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Partial Differential Equations — *The singular set of higher dimensional unstable obstacle type problems,* by John Andersson, Henrik Shahgholian and Georg S. Weiss, communicated on 14 December 2012.

Dedicated to the memory of Gaetano Fichera

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Contents

1. Introduction

In this paper we will investigate the singular points of the following unstable free boundary problem:

$$\Delta u = -\chi_{\{u>0\}} \quad \text{in } B_1(0) \tag{1.1}$$

where $\chi_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$.

This problem was first investigated by G. S. Weiss and R. Monneau [14]. In [14], $C^{1,1}$-regularity locally energy minimising and maximal solutions of (1.1) is shown. There is also some discussion regarding the possibility of the existence of singular points, that is points $x^0 \in B_1(0)$ such that $u \notin C^{1,1}(B_r(x^0))$ for any $r > 0$. Such points are proved to be totally unstable [14].

Let us formally define *singular points* before we proceed.

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DEFINITION 1.1. *Let u be a solution to* (1.1)*. Then we define* $S(u)$*, the set of singular points of u, according to*

$$S(u) = \{\mathbf{x} \in B_1(0); u \notin C^{1,1}(B_r(\mathbf{x})) \text{ for any } r > 0\}.$$

Furthermore we will denote by $S_{n-2}(u)$ *the singular points of co-dimension* 2*:*

$$S_{n-2}(u) = \left\{ \mathbf{y} \in S(u); \lim_{r_j \to 0} \frac{u(r_j\mathbf{x}+\mathbf{y})}{\|u(r_j\mathbf{x}+\mathbf{y})\|_{L^2(B_1(0))}} \right.$$
$$\left. = Q \circ \frac{x_{n-1}^2 - x_n^2}{\|x_{n-1}^2 - x_n^2\|_{L^2(B_1(0))}} \text{ for some } Q \in \mathcal{Q} \text{ and } r_j \to 0 \right\}$$

where $\mathcal{Q}$ *is the matrix group of rotations of* $\mathbb{R}^n$*.*

It was shown in [14] or [3] that if $y \in S(u)$ then

$$\lim_{r_j \to 0} \frac{u(r_j\mathbf{x}+\mathbf{y})}{\|u(r_j\mathbf{x}+\mathbf{y})\|_{L^2(B_1(0))}} \in \mathcal{P}_2$$

if the right hand side is defined, here $\mathcal{P}_2$ is the set of homogeneous second order harmonic polynomials of degree 2. Since the only homogeneous second order harmonic polynomial, up to translations, rotations and multiplicative constants, in $\mathbb{R}^2$ is $x_1^2 - x_2^2$ it follows that S_{n-2} singles out the singular points with co-dimension 2 singularities.

In [4] two of the authors showed rigorously that singular points exists, that is there exist a solution u to (1.1) such that $S(u) \neq \emptyset$. This investigation was followed by the authors in [2] and [3] where we investigated and provided a total classification of singular points in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

In this paper we intend to prove that in $\mathbb{R}^n$ the singular points of smallest co-dimension are locally contained in a C^1-manifold of dimension $n-2$ and that the free boundary Γ_u, defined

$$\Gamma_u = \{\mathbf{x} \in B_1(0); u(\mathbf{x}) = 0\},$$

consists of two C^1 manifolds of dimension $n-1$ intersecting orthogonally at such singular points.

Our main theorem is

THEOREM 1.2. *Let u be a solution to* (1.1) *and assume that*

$$\lim_{r_j \to 0} \frac{u(r_j\mathbf{x})}{\|u(r_j\mathbf{x})\|_{L^2(B_1)}} = \frac{x_{n-1}^2 - x_n^2}{\|x_{n-1}^2 - x_n^2\|_{L^2(B_1)}} \tag{1.2}$$

for some sequence $r_j \to 0$ (*In particular,* $0 \in S_{n-2}(u)$)*. Then*

$$\lim_{r \to 0} \frac{u(r_j\mathbf{x})}{\|u(r_j\mathbf{x})\|_{L^2(B_1)}} = \frac{x_{n-1}^2 - x_n^2}{\|x_{n-1}^2 - x_n^2\|_{L^2(B_1)}} \tag{1.3}$$

and for each $\eta > 0$ *there exists an* $r_\eta > 0$ *such that*

$$S \cap B_{r_0}(0) \cap \left\{ \mathbf{x}; \sum_{i=1}^{n-2} x_i^2 \leq \eta (x_{n-1}^2 + x_n^2) \right\}$$

consists of two C^1 *hypersurfaces intersecting at right angles at the origin.*

Furthermore there is a constant $r_0(u) > 0$ *such that the set*

$$S_{n-2} = \left\{ \mathbf{y}; u(\mathbf{y}) = |\nabla u(\mathbf{y})| = 0 \text{ and } \lim_{r \to 0} \frac{u(r\mathbf{x} + \mathbf{y})}{\|u(r\mathbf{x} + \mathbf{y})\|_{L^2(B_1(0))}} \right.$$
$$\left. = Q \circ \frac{x_{n-1}^2 - x_n^2}{\|x_{n-1}^2 - x_n^2\|_{L^2(B_1(0))}} \text{ for some } q \in \mathcal{Q} \right\}$$

is contained in a C^1 *manifold of dimension* $(n-2)$ *in* $B_{r_0}(0)$.

We would like to place this result in a long tradition of regularity result for parametric non-linear PDE. In particular we may view the free boundary $\Gamma_u = \{\mathbf{x} \in B_1(0); u(\mathbf{x}) = 0\}$ as a parametric surface with singular points in $S(u)$.

Some of the most famous result in this area are the results by Bombieri, De Giorgi, Giusti and Simmons ([6], [17]) that states that no minimal cones exists for minimal surfaces in $n < 8$. We should also mention the result by B. White [18] where uniqueness of tangent cones for 2-dimensional minimal surfaces is proved. From our point of view White's proof is interesting in that he uses a Fourier series expansion in constructing comparison surfaces. However, we work in n-dimensions which means that our Fourier expansions are considerably more subtle and involved than those that appear in [18].

