



Measure and Integration — *On sets of finite perimeter in Wiener spaces: reduced boundary and convergence to halfspaces*, by LUIGI AMBROSIO, ALESSIO FIGALLI and ERIS RUNA, communicated on 11 January 2013.

Dedicated to the memory of Gaetano Fichera

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1. INTRODUCTION

The theory of sets of finite perimeter and BV functions in Wiener spaces, i.e., Banach spaces endowed with a Gaussian Borel probability measure γ , was initiated by Fukushima and Hino in [9, 10, 11], and has been further investigated in [12, 1, 2, 3].

The basic question one would like to consider is the research of infinite-dimensional analogues of the classical fine properties of BV functions and sets of finite perimeter in finite-dimensional spaces. The class of sets of finite Gaussian perimeter E in a Gaussian Banach space (X, γ) is defined by the integration by parts formula

$$\int_E \partial_h \phi d\gamma = - \int_X \phi d\langle D_\gamma \chi_E, h \rangle_H + \int_E \phi \hat{h} d\gamma$$

for all $\phi \in C_b^1(X)$ and $h \in H$. Here H is the Cameron-Martin space of (X, γ) and $D_\gamma \chi_E$ is a H -valued measure with finite total variation in X .

When looking for the counterpart of De Giorgi's and Federer's classical results to infinite-dimensional spaces, it was noticed in [3] that the Ornstein-Uhlenbeck

$$T_t \chi_E(x) := \int_X \chi_E(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y)$$

can be used to rephrase the notion of density, the main result of that paper being

$$(1) \quad \lim_{t \downarrow 0} \int_X \left| T_t \chi_E - \frac{1}{2} d|D_\gamma \chi_E| \right| = 0.$$

According to this formula, we might say that $|D_\gamma \chi_E|$ is concentrated on the set of points of density $1/2$, where the latter set is not defined using volume ratio in balls (as in the finite-dimensional theory), but rather the Ornstein-Uhlenbeck semigroup.

In this paper we improve (1) as follows (we refer to Section 2.5 for the notation relative to halfspaces):

THEOREM 1.1. *Let E be a set of finite perimeter in (X, γ) and let $S(x) = S_{v_E(x)}$ be the halfspaces determined by $v_E(x)$. Then*

$$(2) \quad \lim_{t \downarrow 0} \int_X \int_X |\chi_E(e^{-t}x + \sqrt{1 - e^{-2t}}y) - \chi_{S(x)}(y)| d\gamma(y) d|D_\gamma \chi_E|(x) = 0.$$

A nice interpretation of this result can be obtained stating it in terms of the Gaussian rescaled sets

$$E_{x,t} = \frac{E - e^{-t}x}{\sqrt{1 - e^{-2t}}},$$

namely

$$(3) \quad \lim_{t \downarrow 0} \int_X \|\chi_{E_{x,t}} - \chi_{S(x)}\|_{L^1(\gamma)} d|D_\gamma \chi_E|(x) = 0.$$

Clearly, if we pull the modulus out of the integral in (2) we recover (1), because the measure of halfspaces is $1/2$ and $T_t \chi_E(x) = \gamma(E_{x,t})$. More specifically, (3) formalizes the fact, established by De Giorgi in finite dimensions, that on small scales a set of finite perimeter is close to an halfspace at almost every (w.r.t. surface measure).

The proof of (3) relies mainly on a combination of the careful finite-dimensional estimates of [3] with a variant of the cylindrical construction performed in [12] (with respect to [12], here we use the reduced boundary instead of the essential boundary of the finite-dimensional sections of E).

