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Complex Variable Functions — *Sharp and maximized real-part estimates for derivatives of analytic functions in the disk*, by Gershon Kresin, communicated on 9 November 2012.

Dedicated to the memory of Professor Gaetano Fichera

Abstract. — Representations for the sharp coefficient in an estimate of the modulus of the n-th derivative of an analytic function in the unit disk $\mathbb{D}$ are obtained. It is assumed that the boundary value of the real part of the function on $\partial\mathbb{D}$ belongs to L^p. The maximum of a bounded factor in the representation of the sharp coefficient is found. Thereby, a pointwise estimate of the modulus of the n-th derivative of an analytic function in $\mathbb{D}$ with a best constant is obtained. The sharp coefficient in the estimate of the modulus of the first derivative in the explicit form is found. This coefficient is represented, for $p \in (1, \infty)$, as the product of monotonic functions of $|z|$.

Key words: Analytic functions, estimates for derivatives, real-part theorems.

Mathematics Subject Classification: Primary: 30A10; Secondary: 30H10.

0. Introduction

In this paper we deal with a class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ represented by the Schwarz formula (see, e.g. Levin [7])

$$f(z) = i\Im f(0) + \frac{1}{2\pi}\int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z}\Re f(\zeta)|d\zeta| \tag{0.1}$$

and such that the boundary values on $\partial\mathbb{D}$ of the real part of f belong to the space $L^p(\partial\mathbb{D})$, $1 \le p \le \infty$. Here and henceforth we use the same notation $\mathbb{D}$ for the unit disk in $\mathbb{R}^2$ and $\mathbb{C}$. In what follows, by $h^p(\mathbb{D})$, $1 \le p \le \infty$, we mean the Hardy space of harmonic functions in the real unit disk $\mathbb{D}$ which are represented by the Poisson integral with a density in $L^p(\partial\mathbb{D})$. Thus, we consider analytic functions in $\mathbb{D}$ with $\Re f \in h^p(\mathbb{D})$. We shall adopt the notation $|z| = r < 1$.

We consider the inequality

$$|\Re\{e^{i\alpha} f^{(n)}(z)\}| \le \mathscr{H}_{n,p}(z, \alpha)\|\Re f\|_p \tag{0.2}$$

with the sharp coefficient $\mathscr{H}_{n,p}(z, \alpha)$, where $n \ge 1$, $z \in \mathbb{D}$ and $\|\cdot\|_p$ stands for the norm in $L^p(\partial\mathbb{D})$. Here and in what follows we adopt the notation $\|\Re f\|_p$ for $\|\Re f|_{\partial\mathbb{D}}\|_p$. We find representations for $\mathscr{H}_{n,p}(z, \alpha)$ and for the sharp coefficient

$\mathscr{H}_{n,p}(z)$ in the inequality

$$|f^{(n)}(z)| \le \mathscr{H}_{n,p}(z)\|\Re f\|_p. \tag{0.3}$$

In the case $n = 1$ and $p \in (1, \infty)$, we factorize the coefficient $\mathscr{H}_{1,p}(z)$ with two explicit monotonic functions of r. For higher order derivatives, we find upper estimates for $\mathscr{H}_{n,p}(z)$ with a sharp constant depending only on p.

The present paper extends the topics of the monograph [4] by Kresin and Maz'ya, where explicit formulas

$$\mathscr{H}_{n,1}(z) = \frac{n!}{\pi(1-r)^{n+1}}, \tag{0.4}$$

and

$$\mathscr{H}_{n,2}(z) = \frac{n!}{\sqrt{\pi}(1-r^2)^{(2n+1)/2}}\left\{\sum_{k=0}^{n}\binom{n}{k}^2 r^{2k}\right\}^{1/2} \tag{0.5}$$

were found.

The expression for the sharp coefficient

$$\mathscr{H}_{1,\infty}(z) = \frac{4}{\pi(1-r^2)} \tag{0.6}$$

is due to D. Khavinson [3].

Sharp pointwise estimates for derivatives of analytic functions from the Hardy space $H^p(\mathbb{D})$ were obtained by Makintyre and Rogosinski [8] and Szász [9] for different values of p.

Note that inequalities (0.2) and (0.3) for analytic functions belong to the class of sharp real-part theorems in the disk (see [4] and references there) which go back to Hadamard's real-part theorem [1]. Sharp real-part estimates for derivatives of analytic functions in a half-plane are derived by Kresin and Maz'ya [5, 6].

Now we describe the results of this paper in more detail. Introduction is followed by three sections. The first of them concerns representations for the sharp coefficients $\mathscr{H}_{n,p}(z,\alpha)$ and $\mathscr{H}_{n,p}(z)$. In particular, we show that the sharp coefficient in (0.2) can be written in the form

$$\mathscr{H}_{n,p}(z,\alpha) = \frac{n!}{\pi(1-r)^{(np+1)/p}(1+r)^{(p-1)/p}}K_{n,p}(z,\alpha), \tag{0.7}$$

where

$$K_{n,p}(z,\alpha) = \left\{2\int_{-\pi/2}^{\pi/2}|\Phi_n(\psi;z,\alpha)|^q\left(\frac{1+\left(\frac{1-r}{1+r}\tan\psi\right)^2}{1+\tan^2\psi}\right)^{(n+1)q/2-1}d\psi\right\}^{1/q}, \tag{0.8}$$

and $\Phi_n(\psi; z, \alpha)$ is defined by

$$\Phi_n(\psi; z, \alpha) = \cos\left[(n+1)\psi + (n-1)\arctan\left(\frac{1-r}{1+r}\tan\psi\right) + \alpha - n\vartheta\right] \tag{0.9}$$

with $\vartheta = \arg z$, $1/p + 1/q = 1$.

