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Mathematical Analysis — *Luzin's condition* (N) *and Sobolev mappings*, by Pekka Koskela, Jan Malý and Thomas Zürcher, communicated on 11 May 2012.

Abstract. — For the purpose of change of variables in integral, it is important to know how to verify Luzin's condition (N) for Sobolev mappings. In this article we survey some results on this topic sorted according to the method. We discuss the method of absolute continuity, results obtained via degree, and results based on the interplay between integrability and modulus of continuity.[1]

Key words: Luzin's condition (N), change of variables in integral, area formula, Sobolev mappings.

Mathematics Subject Classification: 26B15 (primary), 28A75, 30C65, 49Q15, 51M25 (secondary).

1. Introduction

An advanced version of the theorem on change of variables in integral is due to Federer [9], [10]. It states that the *area formula*

$$\int_M u(x)|J_f(x)|\,dx = \int_{\mathbb{R}^n}\Big(\sum_{\{x\in M: f(x)=y\}} u(x)\Big)\,dy \tag{1}$$

is valid for all non-negative measurable functions $u:\Omega\to\mathbb{R}$ if $\Omega\subset\mathbb{R}^n$ is an open set, $f:\Omega\to\mathbb{R}^n$ approximately differentiable a.e. (for example, a Sobolev mapping) and $M\subset\Omega$ a suitable set of full measure. In particular, we can take $M=\mathscr{D}_{\mathrm{app}}(f)$, namely, the set where f is approximately Lipschitz continuous. It is desirable to have as small as possible or even empty exceptional set $\Omega\backslash M$. This problem leads us to consider the so-called *Luzin's condition* (N).

Definition 1 (Condition (N), area formula). Let $\Omega\subset\mathbb{R}^n$ be an open set and $f:\Omega\to\mathbb{R}^n$ a mapping. We say that f satisfies (Luzin's) condition (N) on a set $\Omega'\subset\Omega$ if the implication

$$|E|=0 \quad\Rightarrow\quad |f(E)|=0$$

holds for each subset E of Ω'. We say that the *area formula* holds for f on Ω' if (1) holds for each measurable subset M of Ω' with the choice $u=1$ (then it holds

[1] The results of this paper are related to the lecture that the second author gave at the Conference *"Geometric Function Theory"*, which took place at the Accademia dei Lincei on November 3rd 2011.

for an arbitrary measurable $u \geq 0$.) If $\Omega' = \Omega$ above, then we simply say that the area formula holds for f.

Strictly speaking, above we consider a precise representative of f, see e.g. [8]. Now, let us summarize that for a Sobolev mapping (this means $W^{1,1}_{\text{loc}}$), the area formula is equivalent to the condition (N).

The aim of this article is to survey some results on the condition (N) (area formula) and to announce some new progress.

In the scale of Sobolev spaces, the area formula for $W^{1,\infty}$ is just the Lipschitz setting and thus a particular case of Federer's results. It is also classical that the area formula holds for absolutely continuous functions on the real line, which is the case $n = p = 1$. If the space dimension n is strictly larger than 1, the situation is much more complicated. Marcus and Mizel [38] proved that the area formula holds for $W^{1,p}$-mappings if $p > n$, see also [3]. On the other hand, there are planar examples due to Cesari [3] and Reshetnyak [46] according to which the area formula can fail for (continuous) $W^{1,n}$-mappings, for higher dimension see Väisälä [51] and Malý and Martio [37]. The borderline case $p = n$ is actually rather delicate. Indeed, the validity of the area formula can be retrieved if we refine the scale or impose a suitable additional condition. Reshetnyak proved that $W^{1,n}$-homeomorphisms [44] and quasiregular mappings (mappings of bounded distortion) [45] satisfy the area formula. Gol'dshtein and Vodop'yanov [16] extended the latter to the class of mappings of finite distortion. The topological condition of being homeomorphic has been also relaxed in various directions, see e.g. Reshetnyak [46], Martio and Ziemer [39], and Malý and Martio [37] for results in this spirit.