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2. PRELIMINARY RESULTS

We assume that $(X, \|\cdot\|)$ is a separable Banach space and γ is a Gaussian probability measure on the Borel σ -algebra of X . We shall always assume that γ is nondegenerate (i.e., all closed proper subspaces of X are γ -negligible) and centered (i.e., $\int_X x d\gamma = 0$). We denote by H the Cameron-Martin subspace of X , that is

$$H := \left\{ \int_X f(x)x d\gamma(x) : f \in L^2(X, \gamma) \right\},$$

and, for $h \in H$, we denote by $\hat{h} \in L^2(X, \gamma)$ the Fomin derivative of γ along h , namely

$$(4) \quad \int_X \partial_h \phi d\gamma = - \int_X \hat{h} \phi d\gamma$$

for all $\phi \in C_b^1(X)$. Here and in the sequel $C_b^1(X)$ denotes the space of continuously differentiable cylindrical functions in X , bounded and with a bounded gradient. The space H can be endowed with a Hilbertian norm $|\cdot|_H$ that makes the map $h \mapsto \hat{h}$ an isometry; furthermore, the injection of $(H, |\cdot|_H)$ into $(X, \|\cdot\|)$ is compact.

We shall denote by $\tilde{H} \subset H$ the subset of vectors of the form

$$(5) \quad \int_X \langle x^*, x \rangle x d\gamma(x), \quad x^* \in X^*.$$

This is a dense (even w.r.t. to the Hilbertian norm) subspace of H . Furthermore, for $h \in \tilde{H}$ the function $\hat{h}(x)$ is precisely $\langle x^*, x \rangle$ (and so, it is continuous).

Given an m -dimensional subspace $F \subset \tilde{H}$ we shall frequently consider an orthonormal basis $\{h_1, \dots, h_m\}$ of F and the factorization $X = F \oplus Y$, where Y is the kernel of the continuous linear map

$$(6) \quad x \in X \mapsto \Pi_F(x) := \sum_{i=1}^m \hat{h}_i(x) h_i \in F.$$

The decomposition $x = \Pi_F(x) + (x - \Pi_F(x))$ is well defined, thanks to the fact that $\Pi_F \circ \Pi_F = \Pi_F$ and so $x - \Pi_F(x) \in Y$; in turn this follows by $\hat{h}_i(h_j) = \langle \hat{h}_i, \hat{h}_j \rangle_{L^2} = \delta_{ij}$.

Thanks to the fact that $|h_i|_H = 1$, this induces a factorization $\gamma = \gamma_F \otimes \gamma_Y$, with γ_F the standard Gaussian in F (endowed with the metric inherited from H) and γ_Y Gaussian in $(Y, \|\cdot\|)$. Furthermore, the orthogonal complement F^\perp of F in H is the Cameron-Martin space of (Y, γ_Y) .

2.1. BV functions and Sobolev spaces

Here we present the definitions of Sobolev and BV spaces. Since we will consider bounded functions only, we shall restrict to this class for ease of exposition.

Let $u : X \rightarrow \mathbb{R}$ be a bounded Borel function. Motivated by (4), we say that $u \in W^{1,1}(X, \gamma)$ if there exists a (unique) H -valued function, denoted by ∇u , such that $|\nabla u|_H \in L^1(X, \gamma)$ and

$$\int_X u \partial_h \phi d\gamma = - \int_X \phi \langle \nabla u, h \rangle_H d\gamma + \int_X u \phi \hat{h} d\gamma$$

for all $\phi \in C_b^1(X)$ and $h \in H$.

Analogously, following [10, 11], we say that $u \in BV(X, \gamma)$ if there exists a (unique) H -valued Borel measure $D_\gamma u$ with finite total variation in X satisfying

$$\int_X u \partial_h \phi \, d\gamma = - \int_X \phi \, d\langle D_\gamma u, h \rangle_H + \int_X u \phi \hat{h} \, d\gamma$$

for all $\phi \in C_b^1(X)$ and $h \in H$.

In the sequel we will mostly consider the case when $u = \chi_E : X \rightarrow \{0, 1\}$ is the characteristic function of a set E , although some statements are more natural in the general BV context. Notice the inclusion $W^{1,1}(X, \gamma) \subset BV(X, \gamma)$, given by the identity $D_\gamma u = \nabla u \gamma$.

2.2. The OU semigroup and Mehler's formula

In this paper, the Ornstein-Uhlenbeck semigroup $T_t f$ will always be understood as defined by the *pointwise* formula

$$(7) \quad T_t f(x) := \int_X f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\gamma(y)$$

which makes sense whenever f is bounded and Borel. This convention will be important when integrating $T_t f$ against potentially singular measures.