Singularities in parametric problems have appeared in other areas of mathematics as well and our results have some similarities to the theory for harmonic mappings ([16] for a good overview). One could also mention a certain similarity with the theory of singularities that arise for α-uniform measures [13].

Equation (1.1) also arises in several applications for instance in solid combustion (see the references in [14]), the composite membrane problem ([8], [7], [5], [15], [9], [10]), climatology ([11]) and fluid dynamics ([1]).

Our proof will be based on a dynamic systems approach where we project a solution $\frac{u(r\mathbf{x})}{r^2}$ onto the harmonic second order polynomials, call this projection $\Pi(u, r, 0)$ (see Definition 3.2). By a careful analysis of the PDE we will be able to estimate $\Pi(u, r, 0) - \Pi(u, r/2, 0)$. Close to a singular point we have that $\frac{u(r\mathbf{x})}{r^2} \approx \Pi(u, r, 0) + Z_{\Pi(u,r,0)}$ where

$$\begin{aligned}
&\Delta Z_{\Pi(u,r,0)} = -\chi_{\{\Pi(u,r,0)>0\}} \quad \text{in } \mathbb{R}^n \\
&Z_{\Pi(u,r,0)}(0) = |\nabla Z_{\Pi(u,r,0)}(0)| = 0 \\
&\lim_{|\mathbf{x}| \to \infty} \frac{Z_{\Pi(u,r,0)}(\mathbf{x})}{|\mathbf{x}|^3} = 0 \\
&\Pi(Z_{\Pi(u,r,0)}, 1, 0) = 0.
\end{aligned}$$

If we disregard lower order terms we may consider the map $\mathscr{F}(\Pi(u,r,0)) = \Pi(u,r/2,0)$ defined by

$$\mathscr{F}(\Pi(u,r,0)) = \Pi(u,r,0) + \Pi(Z_{\Pi(u,r,0)}, 1/2, 0).$$

The blow-up is unique if $\lim_{k\to\infty} \mathscr{F}^k(\Pi(u,r,0))$ exists.

Since the harmonic second order polynomials form a finite dimensional space. The map $\mathscr{F}$ is a map between finite dimensional vector spaces. The main difficulty is that $\mathscr{F}$ is highly non-linear and we need quite subtle estimates to characterise the map. On the positive side we may write down $\Pi(u,r,0)$ explicitly, modulo lower order terms, by means of Theorem 3.5 by Karp and Margulis [12]. The definition of $\mathscr{F}$ involves a Fourier series expansion of $-\chi_{\Pi(u,r,0)}$ on the unit sphere. Our main effort will be to estimate the Fourier coefficients in this expansion when $\Pi(u,r,0)/\sup_{B_1}|\Pi| \approx x_{n-1}^2 - x_n^2$. For further details on the idea of the proof we refer the reader to [3].

2. List of notation

(1) δ will denote a vector in $\mathbb{R}^{n-2}$, we will always assume that $|\delta| \ll 1$. We also define $\tilde{\delta} = \sum_{i=1}^{n-2} \delta_i$.
(2) $\mathscr{P}_2$ will denote the second order homogeneous polynomials.
(3) $S(u)$ and $S_{n-2}(u)$ are the singular set and the singular set of co-dimension 2 respectively, defined in Definition 1.1.
(4) The mapping F is defined in equation (4.20).
(5) $\Pi(u,r,x^0)$ is defined in Definition 3.2.
(6) The average of u in Ω will be denoted $(u)_\Omega$.
(7) By dA we mean an area element of the surface under considration.
(8) We will use Landau's $O(r)$ notation to indicate a term that is bounded from by Cr for a universal constant C. That is $f(x) = O(r)$ if and only if $|f(x)| \leq Cr$ for a universal constant C. Similarly, $f(r) \geq O(r)$ means that $f(r) \geq Cr$ for some universal constant $C > 0$ etc.
(9) $p_\delta(x) = \sum_{i=1}^{n-2} \delta_i x_i^2 + (1-\tilde{\delta})x_{n-1}^2 - x_n^2$, in particular $p_0(x) = x_{n-1}^2 - x_n^2$.
(10) Z_{p_δ} is defined in (3.9).
(11) $\mathscr{Q}$ is the matrix-group of rotations of $\mathbb{R}^n$.
(12) The functions $B_i(\delta)$, $B(\delta)$, $C_i(\delta)$ and $C(\delta)$ are defined in (4.12), (4.13), Proposition 4.3 and the remark after that Proposition respectively.

3. Background material and general strategy

In this section we will state some of the results of [3] and outline our strategy (which is similar to the strategy of [3]).

Our starting observation is the following proposition (Proposition 5.1 in [14])

PROPOSITION 3.1. *Let u be a solution of* (1.1) *in* $B_1(0)$ *and let us consider a point* $\mathbf{x}^0 \in S(u)$. *Then*

$$\lim_{r_j \to 0} \frac{u(r_j \mathbf{x} + \mathbf{x}^0)}{\|u(r_j x + \mathbf{x}^0)\|_{L^2(B_1(0))}} \in \mathscr{P}_2$$

for each sequence $r_j \to 0$ *such that the limit exists.*

The proof is a fairly standard application of a monotonicity formula.

If u is a solution to (1.1) then $\Delta u \in L^\infty$ which directly implies that $D^2 u \in BMO(B_{1/2}(0))$ which in particular implies, via the Sobolev inequality, that for $\mathbf{x}^0 \in S(u) \cap B_{1/2}(0)$

$$\frac{u(r\mathbf{x} + \mathbf{x}^0) - \frac{1}{2}(\mathbf{x} - \mathbf{x}^0)(D^2 u)_{B_r(\mathbf{x}^0)}(\mathbf{x} - \mathbf{x}^0)}{r^2} \tag{3.4}$$

is locally bounded in L^2 and pre-compact. It will be convenient for some calculations later to subtract a harmonic polynomial in (3.4) instead of the polynomial $\frac{1}{2}(\mathbf{x} - \mathbf{x}^0)(D^2 u)_{B_r(\mathbf{x}^0)}(\mathbf{x} - \mathbf{x}^0)$. We make the following definition.