A consequence of (0.7)–(0.9) is a limit relation for the sharp coefficient $\mathscr{H}_{n,p}(z)$ in a pointwise estimate (0.3) for the modulus of the n-th derivative of an analytic function in a disk as the point approaches the boundary circle:

$$\lim_{r \to 1}(1-r)^{n+1/p}\mathscr{H}_{n,p}(z) = \mathscr{Q}_{n,p},$$

where

$$\mathscr{Q}_{n,p} = \frac{n!}{\pi}\max_{\beta}\left\{\int_{-\pi/2}^{\pi/2}|\cos(\beta + (n+1)\varphi)|^q \cos^{(n+1)q-2}\varphi\, d\varphi\right\}^{1/q}.$$

Besides, we obtain the relation

$$\sup_{|z|<1}\ \sup_{\|\Re f\|_p \le 1}(1-|z|^2)^{n+1/p}|f^{(n)}(z)| \ge 2^{n+1/p}\mathscr{Q}_{n,p}.$$

In Section 2 we find the value

$$\mathscr{K}_p = \max_{\alpha}\max_{|z|\le 1} K_{n,p}(z,\alpha) = \left\{\frac{2\sqrt{\pi}\,\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)}\right\}^{1/q}$$

and thus arrive at the pointwise real-part estimate of the type

$$|f^{(n)}(z)| \le \frac{n!\mathscr{K}_p}{\pi(1-r)^{(np+1)/p}(1+r)^{(p-1)/p}}\|\Re f\|_p \tag{0.10}$$

with the best possible constant. In particular, $\mathscr{K}_1 = 1$, $\mathscr{K}_2 = \sqrt{\pi}$, $\mathscr{K}_\infty = 4$.

Section 3 concerns the case $n = 1$ and the explicit formula for the best coefficient $K_{1,p}(z)$ in the inequality

$$|f'(z)| \le \frac{K_{1,p}(z)}{\pi(1-r)^{(p+1)/p}(1+r)^{(p-1)/p}}\|\Re f\|_p, \tag{0.11}$$

where the factor

$$K_{1,p}(z) = \max_{\alpha} K_{1,p}(z,\alpha)$$

is a decreasing function of $r \in [0,1]$ for any $p \in (1, \infty)$. We prove the formulas

$$K_{1,p}(z) = \frac{(1+r^2)^{1/p}}{(1+r)^{2/p}}\left\{2\sqrt{\pi}\sum_{k=0}^{\infty}\binom{1/(p-1)}{2k}\left(\frac{2r}{1+r^2}\right)^{2k}\frac{\Gamma\left(k+\frac{2p-1}{2p-2}\right)}{\Gamma\left(k+1+\frac{p}{2p-2}\right)}\right\}^{1-1/p}$$

for $1 < p < 2$, and

$$K_{1,p}(z) = \frac{(1+r^2)^{1/p}}{(1+r)^{2/p}} \left\{ 2\Gamma\Big(\frac{2p-1}{2p-2}\Big) \sum_{k=0}^{\infty} \binom{1/(p-1)}{2k} \Big(\frac{2r}{1+r^2}\Big)^{2k} \frac{\Gamma\big(k+\frac{1}{2}\big)}{\Gamma\big(k+1+\frac{p}{2p-2}\big)} \right\}^{1-1/p}$$

for $2 < p < \infty$. The series in k becomes a finite sum for $p = (2m+1)/(2m)$ or $p = (2m+2)/(2m+1)$, where m is a positive integer. The maximum of the function $K_{1,p}(z, \alpha)$ defined by (0.8) with $n = 1$ is attained for $\alpha = \vartheta$ if $1 \le p < 2$ and for $\alpha = \vartheta + (\pi/2)$ if $2 < p < \infty$. The coefficients $K_{1,2}(z, \alpha)$ and $K_{1,\infty}(z, \alpha)$ are independent of α.

A direct corollary of (0.11) is the following inequality with the sharp coefficient

$$|\nabla u(z)| \le \frac{K_{1,p}(z)}{\pi(1-r)^{(p+1)/p}(1+r)^{(p-1)/p}} \|u\|_p.$$

Here u is a harmonic function in the unit disk $\mathbb{D}$ of $\mathbb{R}^2$ from the class $h^p(\mathbb{D})$, $1 \le p \le \infty$, and $z = (x, y) \in \mathbb{D}$. The maximum modulus of the derivative $|(\nabla u(z), \ell)|$ with respect to the direction ℓ and with $\|u\|_p \le 1$ is attained at the direction of the normal for $1 \le p < 2$ and at the tangent direction for $2 < p < \infty$. If $p = 2$ or $p = \infty$, the maximum of $|(\nabla u(z), \ell)|$ with $\|u\|_p \le 1$ and fixed $z \in \mathbb{D}$ is independent of ℓ.

