



Calculus of Variations — *A remark on optimal weighted Poincaré inequalities for convex domains*, by V. FERONE, C. NITSCH and C. TROMBETTI.

ABSTRACT. — We prove a sharp upper bound on convex domains, in terms of the diameter alone, of the best constant in a class of weighted Poincaré inequalities. The key point is the study of an optimal weighted Wirtinger inequality.

KEY WORDS: Poincaré inequalities, p -Laplacian eigenvalues, Wirtinger inequalities.

MSC 2100 CLASSIFICATION: 26D10, 35J92.

1. INTRODUCTION

It is well known that (see for instance [21]), for any given open bounded Lipschitz connected set Ω , a Poincaré inequality holds true, in the sense that there exists a positive constant $C_{\Omega,p}$ such that

$$(1.1) \quad \inf_{t \in \mathbb{R}} \|u - t\|_{L^p(\Omega)} \leq C_{\Omega,p} \|Du\|_{L^p(\Omega)},$$

for all Lipschitz functions u in Ω .

The value of the best constant in (1.1) is the reciprocal of the first nontrivial Neumann eigenvalue of the p -Laplacian over Ω . In [23] (see also [2]), it has been proved that, if $p = 2$, and Ω is convex, in any dimension

$$(1.2) \quad \frac{1}{C_{\Omega,2}} = \min_{\substack{u \in H^1(\Omega) \\ \int_{\Omega} u = 0}} \frac{(\int_{\Omega} |Du|^2)^{1/2}}{(\int_{\Omega} |u|^2)^{1/2}} \geq \frac{\pi}{d},$$

where d is the diameter of Ω . Observe that the last term of (1.2) is exactly the square root of the value achieved, in dimension $n = 1$ on any interval of length d , by the first nontrivial Laplacian eigenvalue (without distinction between the Neumann and the Dirichlet conditions).

The proof of (1.2) in [23, 2] indeed relies on the reduction to a one dimensional problem. At this aim, for any given smooth admissible test function u in the Rayleigh quotient in (1.2), the authors show that it is possible to perform a

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clever slicing of the domain Ω in convex sets which are as tiny as desired in at least $n - 1$ orthogonal directions. On each one of such convex components of Ω , they are able to show that the Rayleigh quotient of u can be approximated by a 1-dimensional weighted Rayleigh quotient. This leads the authors to look for the best constants of a class of one dimensional weighted Poincaré–Wirtinger inequalities. The result was later generalized to $p = 1$ in [1] and only recently to $p \geq 2$ in [13] and for any $p > 1$ in the framework of compact manifolds in [22, 31]. Other optimal Poincaré inequalities can be found in [4, 5, 6, 7, 12, 29, 32].

In this paper we consider a weighted Poincaré inequality, namely, for $p > 1$, given a positive log-concave function ω on an open bounded convex set Ω , there exists a positive constant $C_{\Omega,p,\omega}$ such that, for every Lipschitz function u

$$(1.3) \quad \inf_{t \in \mathbb{R}} \|u - t\|_{L^p_\omega(\Omega)} \leq C_{\Omega,p,\omega} \|Du\|_{L^p_\omega(\Omega)}.$$

Here $\|\cdot\|_{L^p_\omega}$ denotes the weighted Lebesgue norm. The best constant $C_{\Omega,p,\omega}$ in (1.3) is given by

$$(1.4) \quad \frac{1}{C_{\Omega,p,\omega}} = \inf_{\substack{u \text{ Lipschitz} \\ \int_\Omega |u|^{p-2} u \omega = 0}} \frac{(\int_\Omega |Du|^p \omega)^{1/p}}{(\int_\Omega |u|^p \omega)^{1/p}}.$$

Our main result is the following.