DEFINITION 3.2. *By* $\Pi(u, r, \mathbf{x}^0)$ *we will denote the projection operator onto* $\mathscr{P}_2$ *defined as follows:* $\Pi(u, r, \mathbf{x}^0) = \tau_r p$, *where* $\tau_r \in \mathbb{R}^+$ *and* $p \in \mathscr{P}_2$ *satisfies* $\sup_{B_1} |p| = 1$ *as well as*

$$\inf_{h \in \mathscr{P}_2} \int_{B_1(0)} \left| D^2\left(\frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2}\right) - D^2 h \right|^2 = \int_{B_1(0)} \left| D^2\left(\frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2}\right) - \tau D^2 p \right|^2.$$

We will often write $\Pi(u, r)$ *when* $\mathbf{x}^0$ *is either the origin or given by the context. By definition* $\tau_r = \sup_{B_1} |\Pi(u, r)|$ *and* $p_r = \Pi(u, r)/\tau_r$.

It is a simple consequence of the BMO estimate (3.4) that if $\mathbf{x}^0 \in S(u) \cap B_{1/2}(u)$ then (Proposition 3.7 in [3])

$$\left\| \frac{u(r\mathbf{x} + \mathbf{x}^0)}{r^2} - \Pi(u, r, \mathbf{x}^0) \right\|_{C^{1,\alpha}(B_1)} \leq C_\alpha\left(\sup_{B_1} |u|, n\right). \tag{3.5}$$

If $\mathbf{x}^0 \in S(u)$ then

$$\sup_{B_r(\mathbf{x}^0)} |u| > c r^2 \ln(1/r) \tag{3.6}$$

for $0 < r < r_0(u, \mathbf{x}^0)$ and some small $c > 0$. To be more precise it is known that (c.f. Lemma 5.1 in [3]).

LEMMA 3.3. *Let u be a solution to* (1.1) *in* B_1 *such that* $\sup_{B_1} |u| \leq M$ *and* $u(0) = |\nabla u(0)| = 0$. *Then there exist* $\rho_0 > 0$ *and* $r_0 > 0$ *such that if*

$$\sup_{B_1} |\Pi(u, r)| \geq \frac{1}{\rho_0} \tag{3.7}$$

for an $r \leq r_0$ *then*

$$\sup_{B_1} |\Pi(u, r/2)| > \sup_{B_1} |\Pi(u, r)| + \eta_0/2,$$

where η_0 *is a universal constant.*

The Lemma is proved for $n = 3$ in [3] but the proof is the same in arbitrary dimension.

This estimate together with (3.5) implies that $u(\cdot + \mathbf{x}^0) = \Pi(u, r, \mathbf{x}^0) +$ a lower order perturbation. Using the pre-compactness in $C^{1,\alpha}$ (c.f. Equation (3.5)) of

$$\frac{u(r_j\mathbf{x} + \mathbf{x}^0)}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0) \tag{3.8}$$

for some sequence $r_j \to 0$ we may extract a sub-sequence, which we still denote by r_j, such that

$$\lim_{r_j \to 0} \left(\frac{u(r_j\mathbf{x} + \mathbf{x}^0)}{r_j^2} - \Pi(u, r_j, \mathbf{x}^0) \right) = Z_p(\mathbf{x})$$

for some function Z_p. It is not difficult to see that Z_p is the unique solution to

$$\begin{aligned} &\Delta Z_p = -\chi_{\{p(\mathbf{x})>0\}} \quad \text{in } \mathbb{R}^n \\ &Z_p(0) = |\nabla Z_p(0)| = 0 \\ &\lim_{|\mathbf{x}| \to \infty} \frac{Z_p(\mathbf{x})}{|\mathbf{x}|^3} = 0 \\ &\Pi(Z_p, 1) = 0 \end{aligned} \tag{3.9}$$

where

$$p(\mathbf{x}) = \lim_{r_j \to 0} \frac{\Pi(u, r_j, \mathbf{x}^0)}{\|\Pi(u, r_j, \mathbf{x}^0)\|_{L^2(B_1)}}.$$

In order to show regularity for the free boundary near a singular point we would have to control the limit

$$\lim_{r \to 0} \frac{\Pi(u, r, \mathbf{x}^0)}{\|\Pi(u, r, \mathbf{x}^0)\|_{L^2(B_1)}}.$$

If one can show that the limit is unique then it follows that the blow-up

$$\lim_{r \to 0} (u(r\mathbf{x} + \mathbf{x}^0)/r^2 - \Pi(u, r, \mathbf{x}^0)) = Z_p$$

is unique.

The following result, Corollary 7.3 in [3], gives a quantitative measure on how the function $Z_{\Pi(u,r,0)}$ controls the difference between $\Pi(u,r,0)$ and $\Pi(u,r/2,0)$.

PROPOSITION 3.4. *Let u solve (1.1) in $B_1 \subset \mathbb{R}^n$ and assume that $\sup_{B_1}|u| \le M$, $u(0) = |\nabla u(0)| = 0$, and that for some $\rho \le \rho_0$ and $r \le r_0$,*

$$\sup_{B_1} |\Pi(u,r)| \ge \frac{1}{\rho}.$$

Then

$$\sup_{B_1} |\Pi(u,r/2) - \Pi(u,r) - \Pi(Z_{\Pi(u,r)}, 1/2)| \le C(M,n,\alpha)\Big(\sup_{B_1}|\Pi(u,r)|\Big)^{-\alpha}$$

for each $\alpha < 1/4$.

In order to estimate $\sup_{B_1(0)}|\Pi(u,r,0) - \Pi(u,r/2,0)|$ we thus need to be able to calculate $\Pi(Z_{\Pi(u,r,0)}, 1/2, 0)$. We will do this with the help of the following theorem from [12].

THEOREM 3.5. *Let $\sigma \in L^\infty(\mathbb{R}^n)$ be homogeneous of zeroth order, that is $\sigma(\mathbf{x}) = \sigma(r\mathbf{x})$ for all $r > 0$. Assume that σ has the Fourier series expansion*

$$\sigma(\mathbf{x}) = \sum_{i=0}^{\infty} a_i \sigma_i,$$

on the unit sphere, where σ_i is a homogeneous harmonic polynomial of order i.