We note that the best possible constant C_p in the inequality

$$|f'(z)| \le \frac{C_p}{\pi(1-r^2)^{(p+1)/p}} \|\Re f\|_p$$

was obtained in the work by Kalaj and Marković [2]. The last relation is a pointwise real-part estimate of a kind different from (0.10) with $n = 1$. Namely, by (0.11), C_p is the maximal value of $(1+|z|)^{2/p} K_{1,p}(z)$ in $\mathbb{D}$.

1. Representations for sharp coefficients in estimates for derivatives of analytic functions

The next assertion contains representations for the best coefficients in (0.2) and (0.3).

PROPOSITION 1. *Let $\Re f \in h^p(\mathbb{D})$, $1 \le p \le \infty$, and let z be an arbitrary point in $\mathbb{D}$. The sharp coefficient $\mathscr{H}_p(z, \alpha)$ in the inequality*

$$|\Re\{e^{i\alpha} f^{(n)}(z)\}| \le \mathscr{H}_{n,p}(z, \alpha) \|\Re f\|_p \tag{1.1}$$

is given by

$$\mathscr{H}_{n,p}(z, \alpha) = \frac{n!}{\pi(1-r^2)^{(np+1)/p}} H_{n,p}(z, \alpha), \tag{1.2}$$

where

$$H_{n,p}(z,\alpha) = \left\{ \int_{-\pi}^{\pi} |F_n(\varphi; z, \alpha)|^q (1 + 2r\cos\varphi + r^2)^{q-1}\, d\varphi \right\}^{1/q} \tag{1.3}$$

with

$$F_n(\varphi; z, \alpha) = \sum_{k=0}^{n-1} \binom{n-1}{k} r^k \cos[(n-k)\varphi + \alpha - n\vartheta]. \tag{1.4}$$

Alternatively, the sharp coefficient $\mathscr{H}_p(z,\alpha)$ *in inequality* (1.1) *is given by*

$$\mathscr{H}_{n,p}(z,\alpha) = \frac{n!}{\pi(1-r)^{(np+1)/p}(1+r)^{(p-1)/p}} K_{n,p}(z,\alpha), \tag{1.5}$$

where

$$K_{n,p}(z,\alpha) = \left\{ 2\int_{-\pi/2}^{\pi/2} |\Phi_n(\psi; z, \alpha)|^q \left(\frac{1 + \left(\frac{1-r}{1+r}\tan\psi\right)^2}{1+\tan^2\psi} \right)^{(n+1)q/2-1} d\psi \right\}^{1/q}, \tag{1.6}$$

and the function $\Phi_n(\psi; z, \alpha)$ *is defined by formula*

$$\Phi_n(\psi; z, \alpha) = \cos\left[(n+1)\psi + (n-1)\arctan\left(\frac{1-r}{1+r}\tan\psi \right) + \alpha - n\vartheta \right]. \tag{1.7}$$

Here $\vartheta = \arg z$, $1/p + 1/q = 1$.

In particular, the best coefficient in the inequality

$$|f^{(n)}(z)| \le \mathscr{H}_{n,p}(z) \|\Re f\|_p \tag{1.8}$$

is given by

$$\mathscr{H}_{n,p}(z) = \max_{\alpha} \mathscr{H}_{n,p}(z,\alpha). \tag{1.9}$$

PROOF. Differentiating with respect to the parameter z in the right-hand side of (0.1), we obtain

$$f^{(n)}(z) = \frac{n!}{\pi} \int_{|\zeta|=1} \frac{\zeta}{(\zeta - z)^{n+1}} \Re f(\zeta) |d\zeta|,$$

which leads to

$$\Re\{e^{i\alpha} f^{(n)}(z)\} = \frac{n!}{\pi} \int_{|\zeta|=1} \Re\left\{ \frac{\zeta e^{i\alpha}}{(\zeta - z)^{n+1}} \right\} \Re f(\zeta) |d\zeta|. \tag{1.10}$$

Hence the best coefficient $\mathscr{H}_{n,p}(z,\alpha)$ in (1.1) is

$$\mathscr{H}_{n,p}(z,\alpha)=\frac{n!}{\pi}\left\{\int_{|\zeta|=1}\left|\Re\left\{\frac{\zeta e^{i\alpha}}{(\zeta-z)^{n+1}}\right\}\right|^q|d\zeta|\right\}^{1/q}. \tag{1.11}$$

We make the change of variable in (1.11)

$$w=\frac{1-\zeta\bar{z}}{\zeta-z} \tag{1.12}$$

with $|\zeta|=1$. Then $|w|=1$ and

$$\zeta=\frac{1+zw}{w+\bar{z}}. \tag{1.13}$$

Therefore,

$$\zeta-z=\frac{1-|z|^2}{w+\bar{z}},\quad d\zeta=\frac{|z|^2-1}{(w+\bar{z})^2}dw. \tag{1.14}$$

By (1.13) and (1.14), we can write (1.11) as

$$\mathscr{H}_{n,p}(z,\alpha)=\frac{n!}{\pi(1-r^2)^{(np+1)/p}}H_{n,p}(z,\alpha), \tag{1.15}$$

where

$$H_{n,p}(z,\alpha)=\left\{\int_{|w|=1}|\Re\{e^{i\alpha}(1+zw)(w+\bar{z})^n\}|^q\frac{|dw|}{|w+\bar{z}|^2}\right\}^{1/q}. \tag{1.16}$$

The equality (1.15) leads to (1.2). Since $|w|=1$ and

$$(1+zw)(w+\bar{z})=w(\overline{w}+z)(w+\bar{z})=w|w+\bar{z}|^2,$$

(1.16) takes the form

$$H_{n,p}(z,\alpha)=\left\{\int_{|w|=1}|\Re\{e^{i\alpha}w(w+\bar{z})^{n-1}\}|^q|w+\bar{z}|^{2(q-1)}|dw|\right\}^{1/q}. \tag{1.17}$$

From this we conclude that $H_{n,p}(z,\alpha)$ is a bounded function of $z\in\mathbb{D}$ for any $q\in[1,\infty]$.

