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Solid Mechanics — *Parabolic tunnels in a heavy elastic medium*, by M. J. Leitman and P. Villaggio, communicated on 9 November 2012.

Dedicated to the memory of Gaetano Fichera in recognition of his contributions to the Theory of Elasticity

Abstract. — We consider an elastic half-space subject to constant body forces acting perpendicular to its surface. Assume that the medium is perforated by a parabolic cylindrical cavity whose plane of symmetry is perpendicular to the surface. We characterize the state of stress in the medium; in particular, we compute the hoop stresses along the boundary of the cavity. Our solution is obtained by applying the complex variable method in plane elasticity, extending the technique to stress states which do not necessarily vanish at infinity.

Key words: Plane elasticity, complex variables, soil mechanics.

Mathematics Subject Classification: 74B05, 74L10.

1. Introduction

A tunnel is an underground passage through a hill or mountain or beneath the bed of a river. Natural tunnels have served as habitations for humans since the Stone Age. Artificial tunnels, created by superposing stone blocks, represent a brilliant architectural achievement as early as the sixth century BCE (Sparacio [7]). Tunnels were drilled through the Alps in the second half of the 19th Century, though at enormous cost and sacrifice of lives (Orava [3]).

The first mathematical models of tunnels were formulated at the beginning of the 18th Century, but only considering the statics of the arch regarded as a vault loaded by prescribed vertical forces. However, the arch is in contact with the material above, which behaves as an elastic medium subject to gravitational body forces. The problem then arises of determining the interaction between the arch and the heavy material above. Some brilliant solutions were found by Mindlin [2] and recorded in the books of Savin [5] and Poulos and Davis [4]. The latter has a wider collection of particular solutions. The cases in the literature consider a circular cavity in the plane or near the boundary of a half-plane. These solutions, though elegant, are not always realistic since the cross-sections of rail and road tunnels are often parabolic, not circular (see Schleicher [6]).

In this paper we propose an elastic solution for a parabolic cavity piercing an heavy, elastic, half-plane. The solution is obtained by applying the complex variable method in plane elasticity using the conformal map of the exterior of a parabola onto a half-plane. Our model is more realistic, for it takes into account the fact that material weighing on the arch is not infinite but bounded above by a free surface, such as a plane, as shown in Fig. 1. However, the surface tractions on the plane do not vanish in our proposed solution. Therefore we must further discuss in just what sense our solution can be deemed acceptable.

In addition, our elastic solution is physically questionable, since it predicts tensile hoop stresses near the vertex. But tunnels are usually drilled through materials notoriously unable to support tensile stresses. To avoid these tensile stresses, the material near the vertex must be reinforced in some way. An analytic criterion for determining the required superposed stress state is proposed here.

2. The elastic solution for the free cavity

We consider a parabola in the (x, y)-plane described by

$$y \mapsto x = \xi_0^2 - \left(\frac{y}{2\xi_0}\right)^2, \tag{2.1}$$

where $\xi_0 > 0$ is a parameter. The focus is at the origin and the vertex has coordinates $(\xi_0^2, 0)$. The infinite, closed region L, above the parabola and below the half-plane at height $H \geq \xi_0^2$, is occupied by a heavy elastic medium (earth or rock) weighing upon the parabola while the infinite, closed region R, below the parabola and above the half-plane, is empty (see Fig. 1). We assume that the density of the medium, $\gamma > 0$, is constant so that the distribution of the body forces can be represented by

$$b_x = -\gamma \quad \text{and} \quad b_y = 0. \tag{2.2}$$

The parabola in Fig. 1 is the image in the $(z = x + \imath y)$-plane of the line $\xi = \xi_0$ in the $(\zeta = \xi + \imath\eta)$-plane under the conformal map

$$\zeta \mapsto z = m(\zeta) := \zeta^2. \tag{2.3}$$

Equivalently, we have

$$x = \xi^2 - \eta^2 \quad \text{and} \quad y = 2\xi\eta. \tag{2.4}$$

Setting $\xi = \xi_0$ and eliminating η yields Eq. (2.1) (see Figs. 1, 2). The map in Eq. (2.3) is admissible, since the only point at which $m'(\zeta) = 0$ is the origin, which lies in the region R.

