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Mathematical Analysis — *Openness and discreteness for mappings of finite distortion*, by Stanislav Hencl and Kai Rajala, communicated on 9 March 2012.

Abstract. — We give an overview of some recent results concerning openness and discreteness for mappings of finite distortion.

Key words: Mapping of finite distortion, discreteness, openness.

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1. Introduction

In the pioneering works [1] and [2] J. Ball studied a class of mappings that could be used to model nonlinear elasticity and he found weak conditions for regularity and invertibility properties. One of the main properties in the models of nonlinear elasticity is that there is no interpenetration of matter. This in the physically relevant models corresponds to the fact that two parts of the body cannot be mapped to the same place. From the mathematical point of view this means that the map is one-to-one and thus invertible.

Let us consider the holomorphic function $f(z) = z^2$ in the complex plane which can be identified with $\mathbb{R}^2$. We know that $f \in C^\infty$ and its distortion satisfies $K \equiv 1$. On the other hand each nonzero point has two preimages and so f is not invertible. This shows that even for analytically very nice mappings we cannot conclude that the inverse exists without some extra information.

Therefore as a first step one usually studies conditions under which the mapping is open and discrete. Note that for example homeomorphisms are automatically open and discrete.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain. We say that mapping $f : \Omega \to \mathbb{R}^n$ is open if $f(U)$ is open for each open set $U \subset \Omega$, and discrete if the preimage of each point $f^{-1}(y)$ is a discrete set, i.e. does not have an accumulation point in Ω.*

It is known that each open and discrete map which equals to a homeomorphism close to the boundary is necessarily a homeomorphism. Moreover, a discrete and open mapping is locally invertible in the neighborhood of most of the

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points by the following result of Chernavskii [4], [5]. Recall that the branch set of a map is the set of points where it fails to be locally injective.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}^n$ be a discrete and open mapping. Then the topological dimension of the branching set satisfies*

$$\dim B_f = \dim f(B_f) \leq n - 2.$$

The following example shows that the openness and discreteness may fail even for Lipschitz mappings if the integrability of the distortion is not sufficient.

EXAMPLE 1.3 (Ball). *Let $f : (-1,1)^2 \to \mathbb{R}^2$ be defined as*

$$f(x, y) = [x, |x| y].$$

Then f is not open and discrete since $f^{-1}([0,0]) = \{0\} \times (-1,1)$. The derivative equals

$$Df(x_0) = \begin{pmatrix} 1 & 0 \\ \pm y & |x| \end{pmatrix}$$

and therefore it is easy to see that f is Lipschitz and $J_f(x) = |x| \geq 0$. Hence f is a mapping of finite distortion and its distortion

$$K(x) = \frac{1}{|x|}$$

is integrable with any power $p < 1$.

Analogously the mapping $f : (-1,1)^n \to \mathbb{R}^n$ defined as

$$f([x_1, \dots, x_n]) = [x_1, \dots, x_{n-1}, \sqrt{x_1^2 + \cdots + x_{n-1}^2} x_n]$$

is a Lipschitz mapping of finite distortion and its distortion satisfies

$$K(x) = \frac{1}{\sqrt{x_1^2 + \cdots + x_{n-1}^2}}$$

and is integrable with any power $p < n-1$. On the other hand $f^{-1}([0,\dots,0]) = \{0\}^{n-1} \times (-1,1)$ and hence f is not open and discrete. It is moreover possible to extend this mapping to Lipschitz mapping $f : (-2,2)^n \to \mathbb{R}^n$ such that the restriction of f close to the boundary $f|_{(-2,2)^n \setminus [-1,1]^n}$ is a homeomorphism.

The positive results of Ball about invertibility were later extended for example in the works of V. Šverák [28], I. Fonseca and W. Gangbo [10] and S. Müller, S. J. Spector and Q. Tang [21]. The whole theory is nowadays very rich and we recommend the monograph [13] for an overview of the field, discussion of interdisciplinary links and further references.

Let us recall the definition of the class of mappings involved. Suppose that $\Omega \subset \mathbb{R}^n$ is a connected open set and f is a mapping in the Sobolev space

$W^{1,1}(\Omega, \mathbb{R}^n)$ whose pointwise Jacobian J_f is integrable. Then f is said to be a *mapping of finite distortion* K if there is a measurable function $K : \Omega \to [1, \infty)$ such that

$$|Df(x)|^n \leq K(x)J_f(x) \quad \text{a.e. in } \Omega. \tag{1.1}$$

The whole theory reduces to holomorphic functions when $n = 2$ and $K \equiv 1$. An important intermediate step consists of the study of mappings of *bounded distortion*, also called *quasiregular mappings* (for an overview see [26], [27], [13]). In [25], Yu. G. Reshetnyak proves (among other things) the remarkable result that a mapping of bounded distortion is continuous (i.e. has a continuous representative) and either constant or open and discrete. It is worth noticing that Reshetnyak's result gives topological conclusions from purely analytical assumptions.