We shall also use the dual OU semigroup T_t^* , mapping signed measures into signed measures, defined by the formula

$$(8) \quad \langle T_t^* \mu, \phi \rangle := \int_X T_t \phi \, d\mu \quad \phi \text{ bounded Borel.}$$

In the next proposition we collect a few properties of the OU semigroup needed in the sequel (see for instance [4] for the Sobolev case, and [2] for the BV case).

PROPOSITION 2.1. *Let $u : X \rightarrow \mathbb{R}$ be bounded and Borel, and $t > 0$. Then $T_t u \in W^{1,1}(X, \gamma)$ and:*

- (a) *if $u \in W^{1,1}(X, \gamma)$ then, componentwise, it holds $\nabla T_t u = e^{-t} T_t \nabla u$;*
- (b) *if $u \in BV(X, \gamma)$ then, componentwise, it holds $\nabla T_t u \gamma = e^{-t} T_t^*(D_\gamma u)$.*

The next result is basically contained in [4, Proposition 5.4.8], see also [3, Proposition 2.2] for a detailed proof. We state it in order to emphasize that, γ_Y -a.e. $y \in Y$, the regular version of the restriction of $T_t u$ to $y + F$ (provided by the above proposition) is precisely the one pointwise defined in Mehler's formula.

PROPOSITION 2.2. *Let u be a bounded Borel function and $t > 0$. With the above notation, for γ_Y -a.e. $y \in Y$ the map $z \mapsto T_t u(z, y)$ is smooth in F .*

The next lemma provides a rate of convergence of $T_t u$ to u when u belongs to $BV(X, \gamma)$; the proof follows the lines of the proof of Poincaré inequalities, see [3, Lemma 2.3], [4, Theorem 5.5.11].

LEMMA 2.3. *Let $u \in BV(X, \gamma)$. Then*

$$(9) \quad \int_X \int_X |u(x) - u(e^{-t}x + \sqrt{1 - e^{-2t}}y)| d\gamma(x) d\gamma(y) \leq c_t |D_\gamma u|(X)$$

with $c_t := \sqrt{\frac{2}{\pi}} \int_0^t \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} ds$, $c_t \sim 2\sqrt{t/\pi}$ as $t \downarrow 0$. In particular

$$\int_X |T_t u - u| d\gamma \leq c_t |D_\gamma u|(X).$$

Let us now recall the fundamental facts about sets of locally finite perimeter E in \mathbb{R}^m . De Giorgi called *reduced* boundary of E the set $\mathcal{F}E$ of points in the support of $|D\chi_E|$ satisfying

$$\exists v_E(x) := \lim_{r \downarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{and} \quad |v_E(x)| = 1.$$

By Besicovitch theorem, $|D\chi_E|$ is concentrated on $\mathcal{F}E$ and $D\chi_E = v_E |D\chi_E|$. The main result of [6] are: first, the blown-up sets

$$(10) \quad \frac{E - x}{r}$$

converge as $r \downarrow 0$ locally in measure, and therefore in $L^1(G_m \mathcal{L}^m)$, to the half-space $S_{v_E(x)}$ having v_E as inner normal; second, this information can be used to show that $\mathcal{F}E$ is *countably \mathcal{S}^{m-1} -rectifiable*, namely there exist countably many C^1 hypersurfaces $\Gamma_i \subset \mathbb{R}^m$ such that

$$\mathcal{S}^m \left(\mathcal{F}E \setminus \bigcup_i \Gamma_i \right) = 0.$$

In the following results we assume that (X, γ) is an m -dimensional Gaussian space; if we endow X with the Cameron-Martin distance d , then (X, γ, d) is isomorphic to $(\mathbb{R}^m, G_m \mathcal{L}^m, \|\cdot\|)$, $\|\cdot\|$ being the euclidean distance. Under this isomorphism, we have $D_\gamma \chi_E = G_m D\chi_E$ whenever E has finite Gaussian perimeter, so that $|D\chi_E|$ is finite on bounded sets and E has locally finite Euclidean perimeter. Since this isomorphism is canonical, we can and shall use it to define $\mathcal{F}E$ also for sets with finite perimeter in (X, γ) (although a more intrinsic definition along the lines of the appendix of [3] could be given).