The stress state in L may be regarded as the superposition of two stress states: that induced by the body forces of Eq. (2.2), called the *fundamental stress state*, and an *extra stress state* chosen to satisfy the boundary conditions on the parabola.

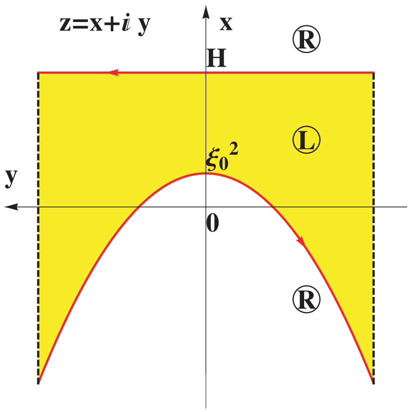

Figure 1: Tunnel
z-Plane: $z = x + \imath y$

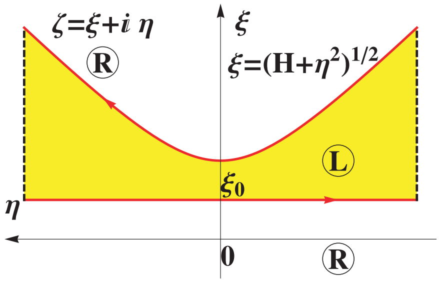

Figure 2: Half-plane
ζ-Plane: $\zeta = \xi + \imath\eta$.

The body forces derive from a potential, say V in the notation of Milne-Thomson [1, §1.72], such that[1]

$$(2.5)\qquad -\gamma = b_x - \imath b_y = \left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right)V = 2\frac{\partial}{\partial z}V.$$

Generalizing Milne-Thomson's method for constructing a class of compatible stresses, we take

$$(2.6)\qquad V(z,\bar{z}) := -\frac{\gamma}{2}(z+\bar{z}) + \gamma H,$$

where $H \geq \xi_0^2$ is the height of the half-plane described above. Now introduce a real-valued function Q such that $\nabla^2 Q = -V$ or

$$(2.7)\qquad 4\frac{\partial^2}{\partial z\partial\bar{z}}Q(z,\bar{z}) = -V(z,\bar{z}) = \frac{\gamma}{2}(z+\bar{z}) - \gamma H.$$

An integral of Eq. (2.7) is

$$(2.8)\qquad Q(z,\bar{z}) = \frac{\gamma}{16}[(z^2\bar{z} + z\bar{z}^2) - 4z\bar{z}H - z^2H].$$

Since Q is bi-harmonic, the strain state induced by the body forces is admissible. If we denote the the stresses in the physical z-plane induced by the body forces,

[1] We systematically use the notation of Milne-Thomson's classic monograph [1].

the fundamental stresses, by σ_x°, σ_y°, τ_{xy}°, the corresponding fundamental stress combinations are given by (see Milne-Thomson [1, §2.10]):

$$\sigma_x^\circ + \sigma_y^\circ = \Theta_0 = (8-4v)\frac{\partial^2}{\partial z \partial \bar{z}} Q(z,\bar{z}) = \gamma(2-v)\Big(\frac{z+\bar{z}}{2} - H\Big), \tag{2.9}$$

$$\sigma_y^\circ - \sigma_x^\circ + 2\imath\tau_{xy}^\circ = \Phi_0 = -4v\frac{\partial^2}{\partial z^2} Q(z,\bar{z}) = \frac{\gamma}{2} v(H-\bar{z}), \tag{2.10}$$

where v is given in terms of Poisson's ratio, η, by

$$v := \frac{1-2\eta}{1-\eta}. \tag{2.11}$$

Note that $0 \leq v < 3/2$ and $v = 0$ implies the material is incompressible.