2. Results

The continuity question for general mappings of finite distortion is nowadays quite well understood. Gol'dstein and Vodop'yanov [11] have shown that mappings of finite distortion in the Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$ are monotone and continuous. Manfredi [18] defined the class of weakly monotone functions and showed that continuity holds more generally for all $W^{1,n}$-functions that are weakly monotone. A generalization to Orlicz-Sobolev spaces below $W^{1,n}$ has been achieved by T. Iwaniec, P. Koskela and J. Onninen [12].

The optimal assumptions for openness and discreteness turned out to be much more challenging. In the fundamental paper [15] T. Iwaniec and V. Šverák proved that the condition $K \in L^1(\Omega)$ is enough to guarantee the openness and discreteness of planar nonconstant $W^{1,2}(\Omega, \mathbb{R}^2)$ mappings of finite distortion. In Example 1.3 we have seen that this condition on the integrability of the distortion is optimal in the plane. In higher dimensions we obtain, that assuming f is Lipschitz and $K \in L^p$ for all $p < n - 1$ is not enough.

It was conjectured in [15] that the conditions $K \in L^{n-1}$ and $f \in W^{1,n}$ are sufficient for openness for all $n \geq 3$. Soon afterwards, different authors conjectured that these conditions are also sufficient for discreteness (openness and discreteness were expected to be equivalent properties under these assumptions; it was known that discreteness implies openness). This problem is well-known and mentioned in several consequent works (see e.g. [13, Conjecture 6.5.1]).

It was shown by J. Manfredi and E. Villamor [19] and [20] that each nonconstant $W^{1,n}$ mapping of finite distortion such that $K \in L^p$ for some $p > n - 1$ is open and discrete. Their proof is based on certain solutions of n-Laplace equation. The proof was shortened and simplified by J. Onninen and X. Zhong [22] who used essentially only the definition of the distributional Jacobian. Moreover, the Sobolev regularity of f can be relaxed to some Orlicz-Sobolev spaces below $W^{1,n}$ as was shown by J. Kauhanen, P. Koskela and J. Malý in [16], but this condition cannot be relaxed for example to $f \in W^{1,p}$ for some $p < n$ (see [17]).

If we a priori require that our mapping equals to a homeomorphism close to the boundary, or that the multiplicity of f is essentially bounded, then the condition $K \in L^{n-1}$ is in fact sufficient for openness and discreteness. This was established in a series of papers of V. Šverák [28], J. Heinonen and P. Koskela [6], S. Hencl and J. Malý [7], S. Hencl and P. Koskela [8] and K. Rajala [23]. These requirements, which are satisfied by Example 1.3, were also used in the original papers [1] and [28]. However, a priori topological assumptions cannot be always required and they are unnatural in some applications.

Recently it was shown by S. Hencl and K. Rajala [9] that, unexpectedly, the conjectures concerning discreteness are false and there is even a Lipschitz mapping of finite distortion $K \in L^{n-1}$ which is not discrete. It follows that infinite twisting around one point in the image may in fact improve the integrability of the distortion. Unfortunately this mapping is open. Therefore it also shows that, surprisingly, openness does not imply discreteness under our assumptions. Here and in the sequel we denote the m-dimensional ball centered at the origin with radius r by $B^m(r)$.

THEOREM 2.1 [9]. *Let $n \geq 3$. There is an open Lipschitz mapping of finite distortion $f : B^{n-1}(1) \times (1,2) \to \mathbb{R}^n$ such that $K \in L^{n-1}(B^{n-1}(1) \times (1,2))$ but $f(\{0\}^{n-1} \times (1,2)) = [0, \ldots, 0]$.*

It is possible to simplify the ideas from this example to obtain a sharp result in the planar case. From Example 1.3 we see that $K = 1/|x|$, and one may ask if the condition $K/\log(e + K) \in L^1$ is sufficient or not (see [14] and [8]). We show that this is not the case and again it is possible to construct a counterexample which is sharper.

THEOREM 2.2 [9]. *There is an open Lipschitz mapping of finite distortion $f : (-1,1) \times (1,2) \to \mathbb{R}^2$ such that $K/\log^{\varepsilon}(e + K) \in L^1((-1,1) \times (1,2))$ for every $\varepsilon > 0$ but $f(\{0\} \times (1,2)) = [0,0]$.*

Let us mention some of the open problems in this area.