Having in mind the Ornstein-Uhlenbeck semigroup, the scaling (10) now becomes

$$(11) \quad E_{x,t} := \frac{E - e^{-t}x}{\sqrt{1 - e^{-2t}}}$$

so that

$$T_t \chi_E(x) = \gamma(E_{x,t}).$$

It corresponds to the scaling (10) with $r = \sqrt{1 - e^{-2t}} \sim \sqrt{2t}$ and with eccentric balls, whose eccentricity equals $x(e^{-t} - 1)$. Since $e^{-t} - 1 = O(t) = o(r)$, this eccentricity has not effect in the limit and allows to rewrite, arguing as in [3, Proposition 3.1], the Euclidean statement in Gaussian terms:

PROPOSITION 2.4. *Let (X, γ) be an m -dimensional Gaussian space and $E \subset X$ of finite Gaussian perimeter. Then, for $|D_\gamma \chi_E|$ -a.e. $x \in X$ the rescaled sets $E_{x,t}$ in (11) converge in $L^2(\gamma)$ to $S_{v_E}(x)$.*

This way, we easily obtain the finite-dimensional version of Theorem 1.1.

As in [3], the following lemma (stated with the outer integral in order to avoid measurability issues) plays a crucial role in the extension to infinite dimensions:

LEMMA 2.5. *Let (X, γ) be a finite-dimensional Gaussian space, let (Y, \mathcal{F}, μ) be a probability space and, for $t > 0$ and $y \in Y$, let $g_{t,y} : X \rightarrow [0, 1]$ be Borel maps. Assume also that:*

- (a) $\{\sigma_y\}_{y \in Y}$ are positive finite Borel measures in X , with $\int_Y^* \sigma_y(X) d\mu(y)$ finite;
- (b) $\sigma_y = G_m \mathcal{S}^{m-1} \llcorner \Gamma_y$ for μ -a.e. y , with Γ_y countably \mathcal{S}^{m-1} -rectifiable.

Then

$$(12) \quad \begin{aligned} & \limsup_{t \downarrow 0} \int_Y^* \int_X T_t g_{t,y}(x) d\sigma_y(x) d\mu(y) \\ & \leq \limsup_{t \downarrow 0} \frac{1}{\sqrt{t}} \int_Y^* \int_X g_{t,y}(x) d\gamma(x) d\mu(y). \end{aligned}$$

The proof, given in detail in [3, Lemma 3.4], relies on the heuristic idea that in an m -dimensional Gaussian space (X, γ) , for the adjoint semigroup T_t^* (i.e. the one acting on measures) we have

$$\sqrt{t} T_t^*(G_m \mathcal{S}^{m-1} \llcorner \Gamma) \leq (1 + o(1))\gamma$$

whenever Γ is a C^1 hypersurface. This is due to the fact in the case when Γ is flat, i.e. Γ is an affine hyperplane, the asymptotic estimate above holds, and that for a non-flat surface only lower order terms appear. In the flat case, using invariance under rotation and factorization of the semigroup (see the next section) one is left to the estimate of $\sqrt{t} T_t^* \sigma$ when $X = \mathbb{R}$ and σ is a Dirac mass. Then, considering for instance $\sigma = \delta_0$, a simple computation gives

$$\sqrt{t} T_t^*(G_m(0)\delta_0) = \frac{1}{2\pi} \frac{\sqrt{t}}{\sqrt{1 - e^{-2t}}} e^{-|y|^2/(1 - e^{-2t})} \mathcal{L}^1 \leq \frac{1}{2\sqrt{2\pi}} \gamma + o(1) \quad \text{as } t \downarrow 0.$$

(See the proof [3, Lemma 3.4] for more details.)

2.3. Factorization of T_t and $D_\gamma u$

Let us consider the decomposition $X = F \oplus Y$, with $F \subset \tilde{H}$ finite-dimensional. Denoting by T_t^F and T_t^Y the OU semigroups in F and Y respectively, it is easy to check (for instance first on products of cylindrical functions on F and Y , and then by linearity and density) that also the action of T_t can be “factorized” in the coordinates $x = (z, y) \in F \times Y$ as follows:

$$(13) \quad T_t f(z, y) = T_t^Y(w \mapsto T_t^F f(\cdot, w)(z))(y)$$

for any bounded Borel function f .