From Eqs. (2.9 and 2.10), we recover the fundamental stresses:

$$\sigma_x^\circ = -\frac{\gamma}{4}(4-v)(H-x) \quad \sigma_y^\circ = -\frac{\gamma}{4}(4-3v)(H-x) \quad \tau_{xy}^\circ = \frac{\gamma}{4} v y. \tag{2.12}$$

This fundamental state satisfies the equilibrium equations. Moreover, on the half-plane $x = H$, we have $\sigma_x^\circ(H,y) = 0$, $\sigma_y^\circ(H,y) = 0$ but *not* the condition $\tau_{xy}^\circ(H,y) = 0$. Consequently, the boundary $x = H$ is *not* a free surface for this stress state. However, we may accept it as an approximation to the true fundamental state when the magnitude of y is small; that is, it should approximate the true fundamental state near the vertex of the parabola. However, this situation may change once an extra stress state is added so that the parabolic surface is free. There are other expressions for the analytical form of the fundamental state (See *e.g.* Worch [8, p. 54].) But Eqs. (2.9 and 2.10) have the advantage of being extendible to more general distributions of body forces which derive from a potential.

From Eqs. (2.9 and 2.10), we can obtain an expression for the fundamental stress vector in L of the ζ-plane:

$$\begin{aligned} 2(\sigma_\xi^\circ + \imath\tau_{\xi\eta}^\circ) &= \Theta_0(m(\zeta)) - \overline{\Phi_0(m(\zeta))}\Big(\frac{\overline{m'(\zeta)}}{m'(\zeta)}\Big) \\ &= -\gamma(2-v)\Big(H - \frac{\zeta^2+\bar{\zeta}^2}{2}\Big) - \frac{\gamma}{2}v(H-\zeta^2)\frac{\bar{\zeta}}{\zeta}. \end{aligned} \tag{2.13}$$

The fundamental stress vector does not vanish on the boundary of the parabola. Indeed, if we evaluate this stress on the boundary of the parabola by setting $\bar{\zeta} = 2\xi_0 - \zeta$, we get

$$\begin{aligned} 2(\sigma_\xi^\circ + \imath\tau_{\xi\eta}^\circ)|_{\bar{\zeta}=2\xi_0-\zeta} &= \frac{\gamma}{2}(4-3v)\zeta^2 - \gamma(4-3v)\xi_0\zeta \\ &\quad - \frac{\gamma}{2}[(4-3v)H - 4(2-v)\xi_0^2] - \gamma v \xi_0 H\frac{1}{\zeta}. \end{aligned} \tag{2.14}$$

Since we require the parabola to be a free surface, it is necessary to add an excess stress state σ_ξ^*, σ_η^*, $\tau_{\xi\eta}^*$ so that, on the parabola,

$$\begin{aligned}(2.15)\qquad 2(\sigma_\xi^* + \imath\tau_{\xi\eta}^*)|_{\bar\zeta=2\xi_0-\zeta} &= -2(\sigma_\xi^\circ + \imath\tau_{\xi\eta}^\circ)|_{\bar\zeta=2\xi_0-\zeta}\\ &= -\frac{\gamma}{2}(4-3\nu)\zeta^2 + \gamma(4-3\nu)\xi_0\zeta\\ &\quad + \frac{\gamma}{2}[(4-3\nu)H - 4(2-\nu)\xi_0^2] + \gamma\nu\xi_0 H\frac{1}{\zeta}.\end{aligned}$$

Since the excess stress state is free of body forces, it can be defined in terms of two holomorphic functions, $W_*(\zeta)$ and $w_*(\zeta)$, related to the stress combinations Θ_* and Φ_* by the formul (See Milne-Thomson [1, §2.20])

$$(2.16)\qquad \sigma_\xi^* + \sigma_\eta^* = \Theta_* = W_*(\zeta) + \overline{W_*(\zeta)},$$

$$(2.17)\qquad \sigma_\eta^* - \sigma_\xi^* - 2\imath\tau_{\xi\eta}^* = \Phi_* = \left(\frac{m(\zeta)}{\overline{m'(\zeta)}}\right)\overline{W_*'(\zeta)} + \overline{w_*(\zeta)}.$$

