



Information Science — *An information-theoretic proof of Nash's inequality*, by GIUSEPPE TOSCANI, communicated on 9 November 2012.

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ABSTRACT. — We show that an information-theoretic property of Shannon's entropy power, known as *concavity of entropy power* [7], can be fruitfully employed to prove inequalities in sharp form. In particular, the *concavity of entropy power* implies the logarithmic Sobolev inequality, and the Nash's inequality with the sharp constant.

KEY WORDS: Entropy power, Fisher information, logarithmic Sobolev inequality, Nash's inequality.

MATHEMATICS SUBJECT CLASSIFICATION: 62B10, 94A15, 94A17.

1. INTRODUCTION

In information theory, inequalities constitute a powerful tool to solve communication theoretic problems. Due to its wide range of application, Shannon's entropy is at the basis of many of these inequalities [8]. Some deeper inequalities were developed by Shannon himself in its pioneering 1948 paper [17]. Among other facts, Shannon stated the entropy power inequality in order to bound the capacity of non-Gaussian additive noise channels.

In its original version, Shannon's entropy power inequality gives a lower bound on Shannon's entropy functional of the sum of independent n -dimensional random variables X, Y with densities

$$(1) \quad \exp\left(\frac{2}{n}H(X+Y)\right) \geq \exp\left(\frac{2}{n}H(X)\right) + \exp\left(\frac{2}{n}H(Y)\right), \quad n \geq 1,$$

with equality if X and Y are Gaussian random variables. In inequality (1), Shannon's entropy of a random variable X with density is defined as

$$(2) \quad H(X) = H(f) = - \int_{\mathbb{R}^n} f(v) \log f(v) dv.$$

Note that Shannon's entropy coincides with Boltzmann's H -functional up to a change of sign [6]. The entropy-power

$$(3) \quad N(X) = N(f) = \exp\left(\frac{2}{n}H(X)\right)$$

(variance of a Gaussian random variable with the same Shannon's entropy functional) is maximum and equal to the variance when the random variable is Gaussian, and thus, the essence of (1) is that the sum of independent random variables tends to be *more Gaussian* than one or both of the individual components.

An interesting property of the entropy power has been discovered in 1985 by Costa [7]. Let $f(v, t)$ denote the solution to the Cauchy problem for the heat equation

$$(4) \quad \frac{\partial f(v, t)}{\partial t} = \Delta f(v, t),$$

posed in the whole space [9], corresponding to the initial value $f(v)$, which we assume to be a probability density function. Note that for $t \geq 0$, the solution to the heat equation (4) can be written as $f(v, t) = f * M_{2t}(v)$, where as usual $*$ denotes convolution, and $M_t(v)$ is the Gaussian density in \mathbb{R}^n of variance nt

$$(5) \quad M_t(v) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|v|^2}{2t}\right).$$

Costa [7] proved that, for any given probability density function f different from the Gaussian density, $N(f * M_{2t})$ is a concave function of time

$$(6) \quad \frac{d^2}{dt^2} N(f * M_{2t}) \leq 0.$$

The *concavity property of entropy power* outlines a new property of Gaussian functions. Indeed, the entropy power of a Gaussian function coincides with its variance, so that the entropy power of the fundamental solution to the heat equation is a linear function of time. This linearity is restricted to Gaussian densities.

Later, the original proof of Costa [7] has been simplified in [10, 11], by an argument based on the Blachman-Stam inequality [3]. More recently, a short and physically relevant proof has been obtained by Villani [22], resorting to some old ideas of McKean [15]. The proof of Villani establishes a deep link between the concavity of entropy power and the logarithmic Sobolev inequality. It is remarkable that the same ideas of McKean have been seminal for a new proof of logarithmic Sobolev inequality published some years ago [19].

The concavity of entropy power involves the solution to the heat equation. This basic fact includes the concavity of entropy power in the set of inequalities which, in alternative to other ways of proof, can be derived by means of the heat equation. Indeed, the linear diffusion equation [20] represents a powerful instrument to obtain a number of mathematical inequalities in sharp form.

This maybe not so well-known property goes back more or less to half a century ago, when independently from each others, researchers from information theory [18, 3], kinetic theory [15], and probability theory [14], established a useful connection between Boltzmann's H -functional and Fisher information exactly by means of the solution to the heat equation.