Let us now discuss the factorization properties of $D_\gamma u$. First of all, we can write $D_\gamma u = v_u |D_\gamma u|$ with $v_u : X \rightarrow H$ Borel vectorfield satisfying $|v_u|_H = 1$ $|D_\gamma u|$ -a.e. Moreover, given a Borel set B , define

$$B_y := \{z \in F : (z, y) \in B\}, \quad B_z := \{y \in Y : (z, y) \in B\}.$$

The identity

$$(14) \quad \int_B |\pi_F(v_u)| |d|D_\gamma u| = \int_Y |D_{\gamma_F} u(\cdot, y)|(B_y) d\gamma_Y(y)$$

is proved in [2, Theorem 4.2] (see also [1, 12] for analogous results), where $\pi_F : H \rightarrow F$ is the orthogonal projection. Along the similar lines, one can also show the identity

$$(15) \quad \int_B |\pi_{F^\perp}(v_u)| |d|D_\gamma u| = \int_F |D_{\gamma_Y} u(z, \cdot)|(B_z) d\gamma_F(z)$$

with $\pi_F + \pi_{F^\perp} = \text{Id}$. In the particular case $u = \chi_E$, with the notation

$$(16) \quad E_y := \{z \in F : (z, y) \in E\}, \quad E_z := \{y \in Y : (z, y) \in E\}$$

the identities (14) and (15) read respectively as

$$(17) \quad \int_B |\pi_F(v_E)| |d|D_\gamma \chi_E| = \int_Y |D_{\gamma_F} \chi_{E_y}|(B_y) d\gamma_Y(y) \quad \text{for all } B \text{ Borel,}$$

$$(18) \quad \int_B |\pi_{F^\perp}(v_E)| |d|D_\gamma \chi_E| = \int_F |D_{\gamma_Y} \chi_{E_z}|(B_z) d\gamma_F(z) \quad \text{for all } B \text{ Borel}$$

with $D_\gamma \chi_E = v_E |D_\gamma \chi_E|$.

REMARK 2.6. Having in mind (17) and (18), it is tempting to think that the formula holds for any orthogonal decomposition of H (so, not only when $F \subset \tilde{H}$), or even when none of the parts is finite-dimensional. In order to avoid merely technical complications we shall not treat this issue here because, in this more general situation, the “projection maps” $x \mapsto y$ and $x \mapsto z$ are no longer continuous.

However, the problem can be solved removing sets of small capacity, see for instance [8] for a more detailed discussion.

2.4. Finite-codimension Hausdorff measures

Following [8], we start by introducing pre-Hausdorff measures which, roughly speaking, play the same role of the pre-Hausdorff measures \mathcal{S}_δ^n in the finite-dimensional theory.

Let $F \subset \tilde{H}$ be a finite-dimensional subspace of dimension m , and for $k \in \mathbb{N}$, $0 \leq k \leq m$, we define (with the notation of the previous section)

$$(19) \quad \mathcal{S}_F^{\infty-k}(B) := \int_Y \int_{B_y} G_m d\mathcal{S}^{m-k} d\gamma_Y(y) \quad \text{for all } B \text{ Borel,}$$

where G_m is the standard Gaussian density in F (so that $\mathcal{S}_F^{\infty-0} = \gamma$). It is proved in [8] that $y \mapsto \int_{B_y} G_m d\mathcal{S}^{m-k}$ is γ_Y -measurable whenever B is Suslin (so, in particular, when B is Borel), therefore the integral makes sense. The first key monotonicity property noticed in [8], based on [7, 2.10.27], is

$$\mathcal{S}_F^{\infty-k}(B) \leq \mathcal{S}_G^{\infty-k}(B) \quad \text{whenever } F \subset G \subset \tilde{H},$$

provided \mathcal{S}^{m-k} in (19) is understood as the *spherical* Hausdorff measure of dimension $m - k$ in F . This naturally leads to the definition

$$(20) \quad \mathcal{S}^{\infty-k}(B) := \sup_F \mathcal{S}_F^{\infty-k}(B), \quad B \text{ Borel,}$$

where the supremum runs among all finite-dimensional subspaces F of \tilde{H} . Notice however that, strictly speaking, the measure defined in (20) does not coincide with the one in [8], since all finite-dimensional subspaces of H are considered therein. We make the restriction to finite-dimensional subspaces of \tilde{H} for the reasons explained in Remark 2.6. However, still $\mathcal{S}^{\infty-k}$ is defined in a coordinate-free fashion.

