

We will use the well known area formula for homeomorphisms in  $W_{\text{loc}}^{1,1}(\Omega)$ , that is

$$(2.2) \quad \int_B \eta(f(x)) |J_f(x)| \, dx \leq \int_{f(B)} \eta(y) \, dy$$

where  $\eta$  is a nonnegative Borel measurable function on  $\mathbb{R}^n$  and  $B \subset \Omega$  is a Borel set (for more details we refer to [6]). The equality

$$(2.3) \quad \int_B \eta(f(x))|J_f(x)| dx = \int_{f(B)} \eta(y) dy$$

is verified if  $f$  is a homeomorphism that satisfies the *Lusin condition N*, i.e. the implication  $|E| = 0 \Rightarrow |f(E)| = 0$  holds for any measurable set  $E \subset \Omega$ .

Note that the function defined in (1.1) satisfies the so-called distortion inequality

$$|Df(x)|^n \leq K_{O,f}(x)J_f(x).$$

Moreover, by virtue of (2.2), we have that  $J_f \in L^1(B)$ . Hence, the definition of homeomorphism with finite distortion coincides with the usual one, given for mappings which are not homeomorphisms (see [18]).

In [11], the authors proved that mappings  $f \in W^{1,n}(\Omega, \mathbb{R}^n)$  of finite distortion satisfy the Lusin condition *N*. A sharp result is contained in [19], where it is shown that if  $f \in W^{1,1}(\Omega, \mathbb{R}^n)$  is a homeomorphism with  $J_f \geq 0$  a.e. in  $\Omega$  and such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} |Df|^{n-\varepsilon} dx = 0$$

then it satisfies the Lusin condition *N*. Here we are interested in mappings of finite distortion whose differential matrices belong to spaces slightly different from  $L^n$ . For this reason let us recall the definitions and some basic properties of these spaces.

### 2.2. Orlicz spaces

Let  $\Phi$  be an increasing function from  $\Phi(0) = 0$  to  $\lim_{t \rightarrow \infty} \Phi(t) = \infty$  and continuously differentiable on  $(0, \infty)$ . The Orlicz class generated by the function  $\Phi(t)$  will be denoted by  $L^\Phi(\Omega)$  and it consists of the functions  $h$  for which there exists a constant  $\lambda = \lambda(h) > 0$  such that  $\Phi\left(\frac{|h|}{\lambda}\right) \in L^1(\Omega)$ .

In particular, we shall work with the Orlicz classes generated by the function  $\Phi(t) \simeq t \mathcal{A}^\alpha(t)$  as  $t \rightarrow \infty$ , where  $\mathcal{A}$  is the function defined at (1.4). Recall that  $\mathcal{P} : [1, +\infty) \rightarrow [0, +\infty)$  is a smooth, non decreasing and onto function such that

$$(2.4) \quad \int_1^{+\infty} \frac{\mathcal{P}(t)}{t^2} dt = +\infty$$

and we will represent it as

$$\mathcal{P}(t) = \frac{t}{\mathcal{L}(t)}$$

where  $\mathcal{L} : [1, +\infty) \rightarrow [0, +\infty)$  is a smooth, non decreasing function, growing at infinity slower than any power. More precisely, we will assume that

$$(2.5) \quad \lim_{t \rightarrow +\infty} \mathcal{L}(t) = +\infty \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\log \mathcal{L}(t)}{\log t} = 0.$$

We extend  $\mathcal{L}(t)$  and  $\mathcal{P}(t)$  for  $0 \leq t \leq 1$  letting  $\mathcal{L}(t) = 1$  and  $\mathcal{P}(t) = 0$ . Recall as well that, for  $t \geq 1$ ,

$$(2.6) \quad \mathcal{A}(t) = 1 + \int_1^t \frac{\mathcal{P}(t)}{t^2} dt.$$

Also the function  $\mathcal{A}(t)$  is extended in  $[0, 1]$  by setting  $\mathcal{A}(t) = 1$ . We shall need the following properties of the functions  $\mathcal{A}$  and  $\mathcal{L}$ , that are proven in Proposition 5.1 of [2].