For other related results see Alberti and Ambrosio [1], Giaquinta, Modica and Souček [13], Hajłasz [20], [21], Kauhanen [28], Swanson [50], Vodop'yanov [52], Vodop'yanov, Gol'dshtein and Reshetnyak [53], and the monographs Reshetnyak [47], Gol'dshtein and Reshetnyak [15], Giaquinta, Modica and Souček [14], and [11]. We also refer to some other sources later in connection with particular methods.

This note is organized as follows. In sections 2–4 we describe three different methods for proving condition (N) and state the corresponding main results. In the final section, Section 5, we pose open questions and make some related comments.

2. The method of absolute continuity and its generalizations

We will consider some n-dimensional concepts of absolute continuity. Let us note that not all classes of functions which reduce to absolutely continuous functions when $n = 1$, or bear the name of "n-dimensional absolutely continuous functions" (e.g. in the sense of Banach or Tonelli) are in our focus of interest.

Following the traditional ε–δ definition of absolutely continuous functions on the real line, the following n-dimensional generalization was introduced in [35].

DEFINITION 2. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f : \Omega \to \mathbb{R}$ is *n-absolutely continuous* (shortly AC^n) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every pairwise disjoint finite collection $\{B_j\}$ of balls in Ω we have

$$\sum_j |B_j| < \delta \quad \Rightarrow \quad \sum_j (\operatorname{osc}_{B_j} f)^n < \varepsilon.$$

The ε–δ definition is not the only way to introduce the class of absolutely continuous functions on the real line. We may say that $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous if there exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that $|f(b) - f(a)| \leq \int_a^b g(x)\, dx$ for each $a < b$. This yields locally the same system of functions. The n-dimensional analogue has been considered by Radó and Reichelderfer [43].

DEFINITION 3. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f : \Omega \to \mathbb{R}$ is *generalized Lipschitz continuous of class RR^n* if there exists a function $\theta \in L^1_{\mathrm{loc}}(\Omega)$ such that for each ball B in Ω we have

$$(\operatorname{osc}_B f)^n \leq \int_B \theta(x)\, dx.$$

By defining $\operatorname{osc}_B f = \sup\{|f(x) - f(y)| : x, y \in B\}$, the two definitions above can be used as well for mappings. It has been observed in [35] that these classes of mappings provide a unified approach to various results on the area formula. Namely, the AC^n condition can be verified in several situations (we will mention them below), and the following theorem holds.

THEOREM 1 [35]. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be an AC^n mapping. Then $f \in W^{1,n}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and f satisfies condition (N). Hence the area formula holds for f.*

For the class RR^n, the following result goes back to Radó and Reichelderfer.

THEOREM 2 [43]. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be generalized Lipschitz continuous of class RR^n. Then the area formula holds for f.*

It is an easy observation that the RR^n property implies AC^n, so that Theorem 2 is a consequence of Theorem 1. Moreover, the converse holds as well (but it is much deeper).

THEOREM 3 (Csörnyei [5]). *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be generalized Lipschitz continuous of class RR^n. Then f is an AC^n mapping.*

2.1. Sufficient conditions for AC^n

We give rather sharp criteria for a mapping to have the AC^n property. Let us begin with the setting of rearrangement invariant spaces (for the definition see e.g. [4]). The following result has been given in [29]. The main achievements were (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). The implications (ii) $\Rightarrow$ (iii)–(vi) are from [35]. For some other implications we refer to [4] and [48]. Recall that $L_{n,1}(\mathbb{R}^n)$ is the Lorentz space of all functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_0^\infty |\{x : |g(x)| > s\}|^{1/n} \, ds < \infty.$$

If Y is a Banach space of functions on $\mathbb{R}^n$, then $W^1(Y)$ is the space of all functions $u \in Y$ such that $\nabla u \in Y$, with the norm $\|u\|_Y + \|\nabla u\|_Y$.