Setting $w=e^{i\psi}$ and $z=re^{i\vartheta}$ in (1.17), we see that

$$H_{n,p}(z,\alpha)=\left\{\int_{-\pi}^{\pi}|\Re\{e^{i(\alpha+\psi)}(e^{i\psi}+re^{-i\vartheta})^{n-1}\}|^q|e^{i\psi}+re^{-i\vartheta}|^{2(q-1)}\,d\psi\right\}^{1/q}.$$

Making the change of variable $\varphi = \psi + \vartheta$ and using the 2π-periodicity of the integrand, we obtain

$$H_{n,p}(z,\alpha) = \left\{ \int_{-\pi}^{\pi} |\Re\{e^{i(\alpha - n\vartheta + \varphi)}(r + e^{i\varphi})^{n-1}\}|^q |r + e^{i\varphi}|^{2(q-1)}\, d\varphi \right\}^{1/q}. \tag{1.18}$$

This implies (1.3) with (1.4).

After the change of variable $\varphi = 2\psi$ in (1.18) we find that

$$H_{n,p}(z,\alpha) = \left\{ 2 \int_{-\pi/2}^{\pi/2} |\Phi_n(\psi; z, \alpha)|^q |r + e^{2i\psi}|^{2(q-1)}\, d\psi \right\}^{1/q}, \tag{1.19}$$

where

$$\Phi_n(\psi; z, \alpha) = \Re\left\{ e^{i(\alpha - n\vartheta + 2\psi)} \left(\frac{r + e^{2i\psi}}{|r + e^{2i\psi}|} \right)^{n-1} \right\}. \tag{1.20}$$

We are looking for the solution $\Psi = \Psi(r, \psi)$ of the equation

$$\frac{r + e^{2i\psi}}{|r + e^{2i\psi}|} = e^{i(\psi + \Psi)}, \tag{1.21}$$

or, equivalently, of the system

$$\frac{r + \cos 2\psi}{\sqrt{1 + 2r\cos 2\psi + r^2}} = \cos(\psi + \Psi), \qquad \frac{\sin 2\psi}{\sqrt{1 + 2r\cos 2\psi + r^2}} = \sin(\psi + \Psi). \tag{1.22}$$

By (1.22) we have $\sin 2\psi \cos(\psi + \Psi) - (r + \cos 2\psi)\sin(\psi + \Psi) = 0$, which can be written as $\sin(\psi - \Psi) - r\sin(\psi + \Psi) = 0$. Hence

$$(1 - r)\sin\psi\cos\Psi - (1 + r)\cos\psi\sin\Psi = 0$$

and therefore

$$\Psi = \arctan\left(\frac{1 - r}{1 + r} \tan\psi \right). \tag{1.23}$$

Combining (1.20) and (1.21), we obtain

$$\Phi_n(\psi; z, \alpha) = \Re\{e^{i(\alpha - n\vartheta + 2\psi)} e^{i(n-1)(\psi + \Psi)}\} = \cos[(n+1)\psi + (n-1)\Psi + \alpha - n\vartheta],$$

which together with (1.19) and (1.23) proves the equality

$$H_{n,p}(z,\alpha) = \left\{ 2 \int_{-\pi/2}^{\pi/2} |\Phi_n(\psi; z, \alpha)|^q (1 + 2r\cos 2\psi + r^2)^{(n+1)q/2 - 1}\, d\psi \right\}^{1/q} \tag{1.24}$$

with

$$\Phi_n(\psi; z, \alpha) = \cos\left[(n+1)\psi + (n-1)\arctan\left(\frac{1-r}{1+r}\tan\psi\right) + \alpha - n\vartheta\right].$$

Expressing $\cos 2\psi$ by $\tan\psi$ in (1.24) and using (1.1) and (1.2), we arrive at (1.5)–(1.7). The relations (1.8) and (1.9) follow from (1.1). □

The next assertion is a direct corollary of the representation (1.5) and formulas (1.6), (1.7) and (1.9) from Proposition 1.

Corollary 1. *The limit relation*

$$\lim_{r\to 1}(1-r)^{n+1/p}\mathcal{H}_{n,p}(z) = \mathcal{Q}_{n,p}, \tag{1.25}$$

holds, where

$$\mathcal{Q}_{n,p} = \frac{n!}{\pi}\max_{\beta}\left\{\int_{-\pi/2}^{\pi/2}|\cos(\beta+(n+1)\varphi)|^q\cos^{(n+1)q-2}\varphi\, d\varphi\right\}^{1/q}. \tag{1.26}$$

Remark 1. This result was proved in a different way by Kresin and Maz'ya [6], where it was shown that $\mathcal{Q}_{n,p}$ is the sharp coefficient in inequality

$$|f^{(n)}(z)| \le \frac{\mathcal{Q}_{n,p}}{(\Im z)^{n+1/p}}\|\Re f\|_p, \tag{1.27}$$

with f being an analytic function in the upper half-plane $\mathbb{C}_+$, $\Re f \in h^p(\mathbb{R}^2_+)$, $1 \le p \le \infty$.