THEOREM 1.1 (Main Theorem). *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set having diameter d and let ω be a positive log-concave function on Ω . For $p > 1$ and in any dimension we have*

$$(1.5) \quad C_{\Omega,p,\omega} \leq \frac{d}{\pi_p}$$

where

$$(1.6) \quad \pi_p = 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1} s^p} ds = 2\pi \frac{(p-1)^{1/p}}{p(\sin(\pi/p))}.$$

Other optimal weighted inequalities can be found in literature (see [21] and the reference therein) but to our knowledge similar explicit sharp bounds were obtained only for $p = 1$ and $p = 2$ (see for instance [8, 9]). We observe that when $\omega = 1$ estimate (1.5) is the optimal estimate already obtained in the unweighted case in [1, 2, 13, 23, 31]. Indeed $\frac{d}{\pi_p}$ is the optimal constant of the one-dimensional unweighted Poincaré–Wirtinger inequality on a segment of length d (see for instance [3], [24]), namely:

$$\frac{\pi_p}{d} = \inf_{w_0^{1,p}(0,d)} \frac{(\int_0^d |u'|^p)^{1/p}}{(\int_0^d |u|^p)^{1/p}}.$$

Explicit expression for π_p (as (1.6)) can be found for instance in [3, 20, 24, 27, 28]. The fact that $\pi_2 = \pi$ is consistent with the classical Wirtinger inequality (see [17]) and obviously also with (1.2).

In the spirit of the proof of (1.2) by Payne and Weinberger [23], our proof of Theorem 1.1 is based on the following estimate on the best constant in a class of weighted Wirtinger inequalities.

PROPOSITION 1.1. *Let f be a positive log-concave function defined on $(0, L)$ and $p > 1$, then*

$$(1.7) \quad \inf_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u f = 0}} \frac{\int_0^L |u'(x)|^p f(x) dx}{\int_0^L |u(x)|^p f(x) dx} \geq \min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u = 0}} \frac{\int_0^L |u'(x)|^p dx}{\int_0^L |u(x)|^p dx} = \left(\frac{\pi_p}{L}\right)^p.$$

For an insight into generalized Wirtinger inequalities, and more generally into weighted Hardy inequalities, we refer to [18, 21, 30], other results can be found for instance in [10, 11, 14, 16, 25].

In Section 2, we prove Proposition 1.1, while in Section 3, we employ a ‘‘slicing argument’’ to pass from the n -dimensional to the one-dimensional case.

2. PROOF OF PROPOSITION 1.1

For the reader convenience we have split the claim in two Lemmata.

In [13] it has been proved the following lemma for which we include the proof for the sake of completeness.

LEMMA 2.1. *Let f be a smooth positive log-concave function defined on $[0, L]$ and $p > 1$. Then there exists $\kappa \in \mathbb{R}$ such that*

$$(2.1) \quad \inf_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u f = 0}} \frac{\int_0^L |u'(x)|^p f(x) dx}{\int_0^L |u(x)|^p f(x) dx} \geq \inf_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u e^{\kappa x} = 0}} \frac{\int_0^L |u'(x)|^p e^{\kappa x} dx}{\int_0^L |u(x)|^p e^{\kappa x} dx}.$$

PROOF. By standard compactness argument (see [18, Theorem 1.5 page 28]) the positive infimum on the left hand side of (2.1) is achieved by some function u_λ belonging to

$$\left\{ u \in W^{1,p}(0, L), \int_0^L |u(x)|^{p-2} u(x) f(x) dx = 0 \right\}.$$

As expected, such a minimizer is also a $C^1(0, L)$ solution to the following Neumann eigenvalue problem

$$(2.2) \quad \begin{cases} (-u'|u'|^{p-2})' = \lambda u|u|^{p-2} + h'(x)u'|u'|^{p-2} & x \in (0, L) \\ u'(0) = u'(L) = 0. \end{cases}$$

Here $h(x) = \log f(x)$ is a smooth bounded concave function and λ is the left hand side of (2.1). We emphasize that the usual derivation of (2.2) as the Euler Lagrange of a Rayleigh quotient is rigorous only to handle the case $p \geq 2$. A refined technique similar to the one worked out in [11, Lemma 2.4] is necessary when $1 < p < 2$.