Moreover assume that $\Delta Z = \sigma$ and that $Z(0) = |\nabla Z(0)| = \lim_{\mathbf{x}\to\infty} Z(\mathbf{x})/|\mathbf{x}|^3 = 0$. Then

$$Z(\mathbf{x}) = q(\mathbf{x})\ln|\mathbf{x}| + |\mathbf{x}|^2\phi(\mathbf{x}),$$

where

$$q = \frac{a_2}{n+2}\sigma_2$$

and

$$\phi(\mathbf{x}) = \sum_{i\neq 2} \frac{a_i}{(n+i)(i-2)}\sigma_i\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big).$$

Our strategy in the rest of the paper will be to use Theorem 3.5 to calculate

$$\Pi(Z_{\Pi(u,r,0)}, 1/2, 0) = -\frac{\ln(2)a_2}{n+2}\sigma_2(\mathbf{x}) \tag{3.10}$$

where σ_2 is the second order term in the Fourier series expansion

$$-\chi_{\{\Pi(u,r,0)>0\}} = \sum_{i=0}^{\infty} a_i \sigma_i(\mathbf{x}) \quad \text{on } \partial B_1(0).$$

Using the expression (3.10) in Proposition 3.4 will give us enough information to deduce that the blow-up of u is unique at all points $\mathbf{x}^0 \in S_{n-2}(u)$.

4. Estimates of the Projections

In order to estimate $\Pi(Z_p, 1/2)$ we need to calculate $a_2\sigma_2$ from Theorem 3.5. That involves calculating the second order Fourier coefficients for $-\chi_{\{p_\delta>0\}}$ on the unit sphere. To that end we choose $nx_i^2 - |\mathbf{x}|^2$ for $i = 1, \dots, n$ and $x_i x_j$ for $i \neq j$ as a basis for the second order harmonic polynomials.

We may choose coordinates so that

$$\text{(4.11)} \qquad \frac{\Pi(u,r,0)}{\sup_{B_1}|\Pi(u,r,0)|} = p_\delta(x) = \delta_1 x_1^2 + \delta_2 x_2^2 + \cdots + \delta_{n-2} x_{n-2}^2 + (1-\tilde{\delta}) x_{n-1}^2 - x_n^2,$$

where $\delta = (\delta_1, \delta_2, \dots, \delta_{n-2})$ and $\tilde{\delta} = \sum_{i=1}^{n-2} \delta_i$. We also define the polynomial p_δ, for a given vector $\delta \in \mathbb{R}^{n-2}$ in equation (4.11). We will assume, for definiteness that $\tilde{\delta} \geq 0$ (this is implicit in the definition of p_δ in equation (4.11)). If $\tilde{\delta} < 0$ then all the following arguments follows through with minor and trivial changes.

It follows from symmetry (i.e. $-\chi_{\{p_\delta>0\}}$ is even and the $x_i x_j$'s are odd on the unit sphere) that the Fourier coefficient of $x_i x_j$ is zero.

Since we are only interested in points $\mathbf{x}^0 \in S_{n-2}(u)$ where

$$\lim_{r_j \to 0} \frac{\Pi(u, r_j, \mathbf{x}^0)}{\sup_{B_1}|\Pi(u, r_j, \mathbf{x}^0)|} = p_0,$$

for some sequence $r_j \to 0$, we may assume that $|\delta| \ll 1$.

We also denote by $B_i(\delta)$ the following integral

$$\text{(4.12)} \qquad B_i(\delta) = -\int_{\partial B_1(0)} \chi_{\{p_\delta>0\}} x_i^2 \, dA$$

and by $B(\delta)$ the following integral

$$\text{(4.13)} \qquad B(\delta) = -\int_{\partial B_1(0)} \chi_{\{p_\delta>0\}} \, dA.$$

Here dA is the surface element. It follows that the Fourier coefficient of $nx_i^2 - |\mathbf{x}|^2$ of $\chi_{\{p_\delta>0\}}$ is

$$\frac{1}{\|nx_i^2 - |\mathbf{x}|^2\|_{L^2(\partial B_1(0))}}(nB_i(\delta) - B(\delta)).$$

Using that $\Pi(Z_{p_\delta}, 1) = 0$ by definition and Theorem 3.5 we may deduce that

$$\Pi(Z_{p_\delta}, 1/2) = -K_0 \sum_{i=1}^{n} (n^2 B_i(\delta) - nB(\delta))x_i^2, \tag{4.14}$$

where

$$K_0 = \frac{\ln(2)}{(n+2)\|nx_i^2 - |\mathbf{x}|^2\|_{L^2(\partial B_1(0))}}.$$

It is clear that we need to estimate the functions $B_i(\delta)$ and $B(\delta)$ in order to estimate

$$\Pi(u, r) - \Pi(u, r/2) = \Pi(Z_{p_\delta}, 1/2) + O(\|\Pi(u, r)\|_{L^\infty(B_1(0))}^{-\alpha}),$$

where the above equality is a direct consequence of Proposition 3.4.

Before we can estimate the integrals in (4.12) and (4.13) we need to introduce some notation for integration on the unit sphere. We parametrise the unit sphere in $\mathbb{R}^2$ according to

$$\partial B_1(0) = \{\bar{\xi}_1(\phi); \phi \in (0, 2\pi)\},$$

where $\bar{\xi}_1(\phi) = (\cos(\phi), \sin(\phi))$. Inductively we define, for $k \geq 2$, the polar coordinates

$$\bar{\xi}_k(\phi, \psi_1, \psi_2, \dots, \psi_{k-1}) = (\sin(\phi_{k-1})\bar{\xi}_{k-1}(\phi, \psi_1, \dots, \psi_{k-2}), \cos(\psi_{k-1})).$$

The unit sphere in $\mathbb{R}^k$ is then defined by

$$\partial B_1(0) = \{\bar{\xi}_{k-1}(\phi, \psi_1, \dots, \psi_{k-2}); \phi \in (0, 2\pi), \psi_j \in (0, \pi)\},$$

modulo a set of measure zero.