Corollary 2. *The relation*

$$\sup_{|z|<1}\ \sup_{\|\Re f\|_p\le 1}(1-|z|^2)^{n+1/p}|f^{(n)}(z)| \ge 2^{n+1/p}\mathcal{Q}_{n,p} \tag{1.28}$$

holds.

Proof. By (1.9)–(1.11) we have

$$\sup_{\|\Re f\|_p\le 1}(1-|z|^2)^{n+1/p}|f^{(n)}(z)| = (1-|z|^2)^{n+1/p}\mathcal{H}_{n,p}(z),$$

which together with (1.5) and (1.9) implies

$$\sup_{\|\Re f\|_p\le 1}(1-|z|^2)^{n+1/p}|f^{(n)}(z)| = \frac{n!}{\pi}(1+|z|)^{n-1+2/p}K_{n,p}(z), \tag{1.29}$$

where

$$K_{n,p}(z) = \max_{\alpha}\mathcal{K}_{n,p}(z,\alpha).$$

We combine this with (1.6), (1.7), and pass to the limit in (1.29) as $|z| \to 1$. This gives

$$\lim_{|z|\to 1} \sup_{\|\Re f\|_p \le 1} (1-|z|^2)^{n+1/p} |f^{(n)}(z)| = 2^{n+1/p} \mathcal{Q}_{n,p},$$

with $\mathcal{Q}_{n,p}$ defined by (1.26). The last relation proves (1.28). □

2. A pointwise real-part estimate for derivatives of analytic functions

Proposition 1 implies the inequality

$$|f^{(n)}(z)| \le \mathcal{H}_{n,p}(z) \|\Re f\|_p \tag{2.1}$$

with the sharp coefficient, where

$$\mathcal{H}_{n,p}(z) = \frac{n! K_{n,p}(z)}{\pi (1-r)^{(np+1)/p} (1+r)^{(p-1)/p}} \tag{2.2}$$

and

$$K_{n,p}(z) = \max_{\alpha} K_{n,p}(z, \alpha). \tag{2.3}$$

The next assertion concerns the value

$$\mathcal{K}_p = \max_{|z| \le 1} K_{n,p}(z) \tag{2.4}$$

and implies (0.10) with the least possible constant.

Proposition 2. *Let $\Re f \in h^p(\mathbb{D})$, $1 \le p \le \infty$, and let z be an arbitrary point in $\mathbb{D}$. The best constant $\mathcal{K}_p$ in the inequality*

$$|f^{(n)}(z)| \le \frac{n! \mathcal{K}_p}{\pi (1-r)^{(np+1)/p} (1+r)^{(p-1)/p}} \|\Re f\|_p \tag{2.5}$$

is given by

$$\mathcal{K}_p = K_{n,p}(0) = \left\{ \frac{2\sqrt{\pi}\,\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)} \right\}^{1/q}. \tag{2.6}$$

In particular,

$$\mathcal{K}_1 = 1, \quad \mathcal{K}_2 = \sqrt{\pi}, \quad \mathcal{K}_\infty = 4. \tag{2.7}$$

Proof. By (1.2) and (1.5) we see that

$$K_{n,p}(z, \alpha) = \frac{H_{n,p}(z, \alpha)}{(1+r)^{n-1+2/p}}, \tag{2.8}$$

which together with (1.3) implies

$$K_{n,p}(z,\alpha) \le \frac{1}{(1+r)^{n-1}} \left\{ \int_{-\pi}^{\pi} |F_n(\varphi; z, \alpha)|^q \, d\varphi \right\}^{1/q}. \tag{2.9}$$

Using (1.4) and the Minkowski inequality, we find

$$\left\{ \int_{-\pi}^{\pi} |F_n(\varphi; z, \alpha)|^q \, d\varphi \right\}^{1/q} \tag{2.10}$$

$$= \left\{ \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n-1} \binom{n-1}{k} r^k \cos[(n-k)\varphi + \alpha - n\vartheta] \right|^q d\varphi \right\}^{1/q}$$

$$\le \sum_{k=0}^{n-1} \binom{n-1}{k} r^k \left\{ \int_{-\pi}^{\pi} |\cos[(n-k)\varphi + \alpha - n\vartheta]|^q \, d\varphi \right\}^{1/q}.$$

The integral in (2.10) can be written as

$$I_k = \int_{-\pi}^{\pi} |\cos[(n-k)\varphi + \alpha - n\vartheta]|^q \, d\varphi = \frac{1}{n-k} \int_{-(n-k)\pi+\alpha-n\vartheta}^{(n-k)\pi+\alpha-n\vartheta} |\cos\psi|^q \, d\psi,$$

which, due to 2π-periodicity of the integrand, leads to

$$I_k = \frac{1}{n-k} \int_{-(n-k)\pi}^{(n-k)\pi} |\cos\psi|^q \, d\psi = \int_0^{2\pi} |\cos\psi|^q \, d\psi = 4 \int_0^{\pi/2} \cos^q \psi \, d\psi.$$