Since it is not difficult to prove that for all $0 < L_1 < L$

$$\inf_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u f = 0}} \frac{\int_0^L |u'(x)|^p f(x) dx}{\int_0^L |u(x)|^p f(x) dx} < \inf_{\substack{u \in W^{1,p}(0,L_1) \\ \int_0^{L_1} |u|^{p-2} u f = 0}} \frac{\int_0^{L_1} |u'(x)|^p f(x) dx}{\int_0^{L_1} |u(x)|^p f(x) dx},$$

then u_λ vanishes in one and only one point namely $x_\lambda \in (0, L)$ and without loss of generality we may assume that $u_\lambda(L) < 0 < u_\lambda(0)$.

We claim that if $\kappa = h'(x_\lambda)$ then

$$\lambda \geq \min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u e^{\kappa x} = 0}} \frac{\int_0^L |u'(x)|^p e^{\kappa x} dx}{\int_0^L |u(x)|^p e^{\kappa x} dx} \equiv \bar{\lambda}.$$

Arguing by contradiction we assume that $\lambda < \bar{\lambda}$. Therefore there exists a function $u_{\bar{\lambda}}$ solution to

$$\begin{cases} (-u'|u|^{p-2})' = \bar{\lambda}u|u|^{p-2} + h'(x_\lambda)u'|u|^{p-2} & x \in (0, L) \\ u'(0) = u'(L) = 0. \end{cases}$$

Standard arguments ensure that $u_{\bar{\lambda}}$ is strictly monotone in $(0, L)$ and therefore vanishes in one and only one point namely $x_{\bar{\lambda}} \in (0, L)$. We assume without loss of generality that $u_{\bar{\lambda}}(L) < 0 < u_{\bar{\lambda}}(0)$. Since h' is non increasing in $[0, L]$, a straightforward consequence of the comparison principle applied to u_λ and $u_{\bar{\lambda}}$ on the interval $[0, x_\lambda]$ enforces $x_{\bar{\lambda}} < x_\lambda$. On the other hand the comparison principle applied to u_λ and $u_{\bar{\lambda}}$ on the interval $[x_\lambda, L]$ enforces $x_{\bar{\lambda}} > x_\lambda$ and eventually a contradiction arises. \square

LEMMA 2.2. For all $\kappa \in \mathbb{R}$ and $p > 1$

$$(2.3) \quad \min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u e^{\kappa x} = 0}} \frac{\int_0^L |u'(x)|^p e^{\kappa x} dx}{\int_0^L |u(x)|^p e^{\kappa x} dx} \geq \left(\frac{\pi_p}{L}\right)^p.$$

PROOF. If u minimizes the left hand side of (2.3) then it solves

$$\begin{cases} (-u'|u|^{p-2})' = \mu u|u|^{p-2} + \kappa u'|u|^{p-2} & x \in (0, L) \\ u'(0) = u'(L) = 0, \end{cases}$$

where

$$\mu = \min_{\substack{u \in W^{1,p}(0,L) \\ \int_0^L |u|^{p-2} u e^{\kappa x} = 0}} \frac{\int_0^L |u'(x)|^p e^{\kappa x} dx}{\int_0^L |u(x)|^p e^{\kappa x} dx}.$$

As in the previous Lemma u' can not vanish inside $(0, L)$ and we may assume without loss of generality that u is an increasing function such that $u(0) < 0 < u(L)$. Then the function $v(x) = \frac{u(x)}{u'(x)}$ is the increasing solution to the following problem

$$(2.4) \quad \begin{cases} v' = 1 + \frac{1}{p-1}(\mu|v|^p + \kappa v) & x \in (0, L) \\ \lim_{x \rightarrow L} v(x) = -\lim_{x \rightarrow 0} v(x) = +\infty, \end{cases}$$

where uniqueness and monotonicity come easily from the fact that the equation in (2.4) is autonomous.