**PROPOSITION 2.1.** *For  $\mathcal{A}$  defined at (2.6), and  $\mathcal{L}$  and  $\mathcal{P}$  satisfying (2.4), (2.5) we have*

(a) *For every  $\gamma > 0$ , there exists a positive constant  $C$  such that*

$$\mathcal{A}(t) \leq Ct^\gamma,$$

*for every  $t \geq 1$ .*

(b)  *$\mathcal{A}$  and  $\mathcal{L}$  do not see powers, i.e. for each  $\alpha > 0$  there exist positive constants  $C_1 = C_1(\alpha)$  and  $C_2 = C_2(\alpha)$  such that*

$$\mathcal{L}(t^\alpha) \leq C_1(\alpha)\mathcal{L}(t)$$

*and*

$$\mathcal{A}(t^\alpha) \leq C_2(\alpha)\mathcal{A}(t).$$

(c) *There exists  $t_0$  such that*

$$\mathcal{A}(s+t) \leq \mathcal{A}(s) + \mathcal{A}(t),$$

*for every  $s, t \geq t_0$ .*

(d) *There exists  $t_0$  such that*

$$\mathcal{A}(st) \leq C(\mathcal{A}(s) + \mathcal{A}(t)),$$

*for every  $t \geq t_0$ .*

(e) *The function  $\frac{\mathcal{A}(t)}{t}$  is decreasing for every  $t \geq 1$ .*

From (e) of previous proposition we deduce the  $\Delta_2$ -property for  $\mathcal{A}$ :

$$\mathcal{A}(2t) \leq 2\mathcal{A}(t), \quad \forall t \geq 1.$$

We shall also need the following Young’s type inequality

LEMMA 2.2. *Let  $\mathcal{A}$  defined at (2.6). For every  $\alpha > 0$  there exists  $c = c(\alpha)$  such that*

$$st \leq s\mathcal{A}^\alpha(s) + (\mathcal{A}^\alpha)^{-1}(ct)$$

for every  $s, t \geq t_0$ .

PROOF. If  $t \leq \mathcal{A}^\alpha(s)$  there is nothing to prove. Otherwise  $s < (\mathcal{A}^\alpha)^{-1}(t)$  which obviously implies

$$(2.7) \quad st \leq t(\mathcal{A}^\alpha)^{-1}(t).$$

Now, for  $\tau = (\mathcal{A}^\alpha)^{-1}(t)$ , we observe that

$$\mathcal{A}^\alpha(\tau\mathcal{A}^\alpha(\tau)) \leq \mathcal{A}^\alpha(c\tau^{\alpha+1}) \leq c(\alpha)\mathcal{A}^\alpha(\tau)$$

where we used (a) and (b) of Proposition (2.1) and the  $\Delta_2$ -property of  $\mathcal{A}$ . Previous estimate is equivalent to the following

$$(2.8) \quad \tau\mathcal{A}^\alpha(\tau) \leq (\mathcal{A}^\alpha)^{-1}(c(\alpha)\mathcal{A}^\alpha(\tau)).$$

Using that  $\tau = (\mathcal{A}^\alpha)^{-1}(t)$  in (2.8), we get

$$(2.9) \quad t(\mathcal{A}^\alpha)^{-1}(t) \leq (\mathcal{A}^\alpha)^{-1}(c(\alpha)t).$$

Combining (2.7) and (2.9) give the conclusion. □

Some examples of functions that we are treating are

$$\begin{aligned} \mathcal{P}(t) = t &\Rightarrow \mathcal{A}(t) \simeq \log(e + t), \\ \mathcal{P}(t) = \frac{t}{\log(e + t)} &\Rightarrow \mathcal{A}(t) \simeq \log \log(e + t), \\ \mathcal{P}(t) = \frac{t}{\log(e + t) \log(e^e + t)} &\Rightarrow \mathcal{A}(t) \simeq \log \log \log(e + t). \end{aligned}$$

For more details on Orlicz spaces, we refer to [20].