Combining this with (2.9) and (2.10), we obtain

$$K_{n,p}(z,\alpha) \le \left\{ 4 \int_0^{\pi/2} \cos^q \psi \, d\psi \right\}^{1/q},$$

which along with (2.3), (2.4) implies

$$\mathcal{K}_p \le \left\{ 4 \int_0^{\pi/2} \cos^q \psi \, d\psi \right\}^{1/q}. \tag{2.11}$$

By (2.3), (2.4), (2.8) and (1.3), (1.4) we get the lower estimate

$$\mathcal{K}_p \ge K_{n,p}(0) = \max_{\alpha} K_{n,p}(0,\alpha) = \max_{\alpha} \left\{ \int_{-\pi}^{\pi} |\cos(n\varphi + \alpha - n\vartheta)|^q \, d\varphi \right\}^{1/q}$$

$$= \left\{ 4 \int_0^{\pi/2} \cos^q \psi \, d\psi \right\}^{1/q},$$

which together with (2.11) leads to (2.6). □

REMARK 2. Note that in view of (2.2),

$$\mathscr{H}_{n,p}(0) = \frac{n!}{\pi} K_{n,p}(0),$$

where $\mathscr{H}_{n,p}(0)$ is the sharp constant in

$$|f^{(n)}(0)| \le \mathscr{H}_{n,p}(0)\|\Re f\|_p.$$

Thus, by (2.6), one can write (2.5) as

$$|f^{(n)}(z)| \le \frac{\mathscr{H}_{n,p}(0)}{(1-r)^{(np+1)/p}(1+r)^{(p-1)/p}}\|\Re f\|_p.$$

The sharp constant $\mathscr{H}_{n,p}(0)$ was found by Kresin and Maz'ya (see [4], Sect. 5.3).

A particular case of (2.5) with $p = \infty$

$$|f^{(n)}(z)| \le \frac{4n!}{\pi(1-r)^n(1+r)}\|\Re f\|_\infty, \tag{2.12}$$

was derived in [4], Sect. 5.6, by a different method and without discussion of sharpness of the constant.

3. THE CASE $n = 1$

By Proposition 1 it follows that the sharp coefficient $\mathscr{H}_{1,p}(z,\alpha)$ in

$$|\Re\{e^{i\alpha} f'(z)\}| \le \mathscr{H}_{1,p}(z,\alpha)\|\Re f\|_p,$$

is given by

$$\mathscr{H}_{1,p}(z,\alpha) = \frac{K_{1,p}(z,\alpha)}{\pi(1-r)^{(p+1)/p}(1+r)^{(p-1)/p}}, \tag{3.1}$$

where

$$K_{1,p}(z,\alpha) = \left\{ 2\int_{-\pi/2}^{\pi/2} |\cos(2\psi+\alpha-\vartheta)|^q \left(\frac{1+\left(\frac{1-r}{1+r}\tan\psi\right)^2}{1+\tan^2\psi}\right)^{q-1} d\psi \right\}^{1/q} \tag{3.2}$$

and $\vartheta = \arg z$, $1/p + 1/q = 1$. By (1.3), (1.4), and (2.8) we can write

$$K_{1,p}(z,\alpha) = \frac{1}{(1+r)^{2/p}}\left\{\int_{-\pi}^{\pi} |\cos(\psi+\alpha-\vartheta)|^q (1+2r\cos\psi+r^2)^{q-1} d\psi\right\}^{1/q}. \tag{3.3}$$

The next assertion contains an explicit expression for $K_{1,p}(z)$ in (2.1) with $n = 1$.

For reader's convenience, we give the proof of (0.4) and (0.5) with $n = 1$ as well as Khavinson's formula (0.6).

COROLLARY 3. *Let* $\Re f \in h^p(\mathbb{D})$, $1 \le p \le \infty$, *and let* z *be an arbitrary point in* $\mathbb{D}$.

(i) *The sharp coefficient* $\mathscr{H}_{1,p}(z)$ *in the inequality*

$$|f'(z)| \le \mathscr{H}_{1,p}(z)\|\Re f\|_p \tag{3.4}$$

is given by

$$\mathscr{H}_{1,p}(z) = \frac{K_{1,p}(z)}{\pi(1-r)^{(p+1)/p}(1+r)^{(p-1)/p}}, \tag{3.5}$$

where the coefficient $K_{1,p}(z)$ *is a decreasing function of* r *on interval* $[0,1]$ *for any* $p \in (1,\infty)$, *and*

$$K_{1,p}(z) = \frac{(1+r^2)^{1/p}}{(1+r)^{2/p}}\left\{2\sqrt{\pi}\sum_{k=0}^{\infty}\binom{1/(p-1)}{2k} \times \left(\frac{2r}{1+r^2}\right)^{2k}\frac{\Gamma\left(k+\frac{2p-1}{2p-2}\right)}{\Gamma\left(k+1+\frac{p}{2p-2}\right)}\right\}^{1-1/p} \tag{3.6}$$

for $1 < p < 2$, *and*

$$K_{1,p}(z) = \frac{(1+r^2)^{1/p}}{(1+r)^{2/p}}\left\{2\Gamma\left(\frac{2p-1}{2p-2}\right)\sum_{k=0}^{\infty}\binom{1/(p-1)}{2k} \times \left(\frac{2r}{1+r^2}\right)^{2k}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+1+\frac{p}{2p-2}\right)}\right\}^{1-1/p} \tag{3.7}$$

for $2 < p < \infty$.