In this note, we proceed along the same lines to show that the concavity of entropy power (a property of the solution to the heat equation) allows to prove

as corollaries important inequalities, like the logarithmic Sobolev inequality and Nash's inequality in sharp form.

Connections between the logarithmic Sobolev inequality and Nash's inequality in sharp form are well known. Beckner [1, 2] used the former to prove the latter inequality with a sharp constant, thus obtaining Nash's inequality from an argument different from the argument used by Carlen and Loss [4]. The best constant for Nash's inequality was indeed calculated by Carlen and Loss, who observed that this inequality is equivalent to the Poincaré inequality in a suitable ball of \mathbb{R}^n .

The next Section is devoted to the proof of the concavity of entropy power. We will be mainly concerned with the key ideas behind this proof, as well as to the analogies between this proof and analogous ones based on the solution to the heat equation.

Section 3 will be devoted to show that the logarithmic Sobolev inequality is a direct consequence of the concavity of entropy power, which in some cases allows to prove the previous inequality with a remainder.

Last, in Section 4 we will show how Nash's inequality with a sharp constant follows from the concavity of entropy power. The proof is very simple, and makes use only of elementary inequalities, as well as of well-known properties of the logarithmic function.

2. THE CONCAVITY OF ENTROPY POWER

In the rest of this Section, for any given probability density function $f(v)$, we will denote by $f(v, t)$ the solution to the Cauchy problem for the heat equation (4), posed in the whole space, such that $f(v, t = 0) = f(v)$.

The proof of concavity then requires to evaluate, for any time $t > 0$, two time derivatives of the entropy power of $f(v, t)$. The first derivative of the entropy power is easily evaluated resorting to the so-called DeBruijn's identity

$$(7) \quad I(f(t)) = \frac{d}{dt} H(f(t)), \quad t > 0,$$

which connects Shannon's entropy functional with the Fisher information of a random variable with density

$$(8) \quad I(X) = I(f) = \int_{\mathbb{R}^n} \frac{|\nabla f(v)|^2}{f(v)} dv.$$

Using identity (7) we get

$$\begin{aligned} \frac{d}{dt} N(f(t)) &= \frac{2}{n} \exp\left\{\frac{2}{n} H(f(t))\right\} \frac{d}{dt} H(f(t)) \\ &= \frac{2}{n} \exp\left\{\frac{2}{n} H(f(t))\right\} I(f(t)). \end{aligned}$$

Hence

$$\frac{d^2}{dt^2} N(f(t)) = \frac{2}{n} \frac{d}{dt} \left[\exp \left\{ \frac{2}{n} H(f(t)) \right\} I(f(t)) \right].$$

Let us set

$$(9) \quad \Upsilon(f) = \exp \left\{ \frac{2}{n} H(f) \right\} I(f).$$

Then, the concavity of entropy power can be rephrased as the decreasing in time property of the functional $\Upsilon(f(t))$ along the solution to the heat equation. If

$$(10) \quad -J(f(t)) = \frac{dI(f(t))}{dt},$$

denotes the derivative of Fisher information along the solution to the heat equation, we obtain

$$\begin{aligned} \frac{d}{dt} \Upsilon(f(t)) &= \exp \left\{ \frac{2}{n} H(f(t)) \right\} \left(\frac{dI(f(t))}{dt} + \frac{2}{n} I(f(t))^2 \right) \\ &= \exp \left\{ \frac{2}{n} H(f(t)) \right\} \left(-J(f(t)) + \frac{2}{n} I(f(t))^2 \right). \end{aligned}$$

Hence, $\Upsilon(f(t))$ is non increasing if and only if

$$(11) \quad J(f(t)) \geq \frac{2}{n} I(f(t))^2.$$

It is interesting to remark that, aiming in proving the old conjecture that subsequent derivatives of Boltzmann's H -functional, evaluated on the solution to heat equation, alternate in sign, the functional $J(f(t))$ was first considered by McKean [15]. In one dimension, inequality (11) is essentially due to him. Let us repeat his highlighting idea. In the one dimensional case one has

$$I(f) = \int_{\mathbb{R}} \frac{f'(v)^2}{f(v)} dv,$$

while

$$(12) \quad J(f) = 2 \left(\int_{\mathbb{R}} \frac{f''(v)^2}{f(v)} dv - \frac{1}{3} \int_{\mathbb{R}} \frac{f'(v)^4}{f(v)^3} dv \right).$$