Theorem 4 [29]. *Let Y be a rearrangement invariant Banach space of functions on $\mathbb{R}^n$. Then the following assertions are equivalent:*

(i) *Y embeds continuously into $L_{n,1}(\mathbb{R}^n)$ (the Lorentz space),*
(ii) *all $W^1(Y)$-functions can be represented as locally AC^n functions,*
(iii) *all continuous $W^1(Y;\mathbb{R}^n)$-mappings satisfy condition (N),*
(iv) *all $W^1(Y)$-functions have a continuous representative,*
(v) *all $W^1(Y)$-functions are in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$,*
(vi) *all continuous $W^1(Y)$-functions are a.e. differentiable,*
(vii) *$W^1(Y)$ embeds into $C_0(\mathbb{R}^n)$,*
(viii) *$W^1(Y)$ embeds into $L^\infty(\mathbb{R}^n)$.*

Notice that this theorem implies the area formula for $W^{1,p}$-mappings for $p > n$ [38].

We can recover some other classical results on the area formula (for references see Introduction) noticing that certain classes of function are contained (locally) in the AC^n class. This is often the case if we consider a $W^{1,n}$-mapping f obeying some additional qualitative properties, like being a homeomorphism, or a continuous and open mapping, or a mapping of finite distortion. The essential information which allows us to prove the AC^n property is a kind of monotonicity condition, as shown by the theorem below.

Definition 4. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a mapping. We say that f is *spherically pseudomonotone* if there exists a constant C such that

$$\operatorname{osc}_B f \leq C \operatorname{osc}_{\partial B} f$$

for each ball $B \subset\subset \Omega$.

Theorem 5 [35]. *Let $f \in W^{1,n}(\Omega, \mathbb{R}^d)$. If f is spherically pseudomonotone, then f is an AC^n mapping.*

3. The Method of Jacobian and degree

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a mapping $f : \Omega \to \mathbb{R}^n$ is *sense preserving*, if f is continuous and for each open set $G \subset\subset \Omega$ and every $y \in f(G) \setminus f(\partial G)$, the topological degree $\deg(f, G, y)$ is strictly positive.

If a Sobolev mapping $f : \Omega \to \mathbb{R}^n$ is sense preserving, then one can achieve condition (N) under the assumptions that the degree can be represented via the pointwise Jacobian of f and this Jacobian is locally integrable. Roughly, the idea is that if $G \subset\subset \Omega$ then

$$|f(G)\backslash f(\partial G)| \leq \int_{f(G)\backslash f(\partial G)} \deg(f, G, y)\, dy \leq \int_G J_f(x)\, dx.$$

Recall that J_f is the pointwise Jacobian computed from the Sobolev derivative ∇f. For smooth f, one may represent the degree via the Jacobian as

$$\deg(f, G, y) = \int_G \varphi(f(x)) J_f(x)\, dx$$

whenever y belongs to a component U of $f(G)\backslash f(\partial G)$ and $\varphi \in C_0^\infty(U)$ satisfies $\int_U \varphi = 1$. Integrating by parts, one ends up with a formula involving the distributional Jacobian $\operatorname{Det} \nabla f$. By approximation, this formula extends to hold for rather general Sobolev mappings g, and one is reduced to the question whether the distributional Jacobian of g can be represented by $J_g = \det \nabla g$. If this is the case, we simply write "$\operatorname{Det} \nabla g = \det \nabla g$".

The following result gives us a setting where the degree approach is applicable.

THEOREM 6 [30]. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n-1$ be a continuous mapping. If f is sense preserving and $\operatorname{Det} \nabla f = \det \nabla f$, then f satisfies condition (N).*

Let us briefly discuss the most important results on validity of the formula

$$\operatorname{Det} \nabla f = \det \nabla f. \tag{2}$$

It has been proved by Müller [40] that (2) holds if $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p \geq \frac{n^2}{n+1}$ and if the distribution $\operatorname{Det} \nabla f$ can additionally be represented as an integrable function. This is a general criterion. However, sometimes it is desirable to make a judgement only based on knowledge of ∇f and its integrability. An important sufficient condition is