These measures have been related for the first time to the perimeter measure $D_\gamma \chi_E$ in [12]. Hino defined the F -essential boundaries (obtained collecting the essential boundaries of the finite-dimensional sections $E_y \subset F \times \{y\}$)

$$(21) \quad \partial_F^* E := \{(z, y) : z \in \partial^* E_y\}$$

and noticed another key monotonicity property (see also [2, Theorem 5.2])

$$(22) \quad \mathcal{S}_F^{\infty-1}(\partial_F^* E \setminus \partial_G^* E) = 0 \quad \text{whenever } F \subset G \subset \tilde{H}.$$

Then, choosing a sequence $\mathcal{F} = \{F_1, F_2, \dots\}$ of finite-dimensional subspaces of \tilde{H} whose union is dense he defined

$$(23) \quad \mathcal{S}_{\mathcal{F}}^{\infty-1} := \sup_n \mathcal{S}_{F_n}^{\infty-1}, \quad \partial_{\mathcal{F}}^* E := \liminf_{n \rightarrow \infty} \partial_{F_n}^* E,$$

and showed that

$$(24) \quad |D_\gamma \chi_E| = \mathcal{L}_F^{\infty-1} \llcorner \partial_F^* E.$$

In order to prove our main result we will follow Hino's procedure, but working with the reduced boundaries in place of the essential boundaries.

2.5. Halfspaces

Let $h \in H$ and \hat{h} be its corresponding element in $L^2(X, \gamma)$. Then there exist a linear subspace $X_0 \subset X$ such that $\gamma(X \setminus X_0) = 0$ and a representative of \hat{h} which is linear in X_0 . Indeed, let $h_n \rightarrow h$ in $L^2(X, \gamma)$ with $\hat{h}_n \in X^*$. It is not restrictive to assume that $\hat{h}_n \rightarrow \hat{h}$ γ -a.e. in X , so if we define

$$X_0 := \{x \in X : \hat{h}_n(x) \text{ is a Cauchy sequence}\}$$

we find that X_0 is a vector space of full γ -measure and that the pointwise limit of \hat{h}_n provides a version of h , linear in X_0 .

Having this fact in mind, it is natural to define halfspaces in the following way.

DEFINITION 2.7. *Given a unit vector $h \in H$ we shall denote by S_h the halfspace having h as "inner normal", namely*

$$(25) \quad S_h := \{x \in X : \hat{h}(x) > 0\}.$$

PROPOSITION 2.8. *For any S_h halfspace it holds $\gamma(S_h) = 1/2$, $P(S_h) = \sqrt{1/(2\pi)}$, and $D\chi_{S_h} = h|D\chi_{S_h}|$. Furthermore, the following implication holds:*

$$\lim_{n \rightarrow \infty} |h_n - h| = 0 \Rightarrow \lim_{n \rightarrow \infty} \chi_{S_{h_n}} = \chi_{S_h}.$$

PROOF. Let us first show that convergence of h_n to h implies convergence of the corresponding halfspaces. Since for all $\varepsilon > 0$ it holds

$$\begin{aligned} \{\hat{h}_n > 0\} \setminus \{\hat{h} > 0\} &\subset (\{\hat{h}_n > 0\} \setminus \{\hat{h} > -\varepsilon\}) \cup \{\hat{h} \in (-\varepsilon, 0)\} \\ &\subset \{|\hat{h}_n - \hat{h}| > \varepsilon\} \cup \{\hat{h} \in (-\varepsilon, 0)\} \end{aligned}$$

and since the convergence of \hat{h}_n to \hat{h} in $L^2(X, \gamma)$ implies $\gamma(\{|\hat{h}_n - \hat{h}| > \varepsilon\}) \rightarrow 0$ we obtain

$$\limsup_{n \rightarrow \infty} \gamma(\{\hat{h}_n > 0\} \setminus \{\hat{h} > 0\}) \leq \gamma(\hat{h}^{-1}(-\varepsilon, 0)).$$