With this parametrisation an area element on the unit sphere becomes

$$dA = \det\begin{bmatrix} \frac{\partial \xi_{k-1}}{\partial \phi} & \frac{\partial \xi_{k-1}}{\partial \psi_1} & \frac{\partial \xi_{k-1}}{\partial \psi_2} & \cdots & \frac{\partial \xi_{k-1}}{\partial \psi_{k-2}} \end{bmatrix} d\phi\, d\psi_1 \dots d\psi_{k-2}, \tag{4.15}$$

where ξ_{k-1} is considered to be a column vector. Somewhat more explicitly the $k \times (k-1)$-matrix in (4.15) is

(4.16)
$$\begin{bmatrix} -\sin(\psi)P_{j=1}^{k-2} & \cos(\phi_1)\cos(\phi)P_{j=2}^{k-2} & \cdots & \cos(\psi_{k-3})\cos(\phi)P_{j=1,j\neq k-3}^{k-2} & \cos(\psi_{k-2})\cos(\phi)P_{j=1}^{k-3} \\ \cos(\phi)P_{j=1}^{k-2} & \sin(\phi_1)\cos(\phi)P_{j=2}^{k-2} & \cdots & \cos(\psi_{k-3})\sin(\phi)P_{j=1,j\neq k-3}^{k-2} & \cos(\psi_{k-2})\sin(\phi)P_{j=1}^{k-3} \\ 0 & -\sin(\psi_1)P_{j=2}^{k-2} & \cdots & \cos(\psi_{k-3})P_{j=1,j\neq k-3}^{k-2} & \cos(\psi_{k-2})P_{j=1}^{k-3} \\ 0 & 0 & \ddots & \cos(\psi_{k-3})P_{j=2,j\neq k-3}^{k-2} & \cos(\psi_{k-2})P_{j=2}^{k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin(\psi_{k-3})\sin(\psi_{k-2}) & \cos(\psi_{k-2})\cos(\psi_{k-3}) \\ 0 & 0 & \cdots & 0 & -\sin(\psi_{k-2}) \end{bmatrix}$$

where we have used the notation

$$P_{j=1}^{k-2} = \Pi_{j=1}^{k-1}\sin(\psi_j).$$

We will denote the matrix in (4.16) by M. By the anti-commutativity of the rows in the determinant function we have the identity

$$(4.17)\quad det(M) = \sin^{k-2}(\psi_{k-3})\sin^{k-2}(\psi_{k-2}) \times \begin{vmatrix} & & & & 0 & 0 \\ & & N & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin(\psi_{k-3})\sin(\psi_{k-2}) & \cos(\psi_{k-2})\cos(\psi_{k-3}) \\ 0 & 0 & 0 & 0 & 0 & -\sin(\psi_{k-2}) \end{vmatrix}$$
$$= \sin^{k-3}(\psi_{k-3})\sin^{k-2}(\psi_{k-2})\det(N)$$

where $N(\phi, \psi_1, \ldots, \psi_{k-4})$ is the $(k-2)\times(k-3)$-matrix satisfying $\sin(\psi_{k-3})\sin(\psi_{k-2})n_{ij} = m_{ij}$ for $1 \le i \le k-2$ and $1 \le j \le k-3$. Notice that N is independent of ψ_{k-3} and ψ_{k-2}.

In order to estimate B_i we will use the identity in (4.17) to write, with $k = n$,

$$(4.18)\qquad B_i(\delta) = -\int_{\partial B_1(0)} \chi_{\{p_\delta>0\}} x_i^2 \, dA_{\partial B_1(0)}$$
$$= -\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \chi_{\{p_\delta>0\}} x_i^2 |det(M)| \, d\psi_{k-2}\, d\psi_{k-3}\ldots d\phi$$
$$= -\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}$$
$$\times \left[\int_0^{\pi}\int_0^{\pi} \chi_{\{p_\delta>0\}} x_i^2 |\sin^{n-1}(\psi_{n-3})\sin^{n}(\psi_{n-2})| \, d\psi_{n-2}\, d\psi_{n-3}\right]$$
$$\times |det(N)| \, d\psi_{n-4}\ldots d\phi.$$

We will need some further simplifications

$$
\begin{aligned}
(4.19)\qquad B_i(\delta) &= -\int_{\partial B_1(0)} \chi_{\{p_\delta>0\}} x_i^2 \, dA_{\partial B_1(0)} \\
&= -\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi} \chi_{\{p_\delta>0\}} x_i^2 |det(M)| \, d\psi_{k-1}\, d\psi_{k-2}\ldots d\phi \\
&= -2^n \int_{(0,\pi/2)^{n-2}} \\
&\quad \times \left[\int_0^{\pi/2}\int_0^{\pi/2} \chi_{\{p_\delta>0\}} x_i^2 S^{n-1,n}(\psi_{n-2},\psi_{n-1})\, d\psi_{n-1}\, d\psi_{n-2}\right] \\
&\quad \times |det(N)|\, d\psi_{n-3}\ldots d\phi \\
&= -2^n \int_{(0,\pi/2)^{n-2}} \left[\int_{A(\mu)} \chi_{\{p_\delta>0\}} x_i^2 S^{n-1,n}(\psi_{n-2},\psi_{n-1})\, d\psi_{n-1}\, d\psi_{n-2}\right] \\
&\quad \times |det(N)|\, d\psi_{n-3}\ldots d\phi \\
&\quad -2^n \int_{(0,\pi/2)^{n-2}} \left[\int_{(0,\pi/2)^2\setminus A(\mu)} \chi_{\{p_\delta>0\}} x_i^2 S(\psi_{n-2},\psi_{n-1})\, d\psi_{n-1}\, d\psi_{n-2}\right] \\
&\quad \times |det(N)|\, d\psi_{n-3}\ldots d\phi \\
&= I_{1,i}(\delta,\mu) + I_{2,i}(\delta,\mu),
\end{aligned}
$$

where

$$S^{n-1,n}(\psi_{n-2},\psi_{n-1}) = |\sin^{n-1}(\psi_{n-2})\sin^n(\psi_{n-1})|,$$

and $A(\mu) = F^{-1}((0,\mu)^2)$ where F is the stereographic projection

$$(4.20)\qquad F(\psi_{n-3},\psi_{n-2}) = \Big(\frac{\cos(\psi_{n-3})}{\sin(\psi_{n-3})}, \frac{\cos(\psi_{n-2})}{\sin(\psi_{n-2})\sin(\psi_{n-3})}\Big).$$

If μ is small then $A(\mu) \approx (\pi/2-\mu, \pi/2)^2$, the exact form of $A(\mu)$ is unimportant as long as $A(\mu)$ contains a small neighbourhood of the point $(\pi/2,\pi/2)$. We choose the particular form of $A(\mu)$ in order to simplify some calculations further on (see equation (4.24)).