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega |\nabla f|^{n-\varepsilon}\, dx = 0,$$

assuming that $J_f(x) = \det \nabla f(x) \geq 0$ almost everywhere. Here one really needs the assumption that the Jacobian has constant sign. This result is due to Iwaniec and Sbordone [25] and Greco [17]. Sharp sufficient conditions in the Orlicz scales are discussed in the papers by Greco [18], Greco, Iwaniec and Moscariello [19], Koskela and Zhong [34], and Kauhanen, Koskela, Malý, Onninen and Zhong [31]. Especially, it suffices that

$$\int_\Omega |\nabla f|^n \log^{-1}(e + |\nabla f|)\, dx < \infty$$

when $J_f(x) \geq 0$ almost everywhere. For an interesting sufficient condition in terms of minors we refer to the paper [12] by Giannetti, Iwaniec, Onninen and Verde. For further studies on this see e.g. the works of Šverák [49], Müller, Qi and Yan [41], Müller and Spector [42], Fonseca and Gangbo [11], Hamburger [23], Iwaniec [24], Jerrard and Soner [26], De Lellis [6], De Lellis and Ghiraldin [7], and Brezis and Nguyen [2].

Remarks 1. 1. Concerning the counterexample by Ponomarev (f is a $W^{1,p}$ homeomorphism with $p < n$ violating the condition (N)), the distributional Jacobian of f has a singular part. In this example, one necessarily has $\int_\Omega |\nabla f|^n \log^{-1}(e + |\nabla f|)\, dx = \infty$.

2. By the degree method, we can obtain results below $W^{1,n}$. Since the class AC^n is locally contained in $W^{1,n}$, these results cannot be obtained through the AC^n condition.

4. Methods involving the modulus of continuity

Let B be a ball in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ be a measurable function. We use the notation

$$\operatorname{med}_{B(x,r)} f = \inf\left\{ s > 0 : |\{x \in B : f(x) > s\}| < \frac{1}{2}|B| \right\}$$

for the upper median of f in B.

Definition 5 (Moduli of continuity). Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^d$ be a measurable function. We define two moduli of continuity for $f : \Omega \to \mathbb{R}^d$ which make sense for x, r such that $B(x,r) \subset \Omega$.

- Classical modulus: $\omega_c(f, x, r) = \sup_{B(x,r)} |f - f(x)|$.
- Median modulus: $\omega(f, x, r) = \operatorname{med}_{B(x,r)} |f - f(x)|$.

Of course, we have $\omega(f, x, r) \leq \omega_c(f, x, r)$, so that the results based on $\omega(f, x, r)$ have weaker assumptions.

We start with the following result, due to Malý and Martio [37].

Theorem 7. *Let $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ be Hölder continuous. Then f satisfies condition (N).*

More generally, condition (N) holds on the set of all points where $\omega(x, f, r) \lesssim r^{\alpha}$ for some α.

Remark 1. Results in spirit of Theorem 7 have a chance to be established in metric measure space setting, see e.g. [36].

Corollary 1 [37]. *If $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ is precisely represented, then there exists a set E of Hausdorff dimension zero such that f satisfies condition (N) on $\Omega \backslash E$.*

4.1. Results in a finer scale

We want to announce our new result in a finer scale of moduli of continuity.

Theorem 8 [33]. *Let $0 < \lambda \leq n-1$ and $\mu > 0$. Set*

$$\psi(t) = \begin{cases} \exp(-\mu \log^{1-\lambda/(n-1)}(1/r)), & \text{if } \lambda < n-1, \\ \log^{-\mu}(1/r), & \text{if } \lambda = n-1. \end{cases}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a precisely represented Sobolev mapping. Assume that $\int_{\Omega} |\nabla f|^n \log^{\lambda}(e + |\nabla f|)\, dx < \infty$. Let $E \subset \Omega$ be a Lebesgue null set. Suppose that $\omega(x, f, r) \lesssim \psi$ at all points of E. Then $|f(E)| = 0$.