### 2.3. Bisobolev mappings

Let us recall that a homeomorphism  $f : \Omega \xrightarrow{\text{onto}} \Omega'$  is said to be a bisobolev map if  $f$  belongs to the Sobolev space  $W_{\text{loc}}^{1,1}(\Omega; \Omega')$  and its inverse  $f^{-1}$  belongs to  $W_{\text{loc}}^{1,1}(\Omega'; \Omega)$ . More specifically, if  $f \in W_{\text{loc}}^{1,p}(\Omega; \Omega')$  and  $f^{-1} \in W_{\text{loc}}^{1,p}(\Omega'; \Omega)$ ,  $1 \leq p < \infty$ , then we say that  $f$  is  $W^{1,p}$ -bisobolev.

The connection between bisobolev mappings and mappings with finite distortion is given by the following results.

**THEOREM 2.3** [17]. *Let  $f : \Omega \rightarrow \mathbb{R}^n$  be a bisobolev map. Suppose that for a measurable set  $E \subset \Omega$  we have  $J_f = 0$  a.e. on  $E$ . Then  $|\text{adj } Df| = 0$  a.e. on  $E$ . If we moreover assume that  $J_f \geq 0$  it follows that  $f$  has finite inner distortion.*

In the opposite direction we have the following

**THEOREM 2.4** [7]. *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism such that*

$$(2.10) \quad |\text{adj } Df(x)|^n \leq K(x)J_f^{n-1}(x)$$

*for some Borel function  $K : \Omega \rightarrow [1, +\infty)$ . Then  $f^{-1}$  is a  $W^{1,1}(\Omega', \Omega)$  map of finite outer distortion. Moreover*

$$|Df^{-1}(y)|^n \leq K(f^{-1}(y))J_{f^{-1}}(y) \quad \text{a.e. in } \Omega$$

and

$$\int_{\Omega'} |Df^{-1}(y)| dy = \int_{\Omega} |\text{adj } Df(x)| dx.$$

Next Theorem relates the regularity of the inner distortion function with the regularity of the inverse mapping

**THEOREM 2.5** [23]. *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism with finite inner distortion such that*

$$K_{I,f} \in L^1(\Omega).$$

Then

$$|Df^{-1}| \in L^n(\Omega')$$

and

$$(2.11) \quad \int_{\Omega'} |Df^{-1}(y)|^n dy = \int_{\Omega} K_{I,f}(x) dx.$$

Moreover we have that

$$(2.12) \quad \log\left(e + \frac{1}{J_f}\right) \in L^1_{\text{loc}}(\Omega).$$



### 3. THE CHAIN RULE

We need to recall the definition of approximate gradient of a Borel map. If  $f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$ , we will say that a point  $x \in \Omega$  is a point of approximate continuity if there exists  $z \in \mathbb{R}^N$  such that

$$(3.1) \quad \lim_{r \rightarrow 0} \int_{B_r(x)} |f(y) - z| \, dy = 0.$$

The precise representative of  $f$  is the function  $f^* : \Omega \rightarrow \mathbb{R}^N$  defined by setting  $f^*(x) = z$ , where  $z$  is the vector appearing in (3.1), if  $x$  is a point of approximate continuity of  $f$  and  $f^*(x) = 0$  otherwise. We note that  $f^*(x) = f(x)$  if  $x$  is a Lebesgue point of  $f$ . We remark that, as proved in [1], the set  $\mathcal{D}_f$  of points where (3.1) does not hold is a  $\mathcal{L}^n$ -negligible Borel set.

Let  $f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^N)$  and let  $x$  a point of approximate continuity of  $f$ . We will say that  $f$  is approximately differentiable at  $x$  if there exists a  $N \times n$  matrix denoted by  $Df(x)$  such that

$$(3.2) \quad \lim_{r \rightarrow 0} \int_{B_r(x)} \frac{|f(y) - f^*(x) - Df(x)(y - x)|}{r} \, dy = 0.$$

The approximate gradient  $Df(x)$  is uniquely determined by (3.2), the set

$$\mathcal{D}_f = \{x \in \Omega : f \text{ is approximately differentiable at } x\}$$

is a Borel set and the map  $Df : \mathcal{D}_f \rightarrow R^{Nn}$  is a Borel map.

If  $f \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^N)$  then is approximately differentiable a.e. in  $\Omega$  and its approximate gradient coincides a.e. with the distributional gradient [1].

A key tool for our aim is the following chain rule formula, which has been proved in [7] for bisobolev mappings (see also [23]).