We can determine $W_*(\zeta)$ and $w_*(\zeta)$ by a slight modification of the standard procedure for solving elastic problems on a half-plane. First write the formal expression for $W_*(\zeta)$ in terms of the Cauchy integral:

$$(2.18)\qquad m'(\zeta)W_*(\zeta) = \frac{1}{2\pi\imath}\int_C \frac{(-2(\sigma_\xi^\circ + \imath\tau_{\xi\eta}^\circ)(\varsigma)|_{\bar\varsigma=2\xi_0-\varsigma})m'(\varsigma)}{\varsigma-\zeta}d\varsigma + P(\zeta),$$

where C is the contour in the complex ζ-plane parameterized by $\tau \mapsto \zeta = \xi_0 + \imath\tau$, for $\infty > \tau > -\infty$, the positively oriented (L on the left) boundary, and $P(\zeta)$ is a polynomial to be determined later. If the Cauchy integral converges, it determines a sectionally holomorphic function with a jump across the contour C. Note that $P(\zeta)$ is holomorphic everywhere and, consequently, does not contribute to the jump. We now proceed to evaluate the integral.

For convenience set $H = \kappa^2$, $\kappa > 0$ and denote the right-hand side of Eq. (2.18) by $I(\zeta, \alpha)$:

$$(2.19)\qquad I(\zeta,\alpha) := \frac{1}{2\pi\imath}\int_{-\alpha}^{\alpha}\frac{(2(\sigma_\xi^\circ(\xi_0+\imath\tau) + \imath\tau_{\xi\eta}^\circ(\xi_0+\imath\tau)))m'(\xi_0+\imath\tau)}{(\xi_0+\imath\tau)-\zeta}\imath d\tau + P(\zeta),$$

where α is a large, positive parameter. Integration yields

$$\begin{aligned}(2.20)\qquad I(\zeta,\alpha) = &\frac{\gamma\alpha}{\pi}(4-3\nu)\left[\zeta^2 - \xi_0\zeta - \left(\kappa^2 - \frac{4-\nu}{4-3\nu}\xi_0^2 - \frac{\alpha^2}{3}\right)\right]\\ &+ \frac{\gamma}{2\pi}\imath[\ln(-(\zeta-\xi_0)-\alpha) - \imath\ln(-(\zeta-\xi_0)+\imath\alpha)]\\ &\cdot[(4-3\nu)\zeta^3 - 2(4-3\nu)\xi_0\zeta^2\\ &\quad - (\kappa^2(4-3\nu) - 4(2-\nu)\xi_0^2)\zeta - 2\nu\kappa^2\xi_0] + P(\zeta).\end{aligned}$$

As $\alpha \to \infty$, the first term in Eq. (2.20) diverges and, hence, the Cauchy integral is not well-defined. Fortunately this term is a polynomial in ζ; so we can choose the polynomial $P(\zeta)$ to cancel it. With

$$P(\zeta) = P(\zeta, \alpha) := -\frac{\gamma\alpha}{\pi}(4-3\nu)\left[\zeta^2 - \xi_0\zeta - \left(\kappa^2 - \frac{4-\nu}{4-3\nu}\xi_0^2 - \frac{\alpha^2}{3}\right)\right], \tag{2.21}$$

Eq. (2.20) becomes

$$\begin{aligned} I(\zeta, \alpha) = \frac{\gamma}{2\pi}\imath[&\ln(-(\zeta-\xi_0) - \imath\alpha) - \ln(-(\zeta-\xi_0)+\imath\alpha)] \\ &\cdot [(4-3\nu)\zeta^3 - 2(4-3\nu)\xi_0\zeta^2 \\ &\quad - (\kappa^2(4-3\nu) - 4(2-\nu)\xi_0^2)\zeta - 2\nu\kappa^2\xi_0]. \end{aligned} \tag{2.22}$$

It is not difficult to show that

$$\lim_{\alpha\to\infty} \imath(\ln(-(\zeta-\xi_0) - \imath\alpha\kappa) - \ln(-(\zeta-\xi_0)+\imath\alpha\kappa)) = \begin{cases} -\pi: & \text{if } \zeta \in L, \\ \pi: & \text{if } \zeta \in R. \end{cases} \tag{2.23}$$