McKean observed that $J(f)$ is positive. In fact, resorting to integration by parts, $J(f)$ can be rewritten as

$$(13) \quad J(f) = 2 \int_{\mathbb{R}} \left(\frac{f''(v)}{f(v)} - \frac{f'(v)^2}{f(v)^2} \right)^2 f(v) dv \geq 0.$$

Having this formula in mind, consider that, for any constant $\lambda > 0$

$$\begin{aligned} 0 &\leq 2 \int_{\mathbb{R}} \left(\frac{f''(v)}{f(v)} - \frac{f'(v)^2}{f(v)^2} + \lambda \right)^2 f(v) dv \\ &= J(f) + 2\lambda^2 + 4\lambda \int_{\mathbb{R}} \left(f''(v) - \frac{f'(v)^2}{f(v)} \right) dv = J(f) + 2\lambda^2 - 4\lambda I(f). \end{aligned}$$

Choosing $\lambda = I(f)$ shows (11) for $n = 1$.

Note that equality in (11) holds if and only if f is a Gaussian density. In fact, the condition

$$\frac{f''(v)}{f(v)} - \frac{f'(v)^2}{f(v)^2} + \lambda = 0,$$

can be rewritten as

$$\frac{d^2}{dv^2} \log f(v) = -\lambda,$$

which corresponds to

$$(14) \quad \log f(v) = -\lambda v^2 + bv + c.$$

Joining condition (14) with the fact that $f(v)$ has to be a probability density, we conclude.

The argument of McKean was used by Villani [22] to obtain (11) for $n > 1$. In the general n -dimensional situation, Villani proved the formula

$$\begin{aligned} (15) \quad J(f) &= 2 \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left[\frac{\partial^2}{\partial v_i \partial v_j} \log f \right]^2 f dv \\ &= 2 \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left[\frac{1}{f} \frac{\partial^2}{\partial v_i \partial v_j} - \frac{1}{f^2} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} \right]^2 f dv. \end{aligned}$$

By means of (15), the nonnegative quantity

$$A(\lambda) = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left[\frac{1}{f} \frac{\partial^2}{\partial v_i \partial v_j} - \frac{1}{f^2} \frac{\partial f}{\partial v_i} \frac{\partial f}{\partial v_j} + \lambda \delta_{ij} \right]^2 f dv,$$

with the choice $\lambda = I(f)/n$, allows to recover inequality (11) for $n > 1$. This proves the concavity property of entropy power.

To show that the concavity of entropy power has significant consequences, we need to outline a further property of the functional $\Upsilon(f)$ [20]. Given a function

$g(v) \geq 0$, $v \in \mathbb{R}^n$, let us consider the scaling (dilation)

$$(16) \quad g(v) \rightarrow g_a(v) = a^n g(av), \quad a > 0,$$

which preserves the total mass of the function g . By direct inspection, it is immediate to conclude that Shannon's entropy (2) is such that, if the probability density f_a is defined as in (16)

$$(17) \quad H(f_a) = H(f) - n \log a.$$

Since Fisher's information (8) scales according to

$$(18) \quad I(f_a) = \int_{\mathbb{R}^n} \frac{|\nabla f_a(v)|^2}{f_a(v)} dv = a^2 \int_{\mathbb{R}^n} \frac{|\nabla f(v)|^2}{f(v)} dv = a^2 I(f),$$

one concludes that the functional $\Upsilon(f(t))$ is invariant with respect to the scaling (16) of the solution $f(v, t)$ of the heat equation. Therefore, for any constant $a > 0$

$$(19) \quad \Upsilon(f(t)) = \Upsilon(f_a(t)).$$

Property (19) allows to identify the long-time behavior of the functional $\Upsilon(f(t))$. Unless the initial value $f(v)$ in the heat equation is a Gaussian function, the functional $\Upsilon(f(t))$ is monotone decreasing, and it will reach its eventual minimum value as time $t \rightarrow \infty$. The computation of the limit value uses in a substantial way the scaling invariance property. In fact, at each time $t > 0$, the value of $\Upsilon(f(t))$ does not change if we scale the argument $f(v, t)$ according to

$$(20) \quad f(v, t) \rightarrow F(v, t) = (\sqrt{1+2t})^n f(v\sqrt{1+2t}, t),$$

which is such that the initial value $f(v)$ is left unchanged. On the other hand, it is well-known that (cfr. for example [5])