In particular we observe that $\mu|y|^p + \kappa y + p - 1 = 0$ can not have solutions $y \in \mathbb{R}$. The fact that v' is bounded away from zero allows us to integrate $\frac{1}{v'}$ with respect to v obtaining

$$\begin{aligned} L &= \int_{-\infty}^{+\infty} \frac{1}{v'} dv = \int_{-\infty}^{+\infty} \frac{1}{1 + \frac{1}{p-1}(\mu|v|^p + \kappa v)} dv \\ &= \int_0^{+\infty} \left(\frac{1}{1 + \frac{1}{p-1}(\mu v^p + \kappa v)} + \frac{1}{1 + \frac{1}{p-1}(\mu v^p - \kappa v)} \right) dv \\ &\geq 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1}\mu v^p} dv, \end{aligned}$$

and the proof is complete observing that rescaling $s = \frac{\pi_p}{L}v$ in (1.6) gives

$$L = 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1} \left(\frac{\pi_p}{L}\right)^p v^p} dv. \quad \square$$

When the function f is smooth, log-concave and bounded away from zero, Proposition 1.1 is a consequence of Lemma 2.1 and Lemma 2.2. In the general case Proposition 1.1 follows by approximation arguments.

3. PROOF OF THEOREM 1.1

The aim of this section is to prove that Theorem 1.1 can be deduced from Proposition 1.1. As we already mentioned the idea is based on a slicing method worked out in [23] and proved in a slightly different way also in [1, 2, 8]. We outline the technique for the sake of completeness.

LEMMA 3.1. *Let Ω be a convex set in \mathbb{R}^n having diameter d , let ω be a positive log-concave function on Ω , and let u be any function such that $\int_{\Omega} |u(x)|^{p-2}u(x)\omega(x) dx = 0$. Then, for all positive ε , there exists a decomposition of the set Ω in mutually disjoint convex sets Ω_i ($i = 1, \dots, k$) such that*

$$\bigcup_{i=1}^k \bar{\Omega}_i = \bar{\Omega}$$

$$\int_{\Omega_i} |u(x)|^{p-2} u(x) \omega(x) dx = 0$$

and for each i there exists a rectangular system of coordinates such that

$$\begin{aligned} \Omega_i \subseteq \{ (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_\ell| \leq \varepsilon, \ell = 2, \dots, n \} \\ (d_i \leq d, i = 1, \dots, k) \end{aligned}$$

PROOF. Among all the $n - 1$ hyperplanes of the form $ax_1 + bx_2 = c$, orthogonal to the 2-plane $\Pi_{1,2}$ generated by the x_1 and x_2 directions, by continuity there exists certainly one that divides Ω into two nonempty subsets on each of which the integral of $u|u|^{p-2}\omega$ is zero and their projections on $\Pi_{1,2}$ have the same area. We go on subdividing recursively in the same way both subset and eventually we stop when all the subdomains $\Omega_j^{(1)}$ ($j = 1, \dots, 2^{N_1}$) have projections with area smaller than $\varepsilon^2/2$. Since the width w of a planar set of area A is bounded by the trivial inequality $w \leq \sqrt{2A}$, each subdomain $\Omega_j^{(1)}$ can be bounded by two parallel $n - 1$ hyperplanes of the form $ax_1 + bx_2 = c$ whose distance is less than ε . If $n = 2$ the proof is complete, provided that we understand $\Pi_{1,2}$ as \mathbb{R}^2 , the projection of Ω on $\Pi_{1,2}$ as Ω itself, and the $n - 1$ orthogonal hyperplanes as lines. If $n > 2$ for any given $\Omega_j^{(1)}$ we can consider a rectangular system of coordinates such that the normal to the above $n - 1$ hyperplanes which bound the set, points in the direction x_n . Then we can repeat the previous arguments and subdivide the set $\Omega_j^{(1)}$ in subsets $\Omega_j^{(2)}$ ($j = 1, \dots, 2^{N_2}$) on each of which the integral of $u|u|^{p-2}\omega$ is zero and their projections on $\Pi_{1,2}$ have the same area which is less than $\varepsilon^2/2$. Therefore, any given $\Omega_j^{(2)}$, can be bounded by two $n - 1$ hyperplanes of the form $ax_1 + bx_2 = c$ whose distance is less than ε . If $n = 3$ the proof is over. If $n > 3$ we can go on considering $\Omega_j^{(2)}$ and rotating the coordinate system such that the normal to the above $n - 1$ hyperplanes which bound $\Omega_j^{(2)}$, points in the direction x_{n-1} and such that the rotation keeps the x_n direction unchanged. The procedure ends after $n - 1$ iterations, at that point we have performed $n - 1$ rotations of the coordinate system and all the directions have been fixed. Up to a translation, in the resulting coordinate system