**LEMMA 3.1.** *Let  $f : \Omega \rightarrow \Omega'$  be a homeomorphism such that  $f$  and  $f^{-1}$  are approximately differentiable a.e.. Set*

$$F = \{y \in \mathcal{D}_{f^{-1}} : |J_{f^{-1}}(y)| > 0\}.$$

*Then there exists a Borel set  $A \subset F$  such that  $|F \setminus A| = 0$ ,  $f^{-1}(A) \subset \{x \in \mathcal{D}_f : |J_f(x)| > 0\}$ , with the following property*

$$(3.3) \quad Df^{-1}(y) = [Df(f^{-1}(y))]^{-1} \quad \forall y \in A.$$

### 4. PROOF OF THEOREMS 1.1 AND 1.2

This section is devoted to the proof of our main results. We begin with

PROOF OF THEOREM 1.1. Since the function  $\mathcal{A}^\alpha$  is diverging at infinity, for  $\alpha \geq 0$ , the assumption  $K_{I,f} \mathcal{A}^\alpha(K_{I,f}) \in L^1(\Omega)$  implies  $K_{I,f} \in L^1(\Omega)$ . Hence we can apply Theorem 2.5 to deduce that  $f^{-1}$  is a homeomorphism in the Sobolev class  $W^{1,n}$  with finite inner distortion. Moreover, we have

$$(4.1) \quad \log\left(e + \frac{1}{J_f}\right) \in L^1_{\text{loc}}(\Omega).$$

Hence by the weak version of the Sard Lemma and by the Lusin property of  $f^{-1}$  we get that  $J_f(x) > 0$  a.e. in  $\Omega$ . Therefore, if  $E$  is a compact subset of  $\Omega$ , denoted by  $A' \subset f(E)$  the set determined by Lemma 3.1, by the area formula at (2.2), we get

$$(4.2) \quad \begin{aligned} & \int_{f(E)} |Df^{-1}(y)|^n \mathcal{A}^\alpha(\log(e + |Df^{-1}(y)|)) dy \\ &= \int_{A'} |Df^{-1}(y)|^n \mathcal{A}^\alpha(\log(e + |Df^{-1}(y)|)) dy \\ &\leq \int_E \frac{|\text{adj } Df(x)|^n}{J_f^{n-1}(x)} \mathcal{A}^\alpha\left(\log\left(e + \frac{|\text{adj } Df(x)|}{J_f(x)}\right)\right) dx \\ &= \int_{E \cap \{|\text{adj } Df| \geq 1\}} K_{I,f}(x) \mathcal{A}^\alpha\left(\log\left(e + \frac{|\text{adj } Df(x)|}{J_f}\right)\right) dx \\ &\quad + \int_{E \cap \{|\text{adj } Df| < 1\}} K_{I,f}(x) \mathcal{A}^\alpha\left(\log\left(e + \frac{|\text{adj } Df(x)|}{J_f}\right)\right) dx \\ &= I + II. \end{aligned}$$

Since  $|\text{adj } Df(x)| \leq |\text{adj } Df(x)|^{n/n-1}$  on the set  $E \cap \{|\text{adj } Df| \geq 1\}$ , we have

$$(4.3) \quad \begin{aligned} I &\leq \int_{E \cap \{|\text{adj } Df| \geq 1\}} K_{I,f}(x) \mathcal{A}^\alpha\left(\log\left(e + \frac{|\text{adj } Df(x)|^{n/n-1}}{J_f}\right)\right) dx \\ &\leq \int_E K_{I,f} \mathcal{A}^\alpha(\log(e + K_{I,f}^{1/(n-1)})) dx \\ &\leq \int_E K_{I,f} \mathcal{A}^\alpha(c(n) \log(e + K_{I,f})) dx \\ &\leq c(n, \alpha) \int_E K_{I,f} \mathcal{A}^\alpha(K_{I,f}) dx \end{aligned}$$

where we used that  $\mathcal{A}$  is an increasing function on  $[1, +\infty)$  and property (b) of Proposition 2.1. We also used the elementary inequality

$$(4.4) \quad \log(e + x) \leq cx^\gamma, \quad \forall \gamma > 0, x \geq 1.$$

Hence we can conclude thank to assumption (1.5) that the integral I is finite.