Remark 2. If $\lambda > n-1$, then the gradient of f is in $L_{n,1}$ and condition (N) follows without any modulus of continuity assumption. The case $\lambda = 0$ corresponds to Theorem 7. If $\lambda < 0$, then we are below $W^{1,n}$ and the method does not seem to work.

5. Further comments and open problems

Recall from Section 3 that condition (N) holds for all Sobolev homeomorphisms with

$$\int_{\Omega} |\nabla f|^n \log^{-1}(e + |\nabla f|)\, dx < \infty;$$

the Jacobian of such a homeomorphism necessarily has constant sign in each connected component. By Section 2, another sufficient condition is that $f \in W^{1,n}(\Omega, \mathbb{R}^d)$ be pseudomonotone. We do not know if a pseudomonotone Sobolev mapping with

$$\int_{\Omega} |\nabla f|^n \log^{-1}(e + |\nabla f|)\, dx < \infty$$

could always satisfy condition (N), even if $J_f(x) \geq 0$ almost everywhere. The obstacle here is that f may well fail to be sense preserving, and thus the technique from Section 3 does not apply.

Secondly, from Section 4, a general $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ satisfies condition (N) outside a set of Hausdorff dimension zero. We do not know if such an exceptional set of Hausdorff dimension strictly less than n could exist for a general Sobolev mapping with

$$\int_\Omega |\nabla f|^n \log^{-\lambda}(e + |\nabla f|)\, dx < \infty$$

for $0 < \lambda < 1$. By [28], this may fail if $\lambda > 1$.

Thirdly, recall the moduli of continuity $\exp(-\mu \log^a(1/r))$ from Section 4. Given $0 \le \lambda < n-1$, one can construct examples [33] of Sobolev mappings f with this modulus of continuity for any $a < \frac{n-1-\lambda}{n}$ for some $\mu > 0$ and with $\int_\Omega |\nabla f|^n \log^{\lambda}(e + |\nabla f|)\, dx < \infty$ so that condition (N) fails. Notice the slight mismatch of a with the exponent from Section 4. We do not know if this is because of the constructions not being optimal or because the arguments are not optimal. When $\lambda = 0$ and $n = 2$, the exponent a above is $\frac{1}{2}$. This very same exponent shows up in the results [27] on the boundary behavior of conformal maps of the unit disk. In fact, any planar $W^{1,2}$-mapping with this modulus of continuity necessarily maps regular Cantor sets to sets of vanishing area [32]. Here regularity requires that the complement of the set be a so-called uniform domain.

Fourthly, in all (continuous) counterexamples to condition (N), one necessarily maps a perfect compact set onto a set of positive measure. For example, in the setting of continuous mappings in $W^{1,p}(\Omega, \mathbb{R}^n)$, where $p > n-1$, one may additionally assume that the set be totally disconnected, see [33]. Hence, at least in this case, one can find a compact set homeomorphic to the ternary Cantor set that gets mapped onto a set of positive measure. We would like to know how generally this phenomenon holds and if an even more regular set that gets blown up would necessarily exist.

For $q > 1$, one can find a continuous f that maps a regular Cantor set onto a cube and so that the gradient of f belongs to the Lorentz space $L_{n,q}(\mathbb{R}^n)$. This issue is related to the problem of covering compact metric spaces by the unit cube. For recent studies on this see [22] for the case of Sobolev spaces and [54] in the Lorentz space setting.

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REFERENCES

[1] G. ALBERTI - L. AMBROSIO, *A geometrical approach to monotone functions in* $\mathbf{R}^n$, Math. Z., 230(2):259–316, 1999.

[2] H. BREZIS - H.-M. NGUYEN, *The Jacobian determinant revisited*, Invent. Math., 185(1):17–54, 2011.

[3] L. CESARI, *Sulle trasformazioni continue*, Ann. Mat. Pura Appl. (4), 21:157–188, 1942.

[4] A. CIANCHI - L. PICK, *Sobolev embeddings into BMO, VMO, and L_∞*, Ark. Mat., 36(2):317–340, 1998.