In the cases $p = 1, 2$, *and* ∞ *we have*

$$\mathscr{H}_{1,1}(z) = \frac{1}{\pi(1-r)^2}, \quad \mathscr{H}_{1,2}(z) = \frac{\sqrt{1+r^2}}{\sqrt{\pi}(1-r^2)^{3/2}}, \quad \mathscr{H}_{1,\infty}(z) = \frac{4}{\pi(1-r^2)}. \tag{3.8}$$

(ii) *The maximum in* α *of the function* $K_{1,p}(z,\alpha)$ *defined by* (3.2) *is attained at* $\alpha = \vartheta$ *if* $1 \le p < 2$ *and at* $\alpha = \vartheta + (\pi/2)$ *if* $2 < p < \infty$. *The coefficients* $K_{1,2}(z,\alpha)$ *and* $K_{1,\infty}(z,\alpha)$ *are independent of* α.

PROOF. 1. *Cases* $p = 1, 2$, *and* $p = \infty$. By (2.3) and (3.3),

$$K_{1,1}(z) = \frac{1}{(1+r)^2}\max_{\alpha}\sup_{|\psi|<\pi}|\cos(\psi+\alpha-\vartheta)|(1+2r\cos\psi+r^2) = 1,$$

which together with (3.1) proves the first formula in (3.8). The maximum value in the last equality is attained for $\psi+\alpha-\vartheta = 0$ and $\psi = 0$, i.e., for $\alpha = \vartheta$.

In view of (3.3),

$$K_{1,2}(z,\alpha) = \frac{1}{1+r}\left\{\int_{-\pi}^{\pi} \cos^2(\psi+\alpha-\vartheta)(1+2r\cos\psi+r^2)\,d\psi\right\}^{1/2} = \frac{\sqrt{\pi(1+r^2)}}{1+r},$$

which together with (3.1) implies the second formula in (3.8) as well as independence of $K_{1,2}(z,\alpha)$ on α.

By (3.3),

$$K_{1,\infty}(z,\alpha) = \int_{-\pi}^{\pi} |\cos(\psi+\alpha-\vartheta)|\,d\psi = \int_{-\pi}^{\pi} |\cos\theta|\,d\theta = 4,$$

which together with (3.1) shows that the third formula in (3.8) holds and that $K_{1,\infty}(z,\alpha)$ is independent of α.

2. *Cases* $1 < p < 2$ *and* $2 < p < \infty$. By (3.2), the coefficient $K_{1,p}(z,\alpha)$ is a decreasing function of $r \in [0,1]$ for any $p \in (1,\infty)$ and any fixed α. Hence the function

$$K_{1,p}(z) = \max_{\alpha} K_{1,p}(z,\alpha)$$

has the same property. Introducing the notation

$$F_q(\alpha) = \int_{-\pi}^{\pi} |\cos(\psi+\alpha)|^q (1+2r\cos\psi+r^2)^{q-1}\,d\psi, \tag{3.9}$$

we can write (3.3) as

$$K_{1,p}(z,\alpha+\vartheta) = \frac{1}{(1+r)^{2/p}}\{F_q(\alpha)\}^{1/q}. \tag{3.10}$$

The function $F_q(\alpha)$ is π-periodic and even. Hence, while looking for its maximum we may take $\alpha \in [0,\pi/2]$.

We differentiate (3.9) in α and take into account that partial derivatives of $|\cos(\psi+\alpha)|^q$ in α and ψ are equal. Integrating by parts in

$$\frac{dF_q}{d\alpha} = \int_{-\pi}^{\pi} (1+2r\cos\psi+r^2)^{q-1} \frac{\partial}{\partial\psi}|\cos(\psi+\alpha)|^q\,d\psi,$$

we obtain

$$\frac{dF_q}{d\alpha} = 2r(q-1)\int_{-\pi}^{\pi} |\cos(\psi+\alpha)|^q (1+2r\cos\psi+r^2)^{q-2}\sin\psi\,d\psi.$$

Let us take here two integrals: over $(0,\pi)$ and over $(-\pi,0)$ and make the change of variable $\varphi = -\psi$ in the second one. Then the sum of those integrals is

$$\frac{dF_q}{d\alpha} = 2r(q-1)\int_0^{\pi} (|\cos(\varphi+\alpha)|^q - |\cos(\varphi-\alpha)|^q)(1+2r\cos\varphi+r^2)^{q-2}\sin\varphi\,d\varphi.$$

Next, we integrate over $(0, \pi/2)$ and $(\pi/2, \pi)$ and make the change of variable $\psi = \pi - \varphi$ in the second one. Their sum will give

$$\frac{dF_q}{d\alpha} = 2r(q-1)\int_0^{\pi/2} (|\cos(\varphi-\alpha)|^q - |\cos(\varphi+\alpha)|^q)\Psi_q(\varphi)\sin\varphi\, d\varphi, \tag{3.11}$$

where

$$\Psi_q(\varphi) = (1 - 2r\cos\varphi + r^2)^{q-2} - (1 + 2r\cos\varphi + r^2)^{q-2}. \tag{3.12}$$