In order to estimate II, we use the Young’s type inequality of Lemma 2.2

$$\begin{aligned} II &\leq \int_{E \cap \{|\text{adj } Df| < 1\}} K_{I,f}(x) \mathcal{A}^\alpha \left( \log \left( e + \frac{1}{J_f} \right) \right) dx \\ &\leq c_\alpha \int_E K_{I,f}(x) \mathcal{A}^\alpha (c_\alpha K_{I,f}(x)) dx \\ &\quad + c_\alpha \int_E (\mathcal{A}^\alpha)^{-1} \left( \mathcal{A}^\alpha \left( \log \left( e + \frac{1}{J_f} \right) \right) \right) dx \\ &\leq c_\alpha \int_E K_{I,f}(x) \mathcal{A}^\alpha (K_{I,f}(x)) dx + c_\alpha \int_E \log \left( e + \frac{1}{J_f} \right) dx \end{aligned}$$

that is finite thanks to the assumption and (4.1). This concludes the proof. □

Now, we provide the

**PROOF OF THEOREM 1.2.** Since  $f \in W^{1,n-1}(\Omega, \Omega')$  is a homeomorphism with finite inner distortion by Theorem 2.4 we have that  $f^{-1} \in W^{1,1}(\Omega'; \Omega)$ . Let  $A'$  be the set determined by Lemma 3.1. Then we can use the chain rule and the area formula, thus getting

$$\begin{aligned} (4.5) \quad &\int_{\Omega'} \frac{|Df^{-1}(y)|^n}{\mathcal{A}^\alpha(|Df^{-1}(y)|)} dy = \int_{A'} \frac{|Df^{-1}(y)|^n}{\mathcal{A}^\alpha(|Df^{-1}(y)|)} dy \\ &\leq \int_{\Omega} \frac{|\text{adj } Df(x)|^n}{J_f(x)^{n-1} \mathcal{A}^\alpha \left( \frac{|\text{adj } Df(x)|}{J_f(x)} \right)} dx = \int_{\Omega} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha \left[ \left( \frac{K_{I,f}(x)}{|\text{adj } Df(x)|} \right)^{1/(n-1)} \right]} dx \\ &= \int_{\{K_{I,f} \leq |\text{adj } Df|^p\}} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha \left[ \left( \frac{K_{I,f}(x)}{|\text{adj } Df(x)|} \right)^{1/(n-1)} \right]} dx \\ &\quad + \int_{\{K_{I,f} > |\text{adj } Df|^p\}} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha \left[ \left( \frac{K_{I,f}(x)}{|\text{adj } Df(x)|} \right)^{1/(n-1)} \right]} dx \\ &\leq c \int_{\Omega} |\text{adj } Df(x)|^p dx + \int_{\Omega} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha (K_{I,f}^{(1-1/p)(1/(n-1))}(x))} dx \\ &\leq c \int_{\Omega} |\text{adj } Df(x)|^p dx + c(n, p) \int_{\Omega} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha (K_{I,f}(x))} dx \end{aligned}$$

where we used that  $\mathcal{A}(t) \geq 1$  for every  $t$  and that

$$\mathcal{A}(t) = \mathcal{A}(t^{(1-1/p)(p/(p-1))((n-1)/(n-1))}) \leq c(n, p)\mathcal{A}(t^{(1-1/p)(1/(n-1))})$$

thank to the property (b) of Proposition 2.1. Since, by our assumptions, the integrals in the right hand side of previous estimate are finite we obtain that  $\frac{|Df^{-1}|^n}{\mathcal{A}^\alpha(|Df^{-1}|)} \in L^1(\Omega')$ . □

We can weaken the regularity assumption on the adjugate matrix of the differential of the homeomorphism  $f$  and we arrive at the same conclusion of Theorem 1.2, slightly improving the regularity assumption on the inner distortion function. In fact we have