Now, since \hat{h} has a standard Gaussian law and ε is arbitrary it follows that $\gamma(\{\hat{h}_n > 0\} \setminus \{\hat{h} > 0\}) \rightarrow 0$. A similar argument (because the laws of all \hat{h}_n are standard Gaussian) yields $\gamma(\{\hat{h} > 0\} \setminus \{\hat{h}_n > 0\}) \rightarrow 0$.

Now, if γ is the standard Gaussian in $X = H = \mathbb{R}^n$ and S_h is a halfspace, it is immediate to check that $\gamma(S_h) = 1/2$. In addition, since $D_\gamma \chi_{S_h} = h|D_\gamma \chi_{S_h}|$ and $\hat{h}(x) = \langle h, x \rangle$, we can use $E = S_h$ and $\phi \equiv 1$ in the integration by parts formula

$$\int_E \partial_h \phi d\gamma + \int_X \phi d\langle h, D_\gamma \chi_E \rangle = \int_E \hat{h} d\gamma$$

to get $|D_\gamma S_h|(X) = \int_{S_h} \langle h, x \rangle dx = \sqrt{1/(2\pi)}$. By a standard cylindrical approximation we obtain that $\gamma(S_h) = \frac{1}{2}$, S_h has finite perimeter, and $D\chi_{S_h} = h|D\chi_{S_h}|$ in the general case. \square

2.6. Convergence to halfspaces

In this section we prove Theorem 1.1. We consider an increasing family of subspaces $F_n \subset \tilde{H}$ and, for any n , we consider the corresponding decomposition $x = (x_1, x_2)$ with $x_1 \in F_n$ and $x_2 \in Y_n$. Denote by $\gamma = \gamma_n \times \gamma_n^\perp$ the corresponding factorization of γ . Then, adapting the definition of boundary given in Hino's work [12] (with reduced in place of essential boundary) we define

$$\mathcal{F}_H E := \liminf_{n \rightarrow \infty} B_n \quad \text{where } B_n = \{x = (x_1, x_2) : x_1 \in \mathcal{F}E_{x_2}\}$$

(recall that $E_{x_2} = \{x_1 \in F_n : (x_1, x_2) \in E\}$). We also set $C_n = \bigcap_{m \geq n} B_m$, so that $C_n \uparrow \mathcal{F}_H E$ as $n \rightarrow \infty$. Recall that by (14) the measure $\sigma_n := |\pi_{F_n}(v_E)| |D_\gamma \chi_E|$ is concentrated on B_n , because by De Giorgi's theorem the derivative of finite-dimensional sets of finite perimeter is concentrated on the reduced boundary. Since σ_n is nondecreasing with respect to n , σ_n is concentrated on all sets B_m with $m \geq n$, and therefore on C_n . It follows that $|D_\gamma \chi_E| = \sup_n \sigma_n$ is concentrated on $\mathcal{F}_H E$, one of the basic observations in [12].

Let us denote by $v_n(x) = v_n(x_1, x_2)$ the approximate unit normal to $E_{x_2}^n$ at x_1 . Notice that, in this way, v_n is pointwise defined at all points $x \in B_n$ and $D_{\gamma_n} \chi_{E_{x_2}^n} = v_n(x) |D_{\gamma_n} \chi_{E_{x_2}^n}|$ (again by De Giorgi's finite-dimensional result). Since the identity (an easy consequence of Fubini's theorem)

$$\pi_F(D_\gamma \chi_E) = D_{\gamma_n} \chi_{E_{x_2}^n} \gamma_n^\perp$$

and the definition of v_n give

$$\pi_{F_n}(v_E) |D_\gamma \chi_E| = D_{\gamma_n} \chi_{E_{x_2}^n} \gamma_n^\perp = v_n |D_{\gamma_n} \chi_{E_{x_2}^n}| \gamma_n^\perp$$