Note that

$$|\cos(\varphi-\alpha)|^q \geq |\cos(\varphi+\alpha)|^q \tag{3.13}$$

for $\alpha, \varphi \in [0, \pi/2]$. In fact, since $|\cos(\varphi-\alpha)| = \cos(\varphi-\alpha)$ for $\alpha, \varphi \in [0, \pi/2]$, $|\cos(\varphi+\alpha)| = \cos(\varphi+\alpha)$ for $\varphi+\alpha \in [0, \pi/2]$, and $|\cos(\varphi+\alpha)| = -\cos(\varphi+\alpha)$ for $\varphi+\alpha \in [\pi/2, \pi]$, it follows that

$$|\cos(\varphi-\alpha)| - |\cos(\varphi+\alpha)| = \begin{cases} 2\sin\varphi\sin\alpha & \text{for } \varphi+\alpha \in [0, \pi/2], \\ 2\cos\varphi\cos\alpha & \text{for } \varphi+\alpha \in [\pi/2, \pi], \end{cases}$$

and hence $|\cos(\varphi-\alpha)| \geq |\cos(\varphi+\alpha)|$ for $\alpha, \varphi \in [0, \pi/2]$. This implies (3.13). Besides, the equality sign in (3.13) holds only for $\alpha = 0$ and for $\alpha = \pi/2$ provided that $\varphi \in (0, \pi/2)$.

Taking into account that the sign of the function (3.12) for $\varphi \in [0, \pi/2)$ is derived by the relations

$$\Psi_q(\varphi) > 0 \quad \text{for } 1 < q < 2, \quad \text{and} \quad \Psi_q(\varphi) < 0 \quad \text{for } q > 2,$$

from (3.11) and (3.13) we obtain

$$\frac{dF_q}{d\alpha} < 0 \quad \text{for } 1 < p < 2, \quad \text{and} \quad \frac{dF_q}{d\alpha} > 0 \quad \text{for } 2 < p < \infty. \tag{3.14}$$

Combining this with (2.3) and (3.10) we see that

$$K_{1,p}(z) = \max_\alpha K_{1,p}(z, \alpha) = \begin{cases} K_{1,p}(z, \vartheta) & \text{for } 1 < p < 2, \\ K_{1,p}\left(z, \vartheta + \dfrac{\pi}{2}\right) & \text{for } 2 < p < \infty, \end{cases} \tag{3.15}$$

which completes the proof of part (ii).

Thus, by (3.3) and (3.15),

$$K_{1,p}(z) = \frac{1}{(1+r)^{2/p}}\left\{2\int_0^\pi |\cos\varphi|^q (1 + 2r\cos\varphi + r^2)^{q-1}\, d\varphi\right\}^{1/q} \tag{3.16}$$

for $1 < p < 2$, and

$$K_{1,p}(z) = \frac{1}{(1+r)^{2/p}} \left\{ 2 \int_0^{\pi} \sin^q \varphi (1 + 2r\cos\varphi + r^2)^{q-1}\, d\varphi \right\}^{1/q} \tag{3.17}$$

for $2 < p < \infty$.

We write the integral in (3.16) as the sum of the integrals over $(0, \pi/2)$ and $(\pi/2, \pi)$, and make the change of variable $\psi = \pi - \varphi$ in the second one. As a result we find

$$\begin{aligned} &\int_0^{\pi} |\cos\varphi|^q (1 + 2r\cos\varphi + r^2)^{q-1}\, d\varphi \\ &\quad = \int_0^{\pi/2} \cos^q \varphi [(1 + 2r\cos\varphi + r^2)^{q-1} + (1 - 2r\cos\varphi + r^2)^{q-1}]\, d\varphi, \end{aligned} \tag{3.18}$$

where $p \in (1, 2)$. Using the series decomposition

$$\begin{aligned} &(1 + 2r\cos\varphi + r^2)^{q-1} + (1 - 2r\cos\varphi + r^2)^{q-1} \\ &\quad = 2(1+r^2)^{q-1} \sum_{k=0}^{\infty} \binom{q-1}{2k} \left(\frac{2r}{1+r^2}\right)^{2k} \cos^{2k}\varphi \end{aligned}$$

and (3.18), we write (3.16) in the form

$$K_{1,p}(z) = \frac{1}{(1+r)^{2/p}} \left\{ 4(1+r^2)^{q-1} \sum_{k=0}^{\infty} \binom{q-1}{2k} \left(\frac{2r}{1+r^2}\right)^{2k} \int_0^{\pi/2} \cos^{2k+q} \varphi\, d\varphi \right\}^{1/q},$$

which implies (3.6).

The integrand in (3.17) can be written with the help of the decomposition

$$(1 + 2r\cos\varphi + r^2)^{q-1} = (1+r^2)^{q-1} \sum_{k=0}^{\infty} \binom{q-1}{k} \left(\frac{2r}{1+r^2}\right)^{k} \cos^k \varphi.$$

Evaluating the integrals in the sum, we arrive at (3.7). □

REMARK 3. Multiplying (3.16) and (3.17) by $(1+r)^{2/p}$, we obtain formulas for the sharp coefficient $C_{1,p}(z) = (1+|z|)^{2/p} K_{1,p}(z)$ in the equality

$$\mathscr{H}_{1,p}(z) = \frac{C_{1,p}(z)}{\pi(1-r^2)^{(p+1)/p}},$$

obtained earlier by Kalaj and Marković [2].

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