**THEOREM 4.1.** *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism with finite inner distortion such that  $|\text{adj } Df| \log(e + |\text{adj } Df|)$  belongs to the space  $L^1(\Omega)$  and that*

$$\frac{K_{I,f}}{\mathcal{A}^\alpha(\log(e + |K_{I,f}|))} \in L^1(\Omega),$$

for some  $\alpha \geq 0$ . Then

$$\frac{|Df^{-1}|^n}{\mathcal{A}^\alpha(|Df^{-1}|)} \in L^1(\Omega').$$

**PROOF.** Since  $f \in W^{1,n-1}(\Omega, \Omega')$  is a homeomorphism with finite inner distortion by Theorem 2.4 we have that  $f^{-1} \in W^{1,1}(\Omega'; \Omega)$ . Then we can use the area formula, thus getting

$$\begin{aligned} (4.6) \quad & \int_{\Omega'} \frac{|Df^{-1}(y)|^n}{\mathcal{A}^\alpha(|Df^{-1}(y)|)} dy \\ & \leq \int_{\Omega} \frac{|\text{adj } Df(x)|^n}{J_f(x)^{n-1} \mathcal{A}^\alpha\left(\frac{|\text{adj } Df(x)|}{J_f(x)}\right)} dx \leq c \int_{\Omega} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha\left(\frac{K_{I,f}(x)}{|\text{adj } Df(x)|}\right)} dx \\ & = \int_{\{K_{I,f} \leq |\text{adj } Df| \log(e + K_{I,f})\}} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha\left(\frac{K_{I,f}(x)}{|\text{adj } Df(x)|}\right)} dx \\ & \quad + \int_{\{K_{I,f} > |\text{adj } Df| \log(e + K_{I,f}(x))\}} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha\left(\frac{K_{I,f}(x)}{|\text{adj } Df(x)|}\right)} dx \\ & \leq \int_{\Omega} |\text{adj } Df(x)| \log(e + K_{I,f}(x)) dx \\ & \quad + \int_{\Omega} \frac{K_{I,f}(x)}{\mathcal{A}^\alpha(\log(e + K_{I,f}(x)))} dx \end{aligned}$$

where we used that  $\mathcal{A}(t)$  is an increasing function such that  $\mathcal{A}(t) \geq 1$  for every  $t$ . Since the second integral in the right hand side of (4.6) is finite thanks to the assumption, it remains to prove that the first integral is finite too. To this aim observe that

$$\begin{aligned}
 (4.7) \quad & \int_{\Omega} |\text{adj } Df(x)| \log(e + K_{I,f}) \, dx \\
 &= \int_{\Omega} |\text{adj } Df(x)| \log\left(e + \frac{|\text{adj } Df(x)|^n}{J_f^{n-1}}\right) \, dx \\
 &\leq c(n) \int_{\Omega} |\text{adj } Df(x)| \log(e + |\text{adj } Df(x)|) \, dx \\
 &\quad + c(n) \int_{\Omega} |\text{adj } Df(x)| \log\left(e + \frac{1}{J_{I,f}}\right) \, dx \\
 &\leq c \int_{\Omega} |\text{adj } Df(x)| \log(e + |\text{adj } Df(x)|) \, dx + c \int_{\Omega} \frac{|\text{adj } Df(x)|}{(J_f(x))^{(n-1)/n}} \, dx \\
 &= c \int_{\Omega} |\text{adj } Df(x)| \log(e + |\text{adj } Df(x)|) \, dx + c \int_{\Omega} K_{I,f}(x)^{(n-1)/n} \, dx
 \end{aligned}$$

where we used again that  $\log(e + x) \leq c(n)x^{(n-1)/n}$ , for  $x \geq 1$ . The first integral in the right hand side is finite by assumptions. In order to estimate the second one we observe that, by (a) of Proposition 2.1,

$$K_{I,f}^{(n-1)/n} = \frac{K_{I,f}}{K_{I,f}^{1/n}} \quad \text{and} \quad \mathcal{A}^{\alpha}(K_{I,f}) \leq c(\alpha, n)K_{I,f}^{1/n}$$

and hence

$$(4.8) \quad \int_{\Omega} K_{I,f}^{(n-1)/n} \, dx \leq c(n, \alpha) \int_{\Omega} \frac{K_{I,f}}{\mathcal{A}^{\alpha}(K_{I,f})} \, dx.$$

Since the integral in the right hand side of previous estimate is finite thank to the assumption, this concludes the proof. □

Remark that previous Theorem still holds if in the assumption the function  $\log(e + x)$  is replaced by any Orlicz function diverging at  $\infty$  slower than any power.