$$(21) \quad \lim_{t \rightarrow \infty} F(v, t) = M_1(v)$$

where, according to (5) $M_1(v)$ is the Gaussian density in \mathbb{R}^n of variance equal to n . Likewise, the limit value of $\Upsilon(f(t))$ does not change if we scale the limit Gaussian function according to (16) in order to have a variance different from one. Therefore, passing to the limit one obtains, for any $\sigma > 0$, the inequality

$$\Upsilon(f) \geq \Upsilon(M_\sigma),$$

or, what is the same

$$(22) \quad \exp\left\{\frac{2}{n} H(f)\right\} I(f) \geq \exp\left\{\frac{2}{n} H(M_\sigma)\right\} I(M_\sigma).$$

3. THE LOGARITHMIC SOBOLEV INEQUALITY

Inequality (22) has various important consequences. First, let us rewrite it in the form

$$(23) \quad \frac{I(f)}{I(M_\sigma)} \geq \exp\left\{-\frac{2}{n}(H(f) - H(M_\sigma))\right\}.$$

Since

$$I(M_\sigma) = \frac{n}{\sigma},$$

while

$$H(M_\sigma) = \frac{n}{2} \log 2\pi\sigma + \frac{n}{2},$$

using that $e^{-x} \geq 1 - x$, we obtain from (23)

$$(24) \quad \int_{\mathbb{R}^n} f(v) \log f(v) dv + n + \frac{n}{2} \log 2\pi\sigma \leq \frac{\sigma}{2} \int_{\mathbb{R}^n} \frac{|\nabla f(v)|^2}{f(v)} dv.$$

Inequality (24) is nothing but the logarithmic Sobolev inequality by Gross [12], written in an equivalent form.

Consider now the case in which the probability density $f(v)$ of the random variable X is such that the second moment of X is bounded. Then, for any σ such that

$$(25) \quad \sigma \geq \frac{1}{n} \int_{\mathbb{R}^n} |v|^2 f(v) dv,$$

it holds

$$\begin{aligned} -H(f) + H(M_\sigma) &= \int_{\mathbb{R}^n} f(v) \log f(v) dv - \int_{\mathbb{R}^n} M_\sigma(v) \log M_\sigma(v) dv \\ &= \int_{\mathbb{R}^n} f(v) \log \frac{f(v)}{M_\sigma(v)} dv + \frac{1}{2\sigma} \int_{\mathbb{R}^n} |v|^2 (M_\sigma - f(v)) dv \\ &\geq \int_{\mathbb{R}^n} f(v) \log \frac{f(v)}{M_\sigma(v)} dv. \end{aligned}$$

By the Csiszar-Kullback inequality [13]

$$(26) \quad 2 \int_{\mathbb{R}^n} f(v) \log \frac{f(v)}{M_\sigma(v)} dv \geq \|f - M_\sigma\|_{L^1}^2.$$

By expanding the right-hand side of inequality (23) up to the second order, we end up with the inequality

$$(27) \quad \frac{\sigma}{2} \int_{\mathbb{R}^n} \frac{|\nabla f(v)|^2}{f(v)} dv - \int_{\mathbb{R}^n} f(v) \log f(v) dv + n + \frac{n}{2} \log 2\pi\sigma \geq \frac{n^2}{8} \|f - M_\sigma\|_{L^1}^4.$$

The right-hand side of (27) constitutes an improvement of the logarithmic Sobolev inequality, in that, at least when the density function involved into inequality (23) has bounded second moment, and it is different from a Gaussian density, it is possible to quantify the positivity of the difference between the right and left sides of (23) in terms of the distance of it from the manifold of the Gaussian densities, with a precise estimate of this distance in terms of the L^1 -norm.

4. NASH'S INEQUALITY REVISITED

A second interesting consequence of the concavity of entropy power is a new proof of Nash's inequality [16]. To this aim, note that the right-hand side of inequality (22), thanks to the scaling invariance property of $\Upsilon(f)$, does not depend of σ . The choice

$$(28) \quad \sigma = \bar{\sigma} = (2\pi e)^{-1},$$

gives

$$I(M_{\bar{\sigma}}) = 2\pi en,$$

and

$$H(M_{\bar{\sigma}}) = 0.$$

Thus, substituting the value $\sigma = \bar{\sigma}$ in (22) we obtain the inequality

$$(29) \quad \exp\left\{\frac{2}{n} H(f)\right\} I(f) \geq 2\pi en.$$

Inequality (29) is known under the name of *Isoperimetric Inequality for Entropies* (cfr. [11] for a different proof).