Upon computing the limit of the expression in Eq. (2.22) as $\alpha \to \infty$ and dividing by $m'(\zeta)$, we recover the sectionally holomorphic function $W_*(\zeta)$ given by

$$\begin{aligned} W_*(\zeta) &= W_*^L(\zeta) \\ &:= -\frac{\gamma}{4}\Big[(4-3\nu)\zeta^2 - 2(4-3\nu)\xi_0\zeta \\ &\qquad - (\kappa^2(4-3\nu) - 4(2-\nu)\xi_0^2) + 2\nu\kappa^2\xi_0\frac{1}{\zeta}\Big], \quad \zeta \in L, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} W_*(\zeta) = W_*^R(\zeta) := \frac{\gamma}{4}\Big[&(4-3\nu)\zeta^2 - 2(4-3\nu)\xi_0\zeta \\ &- (\kappa^2(4-3\nu) - 4(2-\nu)\xi_0^2) + 2\nu\kappa^2\xi_0\frac{1}{\zeta}\Big], \quad \zeta \in R. \end{aligned} \tag{2.25}$$

Of course, for all ζ,

$$W_*^L(\zeta) = -W_*^R(\zeta). \tag{2.26}$$

The jump in $W_*(\zeta)$ across the boundary should equal minus the value of the fundamental stress vector there. Recall that for $\zeta \in L$, the point $2\xi_0 - \bar{\zeta} \in R$ is symmetric to ζ with respect to the boundary line $\xi = \xi_0$. The jump is:

$$\lim_{\zeta\to\xi_0+\imath\eta}(W_*(\zeta) - W_*(2\xi_0 - \bar{\zeta})), \quad \zeta \in L, \tag{2.27}$$

or, in our case,

$$
\begin{aligned}
(2.28)\qquad & \lim_{\zeta\to\xi_0+\imath\eta}(W_*^L(\zeta) - W_*^R(2\xi_0-\bar{\zeta})) \\
&= \lim_{\varepsilon\downarrow 0}(W_*^L(\xi_0+\varepsilon+\imath\eta) - W_*^R(\xi_0-\varepsilon+\imath\eta)) \\
&= \lim_{\varepsilon\downarrow 0}(W_*^L(\xi_0+\imath\eta+\varepsilon) + W_*^L(\xi_0+\imath\eta-\varepsilon)) \\
&= 2W_*^L(\xi_0+\imath\eta).
\end{aligned}
$$

A short computation and the replacement of κ^2 by H shows that

$$
\begin{aligned}
(2.29)\qquad 2W_*^L(\xi_0+\imath\eta) = \frac{\gamma}{2}\left[(4-3\nu)\eta^2 - (4-\nu)\xi_0^2 + H\Big((4-\nu) - 2\nu\frac{\eta^2}{\xi_0^2+\eta^2}\Big)\right] \\
- \imath H\gamma\nu\xi_0\frac{\eta}{\xi_0^2+\eta^2}
\end{aligned}
$$

which, after a bit of arithmetic, is the same as $-(2(\sigma_\xi^\circ(\xi_0+\imath\eta)+\imath\tau_{\xi\eta}^\circ(\xi_0+\imath\eta)))$ given in Eq. (2.14).

As a further check, we can use a version of the formula in Milne-Thomson [1, §6.21(7)] to compute the extra stress vector $2(\sigma_\xi^*(\zeta)-\imath\tau_{\xi\eta}^*(\zeta))$ for $\zeta\in L$:

$$
\begin{aligned}
(2.30)\qquad 2(\sigma_\xi^*(\zeta)-\imath\tau_{\xi\eta}^*(\zeta)) = \frac{1}{m'(\zeta)}\Big[m'(\zeta)W_*^L(\zeta) - m'(2\xi_0-\bar{\zeta})W_*^R(2\xi_0-\bar{\zeta}) \\
+ (m'(\zeta)-m'(2\xi_0-\bar{\zeta}))\overline{W_*^L(\zeta)} \\
+ (m(\zeta)-m(2\xi_0-\bar{\zeta}))\overline{W_*^{L\prime}(\zeta)}\Big].
\end{aligned}
$$