5. INTEGRABILITY OF  $\frac{1}{J_f}$

Theorem 4.1 tells us that  $|Df^{-1}|^n \in L^\Phi(\Omega')$ , where

$$\Phi(t) = \frac{t}{\mathcal{A}^\alpha(t)}$$

and we know by Theorem 2.4 that  $f^{-1}$  has nonnegative Jacobian a.e. in  $\Omega'$ . The higher integrability result for the Jacobian determinant in Orlicz spaces of [22] states that if  $\Phi$  satisfies the  $\Delta_2$ -property and

- a)  $\exists C, t_0 > 0: \Phi(t) \geq C \frac{t}{\log(e+t)}$
- b)  $\exists p > 1: \frac{\Phi(t^{n/(n+1)})}{t^p} \nearrow$

then  $J_{f^{-1}} \in L^\Psi_{loc}(\Omega')$  where

$$(5.1) \quad \Psi(t) = t \int_0^t \frac{\Phi'(s)}{s}.$$

We shall use this regularity result of [22] to deduce a integrability property for  $\frac{1}{J_f}$ . More precisely, we have

**THEOREM 5.1.** *Let  $f \in W^{1,n-1}(\Omega, \Omega')$  be a homeomorphism with finite inner distortion such that  $|\text{adj } Df| \log(e + |\text{adj } Df|)$  belongs to the space  $L^1(\Omega)$  and that*

$$\frac{K_{I,f}}{\mathcal{A}^\alpha(\log(e + |K_{I,f}|))} \in L^1(\Omega),$$

for some  $\alpha \in [0, 1)$ . Then

$$\Theta\left(\frac{1}{J_f(x)}\right) \in L^1_{loc}(\Omega'),$$

where

$$(5.2) \quad \Theta(t) = \frac{\Psi(t)}{t} = \int_0^t \frac{\Phi'(s)}{s}.$$

**PROOF.** For  $\alpha \in [0, 1)$ , one can easily check that the function  $\Phi$  satisfies the assumptions of Theorem 3.1 in [22] that yields  $J_{f^{-1}} \in L^\Psi_{loc}(\Omega')$ , where  $\Psi(t)$  is defined by (5.1). In particular, since  $\Phi$  verifies condition a), Theorem A in [19] implies that  $f^{-1}$  satisfies the Lusin condition N and hence  $|\mathcal{J}_f^0| = 0$ . In fact, as we noticed before, Sard's Lemma yields that  $|f'(\mathcal{D}_f \cap \mathcal{J}_f^0)| = 0$  and hence the N-property of  $f^{-1}$  implies  $|\mathcal{D}_f \cap \mathcal{J}_f^0| = 0$ . So we can argue analogously to before, having

$$\begin{aligned} & \int_E \Theta\left(\frac{1}{J_f(x)}\right) dx \\ &= \int_{E \cap A} \frac{1}{J_f(x)} \Theta\left(\frac{1}{J_f(x)}\right) J_f(x) dx \\ &\leq \int_{f(E \cap A)} J_{f^{-1}}(y) \Theta(J_{f^{-1}}(y)) dy = \int_{f(E \cap A)} \Psi(J_{f^{-1}}(y)) < \infty \end{aligned}$$

where  $A$  is the subset of  $\Omega$  determined by Lemma 3.1. □

We'd like to mention that in [12] it is proven that if  $K_{I,f} \in L^q_{loc}(\Omega)$  then  $\log^q(e + \frac{1}{J_f}) \in L^q_{loc}(\Omega)$ .