[5] M. CSÖRNYEI, *Absolutely continuous functions of Rado, Reichelderfer, and Malý*, J. Math. Anal. Appl., 252(1):147–166, 2000.

[6] C. DE LELLIS, *Some remarks on the distributional Jacobian*, Nonlinear Anal., 53(7–8):1101–1114, 2003.

[7] C. DE LELLIS - F. GHIRALDIN, *An extension of the identity* $\mathbf{Det} = \mathbf{det}$, C. R. Math. Acad. Sci. Paris, 348(17–18):973–976, 2010.

[8] L. C. EVANS - R. F. GARIEPY, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[9] H. FEDERER, *Surface area. II*, Trans. Amer. Math. Soc., 55:438–456, 1944.

[10] H. FEDERER, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969 (Second edition 1996).

[11] I. FONSECA - W. GANGBO, *Degree theory in analysis and applications*, volume 2 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[12] F. GIANNETTI - T. IWANIEC - J. ONNINEN - A. VERDE, *Estimates of Jacobians by subdeterminants*, J. Geom. Anal., 12(2):223–254, 2002.

[13] M. GIAQUINTA - G. MODICA - J. SOUČEK, *Area and the area formula*, In Proceedings of the Second International Conference on Partial Differential Equations (Italian) (Milan, 1992), volume 62, pages 53–87 (1994), 1992.

[14] M. GIAQUINTA - G. MODICA - J. SOUČEK, *Cartesian currents in the calculus of variations. I*, volume 37 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998. Cartesian currents.

[15] V. M. GOL'DSHTEĬN - Y. G. RESHETNYAK, *Quasiconformal mappings and Sobolev spaces*, volume 54 of Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1990. Translated and revised from the 1983 Russian original, Translated by O. Korneeva.

[16] V. M. GOL'DSTEIN - S. VODOP'YANOV, *Quasiconformal mappings and spaces of functions with generalized first derivatives*, Siberian Math. J., 17(3):515–531, 1977.

[17] L. GRECO, *A remark on the equality* $\det Df = \mathrm{Det}\, Df$, Differential Integral Equations, 6(5):1089–1100, 1993.

[18] L. GRECO, *Sharp integrability of nonnegative Jacobians*, Rend. Mat. Appl. (7), 18(3):585–600, 1998.

[19] L. GRECO - T. IWANIEC - G. MOSCARIELLO, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J., 44(2):305–339, 1995.

[20] P. HAJŁASZ, *Change of variables formula under minimal assumptions*, Colloq. Math., 64(1):93–101, 1993.

[21] P. HAJŁASZ, *Sobolev mappings, co-area formula and related topics*, In Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), pages 227–254. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[22] P. HAJŁASZ - J. T. TYSON, *Sobolev Peano cubes*, Michigan Math. J., 56(3):687–702, 2008.

[23] C. Hamburger, *Some properties of the degree for a class of Sobolev maps*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 455(1986):2331–2349, 1999.

[24] T. Iwaniec, *On the concept of the weak Jacobian and Hessian*, In Papers on analysis, volume 83 of Rep. Univ. Jyväskylä Dep. Math. Stat., pages 181–205. Univ. Jyväskylä, Jyväskylä, 2001.

[25] T. Iwaniec - C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal., 119(2):129–143, 1992.

[26] R. L. Jerrard - H. M. Soner, *Functions of bounded higher variation*, Indiana Univ. Math. J., 51(3):645–677, 2002.

[27] P. W. Jones - N. G. Makarov, *Density properties of harmonic measure*, Ann. of Math. (2), 142(3):427–455, 1995.

[28] J. Kauhanen, *Failure of the condition N below* $W^{1,n}$, Ann. Acad. Sci. Fenn. Math., 27(1):141–150, 2002.

[29] J. Kauhanen - P. Koskela - J. Malý, *On functions with derivatives in a Lorentz space*, Manuscripta Math., 100(1):87–101, 1999.