$$\Omega_j^{(n-1)} \subseteq \{ (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_j, |x_\ell| \leq \varepsilon, \ell = 2, \dots, n \}. \quad \square$$

PROOF OF THEOREM 1.1. From (1.4), using the density of smooth functions in Sobolev spaces it will be enough to prove that

$$\frac{\int_{\Omega} |Du|^p \omega}{\int_{\Omega} |u|^p \omega} \geq \left(\frac{\pi_p}{d}\right)^p$$

when u is a smooth function with uniformly continuous first derivatives and

$$\int_{\Omega} |u(x)|^{p-2} u(x) \omega(x) dx = 0.$$

Let u be any such function. According to Lemma 3.1 we fix $\varepsilon > 0$ and we decompose the set Ω in convex domains Ω_i ($i = 1, \dots, k$). We use the notation of Lemma 3.1 and we focus on one of the subdomains Ω_i and fix the reference system such that

$$\Omega_i \subseteq \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, |x_\ell| \leq \varepsilon, \ell = 2, \dots, n\}.$$

For $t \in [0, d_i]$ we denote by $g_i(t)$ the $n - 1$ volume of the intersection of Ω_i with the $n - 1$ hyperplane $x_1 = t$. Since Ω_i is convex, then by Brunn-Minkowski inequality (see [15]) g_i is a log-concave function in $[0, d_i]$.

Then, for any $t \in [0, d_i]$ we denote by $v(t) = u(t, 0, \dots, 0)$, and $f_i(t) = g_i(t)\omega(t, 0, \dots, 0)$. Since $u, \frac{\partial u}{\partial x_1}$ and ω are uniformly continuous in Ω , there exists a modulus of continuity $\eta(\cdot)$ ($\eta(\varepsilon) \searrow 0$ as $\varepsilon \rightarrow 0$) independent of the decomposition of Ω such that

$$(3.1) \quad \left| \int_{\Omega_i} \left| \frac{\partial u}{\partial x_1} \right|^p \omega \, dx - \int_0^{d_i} |v'(t)|^p f_i(t) \, dt \right| \leq \eta(\varepsilon) |\Omega_i|,$$

$$(3.2) \quad \left| \int_{\Omega_i} |u|^p \omega \, dx - \int_0^{d_i} |v(t)|^p f_i(t) \, dt \right| \leq \eta(\varepsilon) |\Omega_i|$$

and

$$(3.3) \quad \left| \int_0^{d_i} |v(t)|^{p-2} v(t) f_i(t) \, dt \right| \leq \eta(\varepsilon) |\Omega_i|.$$

Since $d_i \leq d$, and f_i are positive log-concave functions, applying Proposition 1.1 we have

$$\int_{\Omega_i} |Du|^p \omega \, dx \geq \int_{\Omega_i} \left| \frac{\partial u}{\partial x_1} \right|^p \omega \, dx \geq \left(\frac{\pi_p}{d} \right)^p \int_{\Omega_i} |u(x)|^p \omega \, dx + C\eta(\varepsilon) |\Omega_i|.$$

Here the constant C does not depend on ε . Summing up the last inequality over all $i = 1, \dots, k$

$$\int_{\Omega} |Du|^p \omega \, dx \geq \left(\frac{\pi_p}{d} \right)^p \int_{\Omega} |u(x)|^p \omega \, dx + C\eta(\varepsilon) |\Omega|.$$

and as $\varepsilon \rightarrow 0$ we obtain the desired inequality. □

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