### 6. THE REGULARITY OF THE DISTORTION

In [17], it has been proven that, for  $W^{1,p}$  bisobolev mappings, the case  $p = n$  is somehow critical for what concerns the regularity of the distortion. In fact, if  $p = n$ , then the inner distortion functions of  $f$  and  $f^{-1}$  both belong to  $L^1$ . If  $p > n$ , then they both belong to some  $L^q$ ,  $q > 1$ . In case  $p < n$ , the authors exhibit a counterexample showing that no  $L^1$  integrability of the distortion can be expected. In [10], in the planar case, it is shown that the regularity of the distortion can be obtained in the scale of Orlicz-Zygmund classes for  $W^{1,2}$  bisobolev mappings whose differential matrices belong to  $L^2 \log^\alpha L$ , some  $\alpha \geq 0$ . More precisely, if  $f$  is a  $W^{1,2}$  bisobolev mapping such that  $|Df^{-1}| \in L^2 \log^\alpha L(\Omega')$ , some  $\alpha \geq 0$ , then  $K_f \in L \log^\alpha L(\Omega)$ . Next result is the analogous of that of [10], relative to the  $n$ -dimensional setting and to more general Orlicz functions. More precisely we have that

**THEOREM 6.1.** *Let  $f \in \text{Hom}(\Omega, \Omega')$  be a  $W^{1,n}$  bisobolev map such that*

$$|Df^{-1}|^n \mathcal{A}^\alpha(|Df^{-1}|) \in L^1(\Omega')$$

for some  $\alpha \geq 0$ . Then

$$K_{I,f} \mathcal{A}^\alpha(\log(e + K_{I,f})) \in L^1(\Omega).$$

**PROOF.** First of all, we observe that under our assumptions we can use Theorem 5 in [17], in order to have  $K_{I,f} \in L^1(\Omega)$  and  $K_{I,f^{-1}} \in L^1(\Omega)$ . Hence by Theorem 2.5 we have that

$$\log\left(e + \frac{1}{J_f}\right) \in L^1(\Omega) \quad \text{and} \quad \log\left(e + \frac{1}{J_f^{-1}}\right) \in L^1(\Omega').$$

Moreover, since  $f, f^{-1}$  satisfy the Lusin condition N, they have positive Jacobian a.e.:

$$(6.1) \quad J_f(x) > 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad J_{f^{-1}}(x) > 0 \quad \text{a.e. in } \Omega'.$$

Using the chain rule and the area formula, we get

$$\begin{aligned}
 (6.2) \quad & \int_{\Omega} K_{I,f} \mathcal{A}^{\alpha}(\log(e + K_{I,f})) \, dx \\
 &= \int_{\Omega} \frac{|\text{adj } Df|^n}{J_f^{n-1}} \mathcal{A}^{\alpha}\left(\log\left(e + \frac{|\text{adj } Df|^n}{J_f^{n-1}}\right)\right) \, dx \\
 &= \int_{\Omega'} |Df^{-1}|^n \mathcal{A}^{\alpha}\left(\log\left(e + \frac{|Df^{-1}|^n}{J_{f^{-1}}}\right)\right) \, dy \\
 &\leq c(n, \alpha) \int_{\Omega'} |Df^{-1}|^n \mathcal{A}^{\alpha}(\log((e + |Df^{-1}|))) \, dy \\
 &\quad + c(n, \alpha) \int_{\Omega'} |Df^{-1}|^n \mathcal{A}^{\alpha}\left(\log\left(e + \frac{1}{J_{f^{-1}}}\right)\right) \, dy \\
 &= I + II
 \end{aligned}$$

where we used (c) of Proposition 2.1. In order to estimate the integral I in (6.2), we use again the elementary inequality (4.4) to obtain that

$$(6.3) \quad I \leq c(n, \alpha) \int_{\Omega'} |Df^{-1}|^n \mathcal{A}^{\alpha}(|Df^{-1}|) \, dx$$

and hence the integral I is finite thank to the assumption. In order to estimate the second integral, we use the Young's type inequality of Lemma 2.2 to have

$$\begin{aligned}
 (6.4) \quad & II \leq c(\alpha, n) \int_{\Omega'} |Df^{-1}|^n \mathcal{A}^{\alpha}(|Df^{-1}|) \, dx \\
 & \quad + c(\alpha) \int_{\Omega'} (\mathcal{A}^{\alpha})^{-1}\left(\mathcal{A}^{\alpha}\left(\log\left(e + \frac{1}{J_{f^{-1}}}\right)\right)\right) \, dy.
 \end{aligned}$$

Then we conclude by using the assumption on  $Df^{-1}$  and (6.1).  $\square$

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