We will estimate $I_{1,i}(\delta,\mu)$ and $I_{2,i}(\delta,\mu)$ separately for $|\delta|$ small. Fix a $\mu>0$ such that $|\delta| \ll \mu \ll 1$. The value of μ is not very important and can be chosen universal, depending only on n in particular $\mu < c_L$ in (4.32).

To estimate $I_{2,i}(\delta,\mu)$ we notice that

$$\nabla p_\delta = 2(\delta_1 x_1, \delta_2 x_2, \ldots, \delta_{n-2}x_{n-2}, (1-\tilde{\delta})x_{n-1}, -x_n).$$

By our choice of polar coordinates we have that when $\psi_{n-1} \in (0, \pi/2 - \mu)$ then

$$x_n = \cos(\psi_{n-1}) \geq c\mu.$$

This means that the gradient of p_δ is bounded from below by a constant times μ on its zero level set. It is therefore very easy to estimate $I_{2,i}(\delta, \mu)$ by means of the co-area formula.

By the co-area formula it follows that for $t \in (0,1)$ and with the notation $q_\delta = \sum_{j=1}^{n-2} \delta_j x_j^2$

$$\left|\frac{d}{dt} I_{2,i}(\delta, t\delta)\right| = \left| \int_{\{(x_{n-2}^2 - x_n^2)/(q_\delta - \tilde{\delta} x_{n-1}^2) = t\}} \frac{1}{\left|\nabla \frac{x_{n-2}^2 - x_n^2}{q_\delta - \tilde{\delta} x_{n-1}^2}\right|} dA_{\{(x_{n-2}^2 - x_n^2)/(q_\delta - \tilde{\delta} x_{n-1}^2) = t\}} \right| \leq C \frac{|\delta|}{\mu}.$$

In particular

$$|I_{2,i}(\delta, \mu) - I_{2,i}(\delta, 0)| \leq C \frac{|\delta|}{\mu}. \tag{4.21}$$

We need to work a little harder in order to estimate $I_1(\delta, \mu)$. We begin to prove a simple lemma that will allow us to do some integrations explicitly module $O(|\delta|)$-terms.

LEMMA 4.1. *Let $\phi^0, \psi_1^0, \psi_2^0, \dots, \psi_{n-4}^0$ be fixed. Furthermore we let $\mu > 0$ be a small constant and $1 \leq i \leq n$. We use polar coordinates $x_i(\phi, \psi_1, \psi_2, \dots, \psi_{n-2})$.*

We also assume that

$$\sum_{j=1}^{n-2} \delta_j x_j^2 (\phi^0, \psi_1^0, \psi_2^0, \dots, \psi_{n-4}^0, \pi/2, \pi/2)^2 \geq 0. \tag{4.22}$$

Then there exist a constant $c > 0$ such that

$$\begin{aligned}
(1 - c\mu) \int_{A(\mu)} & x_i^2 (\phi^0, \psi_1^0, \dots, \psi_{n-2})^2 \\
& \times (\chi_{\{p_\delta > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-2}) \\
& - \chi_{\{p_0 > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-2})) S^{n-1,n} \, d\psi_{n-2} \, d\psi_{n-1} \\
\leq & \int_0^\mu \int_0^\mu \tilde{x}_i^2 (\chi_{\{p_\delta(\tilde{x}) > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-3}, \psi_{n-2}) \\
& - \chi_{\{p_0 > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-2})) \, d\tilde{x}_{n-1} \, d\tilde{x}_n \\
\leq & (1 + c\mu) \int_{A(\mu)} x_i^2 (\phi^0, \psi_1^0, \dots, \psi_{n-3}, \psi_{n-2})^2 \\
& \times (\chi_{\{p_\delta > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-3}, \psi_{n-2}) \\
& - \chi_{\{p_0 > 0\}} (\phi^0, \psi_1^0, \dots, \psi_{n-2})) S^{n-1,n} \, d\psi_{n-3} \, d\psi_{n-2},
\end{aligned}$$

where

$$S^{i,j} = |\sin^i(\psi_{n-3})\sin^j(\psi_{n-2})|,$$

$$\tilde{x}_i(\phi, \psi_1, \psi_2, \ldots, \psi_{n-2}) = \frac{x_i}{\sqrt{\sum_{j=1}^{n-2} x_j^2}}.$$

and the set A is the stereographic projection of the two dimensional spherical area

$$\{x(\phi^0, \psi_1^0, \ldots, \psi_{n-3}, \psi_{n-2}); (\phi_{n-3}, \phi_{n-2}) \in (\pi/-\mu, \pi/2)^2\}$$

under the projection $x \to \tilde{x}$.