Rather than explicitly computing the extra stress vector in L, we merely observe that, on the boundary, Eq. (2.30) reduces to:

$$
(2.31)\qquad 2(\sigma_\xi^*(\xi_0+\imath\eta)-\imath\tau_{\xi\eta}^*(\xi_0+\imath\eta))|_{\bar{\zeta}=2\xi_0-\zeta} = 2W_*^L(\xi_0+\imath\eta).
$$

Therefore, the total stress vector on the boundary is zero and, hence, the *total stress state:*

$$
(2.32)\qquad \sigma_\xi^{\text{tot}} := \sigma_\xi^\circ + \sigma_\xi^* \quad \sigma_\eta^{\text{tot}} := \sigma_\eta^\circ + \sigma_\eta^* \quad \tau_{\xi\eta}^{\text{tot}} := \tau_{\xi\eta}^\circ + \tau_{\xi\eta}^*
$$

is an admissible stress state which satisfies the free boundary condition on the parabolic boundary.

We could at this point obtain $w_*(\zeta)$, the second complex stress for the extra state. However, we are primarily concerned with determining the the hoop stress on the boundary of the parabola at and near the vertex. In view of the fact that $\sigma_\xi^{\text{tot}} = 0$ on the entire boundary, the hoop stress is just

$$
\begin{aligned}
\sigma_\eta^{\text{tot}} &:= \sigma_\eta^\circ + \sigma_\eta^* \\
&= [\Theta_0(m(\zeta)) + \Theta_*(m(\zeta))]|_{\bar{\zeta}=2\xi_0-\zeta} \\
&= [\Theta_0(m(\zeta)) + W_*^L(\zeta) + \overline{W_*^L(\zeta)}]|_{\bar{\zeta}=2\xi_0-\zeta}.
\end{aligned}
\tag{2.33}
$$

We can recover $\Theta_0(m(\zeta))$ from Eq. (2.9) and $W_*^L(\zeta)$ from Eq. (2.24). The hoop stress on the boundary $\zeta = \xi_0 + \imath\eta$ is then

$$
\eta \mapsto \sigma_\eta^{\text{tot}}(\xi_0 + \imath\eta) = \gamma\nu\left[(H - \xi_0^2) - \eta^2\left(1 + \frac{2H}{\eta^2 + \xi_0^2}\right)\right]. \tag{2.34}
$$

Observe first that the hoop stress at the vertex $\eta = 0$ is

$$
\sigma_\eta^{\text{tot}}(\xi_0) = \gamma\nu(H - \xi_0^2), \tag{2.35}
$$

which, for all values of Poisson's ratio, is non-negative (tension) and proportional to the distance between the vertex and the bounding plane.[2] Second, the hoop stress is even in η and strictly decreasing to $-\infty$ as $|\eta|$ gets large. Therefore, the hoop stress must become compressive at some critical value of $|\eta|$. This critical value increases with the distance between the vertex and the bounding plane from zero to ξ_0 as this distance becomes large. It is interesting to note that the location of the inversion from tension to compression is bounded; moreover, the bound is strictly geometric, for it depends only on the shape of the parabola.

3. A reinforcement problem

The value of the hoop stress given in Eq. (2.34) stems from the implicit assumption that the material is perfectly elastic and infinitely resistant in tension and compression. This assumption is not valid if the medium through which the tunnel is bored is soil or rock, both of which are strong in compression but notoriously weak in tension. So our result would predict that collapse of the tunnel is likely near the vertex.

A device traditionally adopted by builders since remote times is to bind or reinforce the soil or rock near the vertex where tensile stresses are expected. Since the procedure has often been informed by empirical criteria, it naturally begs the question as to whether a binding level can be determined analytically. A simple way to do this is the following.