we can use (14) once more to get

$$\pi_{F_n}(v_E) |D_\gamma \chi_E| = v_n |\pi_{F_n}(v_E)| |D_\gamma \chi_E|,$$

so that $v_n = \pi_{F_n}(v_E) / |\pi_{F_n}(v_E)|$ σ_n -a.e. in X . Since $\sigma_n \uparrow |D_\gamma \chi_E|$ as $n \rightarrow \infty$, it follows that on each set C_n the function v_m is defined for $m \geq n$, and converges to

v_E as $m \rightarrow \infty$ $|D_\gamma \chi_E|$ -a.e. on C_n . Then, Proposition 2.8 and the convergence of v_n give

$$(26) \quad \lim_{n \rightarrow \infty} \int_X \int_X |\chi_{S_{v_n}} - \chi_{S_{v_E}}| d\gamma d\sigma_n = 0.$$

In addition, by the finite-dimensional result of convergence to half spaces, we get

$$(27) \quad \lim_{t \downarrow 0} \int_X \int_{F_n} |\chi_{E_{x_2}}(e^{-t}x_1 + \sqrt{1 - e^{-2t}}x'_1) - \chi_{\tilde{S}_{v_n(x)}}(x'_1)| d\gamma_n(x'_1) d\sigma_n(x) = 0,$$

where \tilde{S}_{v_n} is the projection of S_{v_n} on F_n . Now, notice that $S_{v_n} = \tilde{S}_{v_n} \times Y_n$, since $v_n \in F$. This observation, in combination with (26), gives that

$$\limsup_{t \downarrow 0} \int_X \int_X |\chi_{E_{x_2}}(e^{-t}x_1 + \sqrt{1 - e^{-2t}}x'_1) - \chi_{S_{v_E(x)}}(x')| d\gamma(x') d\sigma_n(x)$$

is infinitesimal as $n \rightarrow \infty$. Therefore to prove (2) it suffices to show that

$$(28) \quad \limsup_{t \downarrow 0} \int_X \int_X |\chi_{E_{x_2}}(e^{-t}x_1 + \sqrt{1 - e^{-2t}}x'_1) - \chi_E(e^{-t}x + \sqrt{1 - e^{-2t}}x')| d\gamma(x') d\sigma_n(x)$$

is infinitesimal as $n \rightarrow \infty$.

In order to show this last fact, using again $\sigma_n = |D_{\gamma_n} \chi_{E_{x_2}}| \gamma_n^\perp$, we can write the expression as

$$\limsup_{t \downarrow 0} \int_{Y_n} \int_{F_n} T_t^{F_n} g_t(x_1, x_2) d|D_{\gamma_n} \chi_{E_{x_2}}|(x_1) d\gamma_n^\perp(x_2)$$

with $g_t(x_1, x_2) := \int_{Y_n} |\chi_E(x_1, x'_2) - \chi_E(x_1, e^{-t}x_2 + \sqrt{1 - e^{-2t}}x'_2)| d\gamma_n^\perp(x'_2)$. As in [3] we now use Lemma 2.5 and the rectifiability of the measures $|D_{\gamma_n} \chi_{E_{x_2}}|$ to bound the limsup above by

$$(29) \quad \limsup_{t \downarrow 0} \int_{Y_n} \int_{F_n} \frac{g_t(x_1, x_2)}{\sqrt{t}} d\gamma_n(x_1) d\gamma_n^\perp(x_2).$$

Now we integrate w.r.t. γ_n the inequality (ensured by (9))

$$\int_{Y_n} g_t(x_1, x_2) d\gamma_n^\perp(x_2) \leq c\sqrt{t} |D_{\gamma_n^\perp} \chi_{E_{x_1}}|(Y_n),$$

valid for all x_1 such that E_{x_1} has finite perimeter in (Y_n, γ_n^\perp) , to bound the lim sup in (29) by

$$c \int_{F_n} |D_{\gamma_n^\perp} \chi_{E_{x_1}}|(Y_n) d\gamma_n(x_1) = c \int_X |\pi_{F_n}^\perp(v_E)| d|D_\gamma \chi_E|.$$

Since $|\pi_{F_n}^\perp v_E| \downarrow 0$ as $n \rightarrow \infty$, this concludes the proof.

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