REMARK. *Assumption* (4.22) *is non-essential and only made for definiteness and the result still holds if*

$$\sum_{j=1}^{n-2} \delta_j x_j^2(\phi^0, \psi_1^0, \psi_2^0, \ldots, \psi_{n-4}^0, \pi/2, \pi/2)^2 < 0.$$

PROOF. It is trivial that $1 - c\mu \leq \sin(\psi_{n-3}) \leq 1$ and that $1 - c\mu \leq \sin(\psi_{n-3}) \leq 1$. Therefore

$$1 - c_{i,j}\mu \leq S^{i,j} \leq 1. \tag{4.23}$$

Use the change of variables

$$(\psi_{n-3}, \psi_{n-2}) \to \left(\frac{\cos(\psi_{n-3})}{\sin(\psi_{n-3})}, \frac{\cos(\psi_{n-2})}{\sin(\psi_{n-2})\sin(\psi_{n-3})}\right) = (\tilde{x}_{n-1}, \tilde{x}_{n-2})$$

in

$$\begin{aligned} &\int_{A(\mu)} x_i^2(\phi^0, \psi_1^0, \ldots, \psi_{n-3}, \psi_{n-2})^2 \\ &\qquad \times (\chi_{\{p_\delta>0\}}(\phi^0, \psi_1^0, \ldots, \psi_{n-3}, \psi_{n-2}) \\ &\qquad - \chi_{\{p_0>0\}}(\phi^0, \psi_1^0, \ldots, \psi_{n-2}))S^{n-1,n}\, d\psi_{n-3}\, d\psi_{n-2} \\ &\quad = \int_0^\mu \int_0^\mu x_i^2(\phi^0, \psi_1^0, \ldots, \psi_{n-3}, \psi_{n-2})^2 \\ &\qquad \times (\chi_{\{p_\delta(\tilde{x})>0\}} - \chi_{\{p_0(\tilde{x})>0\}})S^{n-4,n-2}\, d\tilde{x}_{n-1}\, d\tilde{x}_n, \end{aligned} \tag{4.24}$$

it is in this change of variables that we use the rather awkward definition of $A(\mu)$ in order to get a nice area of integration to the right.

Since $\sqrt{\sum_{j=1}^{n-2} x_j^2} = \sin(\psi_{n-3})\sin(\psi_{n-2})$ we may estimate

$$(1 - c\mu)\tilde{x}_i \leq x_i \leq \tilde{x}_i \tag{4.25}$$

Notice that since

$$\sum_{j=1}^{n-2} \delta_j x_j(\phi^0, \psi_1^0, \psi_2^0, \dots, \psi_{n-4}^0, \pi/2, \pi/2)^2 \geq 0.$$

the integrand is non-negative so we may use (4.23) and (4.25) in (4.24) to deduce the desired estimates. □

LEMMA 4.2. *Let $|\delta| \ll \mu \ll 1$. Also denote*

$$q_\delta = \sum_{j=1}^{n-2} \delta_j x_j^2$$

and $\phi^0, \psi_1^0, \dots, \psi_{n-4}^0$ fixed constants. Then, for $i = 1, \dots, n-2$,

$$\begin{aligned}
&\int_{(\pi/2-\mu,\pi/2)^2} x_i^2(\chi_{\{p_\delta>0\}}(\phi, \psi_1, \dots, \psi_{n-2}) - \chi_{\{p_0>0\}})S^{n-1,n}\, d\psi_{n-3}\, d\psi_{n-2} \\
&\quad = -\frac{1+O(\mu)}{4}\, \frac{q_\delta(\phi^0, \psi_1^0, \dots, \pi/2, \pi/2)}{1-\tilde{\delta}} |\ln(|q_\delta(\phi^0, \psi_1^0, \dots, \pi/2, \pi/2)|)| \\
&\qquad + O(|\delta|/\mu + \mu|\delta|\, |\ln(|\delta|)|)
\end{aligned}$$

and for $i = n-1, n$ we have

$$\int_{(\pi/2-\mu,\pi/2)^2} x_i^2(\chi_{\{p_\delta>0\}}(\phi, \psi_1, \dots, \psi_{n-2}) - \chi_{\{p_0>0\}})S^{n-1,n}\, d\psi_{n-3}\, d\psi_{n-2} = O(|\delta|)$$

PROOF. By Lemma 4.1 it is enough to prove the estimate for

$$\int_0^\mu \int_0^\mu x_i^2(\chi_{\{p_\delta(x)\}} - \chi_{p_0(x)})\, dx_{n-1}\, dx_n, \tag{4.26}$$

where $\sum_{j=1}^{n-2} x_j^2 = 1$.

To simplify notation we will write

$$\kappa = q_\delta(x).$$

And we will assume that $\kappa > 0$, if $\kappa = 0$ then the argument is simple and the case $\kappa < 0$ is treated analogously.

Notice that

$$\chi_{\{p_\delta(\tilde{x})>0\}} = \begin{cases} 1 & \text{if } 0 < \tilde{x}_n < \sqrt{\kappa + (1+\tilde{\delta})\tilde{x}_{n-1}^2} \\ 0 & \text{else.} \end{cases}$$

For $i = 1, \dots, n-2$ we may write (4.26) as

$$\int_0^{\mu}\left(\sqrt{\kappa+(1-\tilde{\delta})x_{n-1}^2)}-\sqrt{x_{n-1}^2}\right)\right)\tilde{x}_i^2\,dx_{n-1}$$
$$=\frac{1}{4}\frac{\kappa}{1-\tilde{\delta}}\ln(\kappa)\tilde{x}_i^2+\frac{\mu}{2(1+\mu^2)}+O(|\delta|/\mu+\mu|\delta|\,|\ln(|\delta|)|),$$

where we have used the identity

$$\int\sqrt{1+x^2}\,dx=\frac{1}{2}x\sqrt{1+x^2}+\frac{1}{2}\ln(x+\sqrt{1+x^2})$$

to evaluate the integral.

For $i=n-1$ we can calculate

$$\int_0^{\mu}\left(\sqrt{\kappa+(1+\tilde{\delta})x_{n-1}^2}-\sqrt{x_{n-1}^2}\right)\tilde{x}_{n-1}^2\,dx_{n-1}=O(\mu^2\kappa).$$

Finally, for $i=n$ we get

$$\int_0^{\mu}\int_0^{\mu}x_n^2(\chi_{\{p_\delta(x)\}}-\chi_{p_0(x)})\,dx_{n-1}\,dx_n$$
$$=\int_0^{\mu}\left[\int_0^{\sqrt{\kappa+(1-\tilde{\delta})x_{n-1}^2}}x_n^2\,dx_n-\int_0^{x_{n-1}}x_n^2\,dx_n\right]dx_{n-1}=O(\kappa\mu^2).\qquad\square$$

Proposition 4.3. *If $|\delta|$ is small enough and $C_i(\delta)$ is defined according to*

$$C_i(\delta)=B_i(\delta)-B_i(0)$$

then there exists a universal constant c such that

$$\frac{1}{c}|\delta\ln(|\delta|)|\le\sum_{j=1}^{n-2}|C_i(\delta)|\le c|\delta\ln(|\delta|)|.$$

Moreover, if $\delta_i>\delta_j$ then $C_i(\delta)<C_j(\delta)$.