We suppose that a binding procedure produces an additional *bound stress state*, vanishing at infinity, characterized by the complex stresses $W_b(\zeta)$ and $w_b(\zeta)$. For all ζ we take

$$
W_b(\zeta) := \frac{A}{\zeta}, \tag{3.1}
$$

[2] At the extreme value $\frac{1}{2}$ of Poisson's Ratio, at which the material is incompressible, ν and, hence, $\sigma_\eta^{\text{tot}}(\xi_0)$ is zero.

where A is a real parameter to be determined. The second complex stress can then be determined by reflection across the line $\xi = \xi_0$ in the ζ-plane (see Milne-Thomson [1, §6.21]).

$$w_b(\zeta) := A\left[\frac{m'(2\xi_0 - \zeta)}{m'(\zeta)}\left(\frac{1}{\zeta} + \frac{1}{2\xi_0 - \zeta}\right) + \frac{m(2\xi_0 - \zeta)}{m'(\zeta)}\frac{1}{\zeta^2} = \frac{1}{2\zeta} + \frac{2\xi_0^2}{\zeta^3}\right]. \tag{3.2}$$

However, since we are interested in the induced hoop stress on the boundary of the parabola, $\sigma_\eta^b(\xi_0 + \imath\eta)$, we get directly

$$\sigma_\eta^b(\xi_0 + \imath\eta) = \frac{2\xi_0 A}{\xi_0^2 + \eta^2}. \tag{3.3}$$

It also turns out that, on the entire boundary of the parabola, we have

$$\sigma_\xi^b(\xi_0 + \imath\eta) = 0 \quad \tau_{\xi\eta}^b(\xi_0 + \imath\eta) = 0; \tag{3.4}$$

so the addition of this binding stress state will still yield an admissible stress state which satisfies the the free boundary condition on the parabola.

To avoid tensile hoop stresses on the free surface of the parabola we need to have

$$\sigma_\eta^{\mathrm{tot}}(\xi_0 + \imath\eta) + \sigma_\eta^b(\xi_0 + \imath\eta) \leq 0. \tag{3.5}$$

We can achieve this by choosing A so that at the vertex of the parabola

$$\sigma_\eta^{\mathrm{tot}}(\xi_0) + \sigma_\eta^b(\xi_0) \leq 0. \tag{3.6}$$

From Eqs. (2.35) and (3.3) we conclude that

$$A \leq \hat{A} := -\frac{\gamma v}{2}\xi_0(H - \xi_0^2) \leq 0. \tag{3.7}$$

This confirms the intuitive expectation that once ξ_0 is prescribed, the level of the binding constraint characterized by $\hat{A}$ is proportional to the distance between the vertex of the parabola and the bounding plane.

4. Comments and conclusions

The present solution exactly satisfies the free boundary condition on the parabolic contour, even with the additional binding forces of Sec. 3, but *not* along the plane at $x = H \geq \xi_0^2$. In principle, that surface should be free. Therefore, we must determine in what sense our solution is acceptable. Consider the following argument.

The fundamental stress state given in Eqs. (2.10) satisfies the the free boundary condition on $x = H \geq \xi_0^2$ only partially, since the tangential stress τ_{xy}° does not vanish there. Yet we considered it satisfactory near the x-axis, where its values are small. Indeed, all three fundamental stress components vanish at the point $(x, y) = (H, 0)$. As for the additional and binding stress states, an estimate of their influence near the x-axis can be obtained by evaluating the stress combinations $\Theta_* = \sigma_\xi^* + \sigma_\eta^*$ and $\Theta_b = \sigma_\xi^b + \sigma_\eta^*$ at $\xi = \kappa$. From Eqs. (2.24) and (3.1) we obtain:

$$\Theta_* = W_*^L(\zeta) + \overline{W_*^L(\zeta)}|_{\zeta=\kappa} = 2\gamma(2-\nu)\xi_0(\kappa - \xi_0), \tag{4.1}$$

$$\Theta_b = W_b^L(\zeta) + \overline{W_b^L(\zeta)}|_{\zeta=\kappa} = 2\frac{\hat{A}}{\kappa} = -\frac{\gamma\nu\xi_0}{\kappa}(\kappa^2 - \xi_0^2), \tag{4.2}$$

where $\hat{A}$ is given in Eq. (3.7). This could be deemed unsatisfactory, since the sum $\Theta_* + \Theta_b$ does not even vanish at $(\xi, \eta) = (\kappa, 0)$, which corresponds to $(x, y) = (H, 0)$. However, inspection of Eqs. (4.1) and (4.2) shows that once $H = \kappa^2$ is fixed, the sum is negligible near the x-axis for deep tunnels, characterized by small values of the ratio ξ_0/κ.

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