[30] J. Kauhanen - P. Koskela - J. Malý, *Mappings of finite distortion: condition N*, Michigan Math. J., 49(1):169–181, 2001.

[31] J. Kauhanen - P. Koskela - J. Malý - J. Onninen - X. Zhong, *Mappings of finite distortion: sharp Orlicz-conditions*, Rev. Mat. Iberoamericana, 19(3):857–872, 2003.

[32] A. Kauranen - P. Koskela - A. Zapadinskaya, *Planar Sobolev mappings and area*, In preparation, 2011.

[33] P. Koskela - J. Malý - T. Zürcher, *Lusin's condition* (N) *and modulus of continuity*, In preparation, 2011.

[34] P. Koskela - X. Zhong, *Minimal assumptions for the integrability of the Jacobian*, Ricerche Mat., 51(2):297–311 (2003), 2002.

[35] J. Malý, *Absolutely continuous functions of several variables*, J. Math. Anal. Appl., 231(2):492–508, 1999.

[36] J. Malý, *Sufficient conditions for change of variables in integral*, In Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), pages 370–386. Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[37] J. Malý - O. Martio, *Lusin's condition* (N) *and mappings of the class* $W^{1,n}$, J. Reine Angew. Math., 458:19–36, 1995.

[38] M. Marcus - V. J. Mizel, *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems*, Bull. Amer. Math. Soc., 79:790–795, 1973.

[39] O. Martio - W. P. Ziemer, *Lusin's condition* (N) *and mappings with nonnegative Jacobians*, Michigan Math. J., 39(3):495–508, 1992.

[40] S. Müller, Det $=$ det. *A remark on the distributional determinant*, C. R. Acad. Sci. Paris Sér. I Math., 311(1):13–17, 1990.

[41] S. Müller - T. Qi - B. S. Yan, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11(2):217–243, 1994.

[42] S. Müller - S. J. Spector, *An existence theory for nonlinear elasticity that allows for cavitation*, Arch. Rational Mech. Anal., 131(1):1–66, 1995.

[43] T. Rado - P. V. Reichelderfer, *Continuous transformations in analysis. With an introduction to algebraic topology*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. LXXV. Springer-Verlag, Berlin, 1955.

[44] J. G. Rešetnjak, *Certain geometric properties of functions and mappings with generalized derivatives*, Sibirsk. Mat. Ž., 7:886–919, 1966.

[45] J. G. Rešetnjak, *Spatial mappings with bounded distortion*, Sibirsk. Mat. Ž., 8:629–658, 1967.

[46] Y. G. Reshetnyak, *The N condition for spatial mappings of the class $W^1_{n,\mathrm{loc}}$*, Sibirsk. Mat. Zh., 28(5):149–153, 1987.

[47] Y. G. Reshetnyak, *Space mappings with bounded distortion*, volume 73 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1989. Translated from the Russian by H. H. McFaden.

[48] E. M. Stein, *Editor's note: the differentiability of functions in* $\mathbf{R}^n$, Ann. of Math. (2), 113(2):383–385, 1981.

[49] V. Šverák, *Regularity properties of deformations with finite energy*, Arch. Rational Mech. Anal., 100(2):105–127, 1988.

[50] D. Swanson, *Area, coarea, and approximation in* $W^{1,1}$, Ark. Mat., 45(2):381–399, 2007.

[51] J. Väisälä, *Quasiconformal maps and positive boundary measure*, Analysis, 9(1–2):205–216, 1989.

[52] S. K. Vodop'yanov, *Topological and geometric properties of mappings with an integrable Jacobian in Sobolev classes. I*, Sibirsk. Mat. Zh., 41(1):23–48, i, 2000.

[53] S. K. Vodop'yanov - V. M. Gol'dshteĭn - Y. G. Reshetnyak, *The geometric properties of functions with generalized first derivatives*, Uspekhi Mat. Nauk, 34(1(205)):17–65, 1979.

[54] K. Wildrick - T. Zürcher, *Peano cubes with derivatives in a Lorentz space*, Illinois J. Math., 53(2):365–378, 2009.

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