OPEN PROBLEM 1 [15]. Suppose that $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, $n \geq 3$, is a nonconstant mapping of finite distortion such that $K \in L^{n-1}(\Omega)$. Is f open?

OPEN PROBLEM 2. Suppose that $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, $n \geq 3$, is a nonconstant mapping of finite distortion such that the inner distortion satisfies $K_I \in L^p(\Omega)$ for some $p > 1$ (see [13] for the definition of the inner distortion). Is f open and discrete?

Let us note that some results in this direction were obtained by K. Rajala [23] and [24].

OPEN PROBLEM 3. Suppose that $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, $n \geq 3$, is a nonconstant mapping of finite distortion such that $K \in L^{n-1}(\Omega) \log^{\alpha} L$ for some $\alpha \in [n-2, n(n-2)]$. Is f open and discrete?

If $\alpha < n-2$ then it is possible to use the counterexample from [9]. If $\alpha > n(n-2)$ then the answer is positive as was shown by J. Björn [3].

3. Positive result below $K \in L^1$

Let us close this note by a small observation that there is some positive result in the plane under weaker assumptions than $K \in L^1$.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $f : \Omega \to \mathbb{R}^2$ be a non-constant mapping of finite distortion. Suppose that f is Lipschitz and that $K/\log(\log(e+K)) \in L^1(\Omega)$. Then f is open and discrete.*

Proof. It is well-known that it is enough to show that $f^{-1}(y)$ is totally disconnected for all $y \in \mathbb{R}^n$, that is, $f^{-1}(y)$ does not contain an arc. Therefore following the proof in [22] we can use a translation and it is enough to show that

$$\mathscr{H}^1(f^{-1}(0)) = 0.$$

Suppose for contrary that

$$\mathscr{H}^1(f^{-1}(0)) > 0.$$

Then we can use [8, Theorem 3.2] to conclude

$$\int_{0<|f|<\frac{1}{e^2}} \frac{|Df(x)|}{|f(x)|\log(1/|f(x)|)\log(\log(1/|f(x)|))}\,dx = \infty. \tag{3.1}$$

We claim that it is possible to use the computations analogous to [22, proof of Theorem 1.1] to show that

$$\int_{0<|f|<\frac{1}{e^2}} \frac{|J_f(x)|}{|f(x)|^2\log^2(1/|f(x)|)\log(\log(1/|f(x)|))}\,dx < \infty. \tag{3.2}$$

Then we can use Young inequality for $A(t) \sim t^2\log^{-1}(\log(e^2+t))$ and $B(t) \sim t^2\log(\log(e^2+t))$ and we obtain

$$\begin{aligned} &\int_{0<|f|<\frac{1}{e^2}} \frac{\sqrt{K(x)J_f(x)}}{|f(x)|\log(1/|f(x)|)\log(\log(1/|f(x)|))}\,dx \\ &\quad\le C\int_{0<|f|<\frac{1}{e^2}} \frac{K(x)}{\log(\log(e^2+K(x)))}\,dx \\ &\quad\quad + C\int_{0<|f|<\frac{1}{e^2}} \frac{J_f(x)\log\Big(\log\Big(e^2+\frac{\tilde{C}}{|f(x)|^2}\Big)\Big)}{|f(x)|^2\log^2(1/|f(x)|)\log^2(\log(1/|f(x)|))}\,dx \end{aligned} \tag{3.3}$$

where we have estimated J_f by $\tilde{C}$ in the last integral since f is Lipschitz. The first integral is bounded by $K/\log(\log(e+K)) \in L^1$ and the second one is bounded by a constant multiple of (3.2). Therefore this integral is finite which together with the distortion inequality $|Df(x)|^2 \le K(x)J_f(x)$ contradict (3.1).

It remains to show the claim (3.2). This is analogous to [22, proof of Theorem 1.1] and therefore we will only sketch it and point out the differences. We use [22, Lemma 2.1] for

$$\Psi(t) = \frac{1}{2t}\int_0^t \frac{\varphi_\varepsilon(s)}{s\log^2\left(\frac{1}{s}\right)\log\left(\log\left(\frac{1}{s}\right)\right)}\,ds$$

and on the right hand side we use the estimate

$$\int_0^{|f(x)|^2} \frac{1}{s\log^2\left(\frac{1}{s}\right)\log\left(\log\left(\frac{1}{s}\right)\right)}\,ds \le C\frac{1}{\log\frac{1}{|f(x)|^2}\log\log\frac{1}{|f(x)|^2}}.$$

At the end we can estimate the right hand side by Young's inequality analogously to (3.3) and we get the claim. □

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