The case in which $f(v) \geq 0$ is a density of mass different from 1, leads to a modified inequality. Let us set

$$\mu = \int_{\mathbb{R}^n} f(v) dv.$$

Then, the function $\phi(v) = f(v)/\mu$ is a probability density, which satisfies (29). Therefore

$$\begin{aligned}
(30) \quad I(\mu\phi) &= \mu I(\phi) \geq \mu I(M_\sigma) \exp\left\{\frac{2}{n}H(M_\sigma)\right\} \exp\left\{-\frac{2}{n}H(\phi)\right\} \\
&= \mu I(M_\sigma) \exp\left\{\frac{2}{n}(H(M_\sigma) - \log \mu)\right\} \exp\left\{-\frac{2}{n}(H(\phi) - \log \mu)\right\} \\
&= \mu I(M_\sigma) \exp\left\{\frac{2}{n} \frac{1}{\mu} H(\mu M_\sigma)\right\} \exp\left\{-\frac{2}{n} \frac{1}{\mu} H(\mu\phi)\right\}.
\end{aligned}$$

In (30) we used the identity

$$H(\mu\phi) = \mu H(\phi) - \mu \log \mu.$$

Setting now $\sigma = \bar{\sigma}$, as given by (28), we conclude with the inequality

$$(31) \quad I(f) \geq 2\pi en \|f\|_{L^1} \exp\left\{-\frac{2}{n\|f\|_{L^1}} [H(f) - \|f\|_{L^1} \log \|f\|_{L^1}]\right\},$$

which clearly holds for any integrable function $f(v) \geq 0$.

Given a probability density function $g(v)$, let us set $f(v) = g^2(v)$. In this case

$$H(f) = H(g^2) = -\int_{\mathbb{R}^n} g^2(v) \log g^2(v) dv = -2 \int_{\mathbb{R}^n} (g(v) \log g(v)) g(v) dv.$$

Since the function $h(r) = r \log r$ is convex, and $\|g\|_{L^1} = 1$, Jensen's inequality implies

$$(32) \quad -H(g^2) \geq 2 \int_{\mathbb{R}^n} g^2(v) dv \log \int_{\mathbb{R}^n} g^2(v) dv.$$

Using (32) into (31) gives

$$(33) \quad I(g^2) \geq 2\pi en \int_{\mathbb{R}^n} g^2(v) dv e^{(2/n) \log \int_{\mathbb{R}^n} g^2(v) dv} = \left(\int_{\mathbb{R}^n} g^2(v) dv\right)^{1+2/n}.$$

Using the identity

$$I(g^2) = 4 \int_{\mathbb{R}^n} |\nabla g(v)|^2 dv$$

we obtain from (33) the classical Nash's inequality in sharp form

$$(34) \quad \left(\int_{\mathbb{R}^n} g^2(v) dv\right)^{1+2/n} \leq \frac{2}{\pi en} \int_{\mathbb{R}^n} |\nabla g(v)|^2 dv$$

Inequality (34) clearly holds for all probability density functions $g(v)$. Note that, if $\|g\|_{L^1} \neq 1$, (34) implies

$$(35) \quad \left(\int_{\mathbb{R}^n} g^2(v) dv \right)^{1+2/n} \leq \frac{2}{\pi en} \left(\int_{\mathbb{R}^n} |g(v)| dv \right)^{4/n} \int_{\mathbb{R}^n} |\nabla g(v)|^2 dv.$$

The constant $2/(\pi en)$ in (35) is sharp.

5. CONCLUSIONS

The concavity of entropy power is a property of Shannon's entropy which has unexpected consequences in terms of functional inequalities. In this paper we made explicit the links between this property and the logarithmic Sobolev inequality by Gross [12], as well as Nash's inequality [16]. In both cases, the concavity of entropy power allows to obtain these inequalities in sharp form. Moreover, in the case of the logarithmic Sobolev inequality, it is shown that, for densities with bounded second moment, it is possible to give a precise estimate of the distance between the density and the manifold of Gaussian functions, which are known to saturate the inequality. Also, the clearness of the physical idea, and the relative simplicity of the underlying computations, are in favor of the information-theoretic proof of these inequalities.

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