Proof. In (4.19) we showed that we can write

$$B_i(\delta)-B_i(0)=[I_{2,i}(\delta,\mu)-I_{2,i}(0,\mu)]+[I_1(\delta,\mu)-I_1(0,\mu)].$$

We also showed, (4.21), that

$$[I_{2,i}(\delta,\mu)-I_{2,i}(0,\mu)]=O(|\delta|/\mu).$$

Also in (4.19) we showed that we can write

$$I_1(\delta,\mu) - I_1(0,\mu) = \int_{B_1^{n-2}} \left[\int_A (\chi_{\{p_\delta>0\}} - \chi_{p_0>0}) S^{n-1,n} \, d\psi_{n-3} \, d\psi_{n-2} \right] \times det(N) \, dA_{\partial B_1^{n-2}}(\phi, \ldots, \psi_{n-4}). \tag{4.27}$$

Furthermore we showed, in Lemmas 4.1 and 4.2, that the inner integral in (4.27) satisfies

$$\begin{aligned} &\int_A (\chi_{\{p_\delta>0\}} - \chi_{p_0>0}) S^{n-1,n} \, d\psi_{n-3} \, d\psi_{n-2} \\ &\quad = (1 + O(\mu)) \int_A x_i^2 (\chi_{\{p_\delta(\tilde{x})>0\}} - \chi_{\{p_0>0\}}) \, dx_{n-1} \, dx_n \\ &\quad = -\frac{1+O(\mu)}{4} q_\delta(x_1, \ldots, x_{n-2}) \| \ln(|q_\delta(x_1, \ldots, x_{n-2}|) \| \\ &\qquad + O(|\delta|/\mu + \mu|\delta| \, |\ln(|\delta|)|) \end{aligned}$$

for $(x_1, \ldots, x_{n-2}) \in \partial B_1^{n-2}$. Disregarding lower order terms we may conclude that

$$\begin{aligned} &I_{1,i}(\delta,\mu) - I_{1,i}(0,\mu) \\ &\quad = -\frac{1}{4} \int_{\partial B_1^{n-2}} q_\delta |\ln(q_\delta)| \, det(N) \, dA_{\partial B_1^{n-2}} + O(|\delta|/\mu + \mu|\delta| \, |\ln(|\delta|)|). \end{aligned} \tag{4.28}$$

Let us denote the integrand $F(q_\delta)$, that is $F(t) = t|\ln(|t|)|$. We may estimate

$$|F(q_\delta) - |\delta| \, |\ln(|\delta|)\bar{q}_\delta| \, | \le |\delta \bar{q}_\delta \ln(|\bar{q}_\delta|)| \tag{4.29}$$

where $\bar{q}_\delta = \frac{1}{|\delta|} q_\delta$. Since $\bar{q}_\delta$ is a second order polynomial with coefficients bounded by one it directly follows that

$$\left| \int_{\partial B_1^{n-2}} |\delta \bar{q}_\delta \ln(|\bar{q}_\delta|)| \, det(N) \, dA_{\partial B_1^{n-2}} \right| = O(|\delta|). \tag{4.30}$$

By (4.30), (4.29) and (4.28) we may estimate

$$\begin{aligned} &I_{1,i}(\delta,\mu) - I_{1,i}(0,\mu) \\ &\quad = -\frac{|\ln(|\delta|)|}{4} \int_{\partial B_1^{n-2}} \bar{q}_\delta \, det(N) x_i^2 \, dA_{\partial B_1^{n-2}} + O(|\delta|/\mu + \mu|\delta| \, |\ln(|\delta|)|). \end{aligned} \tag{4.31}$$

We define the linear functional $L : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ by

$$L\delta = \begin{bmatrix} \int_{\partial B_1^{n-2}} \bar{q}_\delta x_1^2 \, dA_{\partial B_1^{n-2}} \\ \vdots \\ \int_{\partial B_1^{n-2}} \bar{q}_\delta x_{n-2}^2 \, dA_{\partial B_1^{n-2}} \end{bmatrix}.$$

Writing L in matrix form we get

$$L = \lambda_1 I + \lambda_2 J$$

where $\lambda_1, \lambda_2 > 0$, I is the identity matrix and

$$J = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

It is easy to see that $v^i = [1, 1, 1, \dots, 1]^T$ is an eigenvector corresponding to the eigenvalue $\lambda_1 + (n-2)\lambda_2$ and that $v^j = e_1 - e_j$ for $j = 2, \dots, n-2$ are eigenvectors corresponding to the eigenvalue λ_1. In particular L have $(n-2)$-linearly independent eigenvectors that correspond to strictly positive eigenvalues. We may conclude that $\det(L) > 0$. It follows that there exist a universal constant $c_L > 0$ such that

$$|L\delta| > c_L|\delta|. \tag{4.32}$$

To finish the proof we notice that

$$\begin{aligned}\sum_{j=1}^{n-2} |C_i(\delta)| = \sum_{j=1}^{n-2} |B_i(\delta) - B_i(0)| &= |\ln(|\delta|)|\,|L\delta| + O(|\delta|/\mu + \mu|\delta|\,|\ln(|\delta|)|) \\ &> \frac{1}{c}|\delta|\,|\ln(|\delta|)| + O(|\delta|/\mu + \mu|\delta|\,|\ln(|\delta|)|).\end{aligned}$$

And

$$\begin{aligned}\sum_{j=1}^{n-2} |C_i(\delta)| = \sum_{j=1}^{n-2} |B_i(\delta) - B_i(0)| &= |\ln(|\delta|)|\,|L\delta| + O(|\delta|/\mu + \mu|\delta|\,|\ln(|\delta|)|) \\ &< c|\delta|\,|\ln(|\delta|)| + O(|\delta|/\mu + \mu|\delta|\,|\ln(|\delta|)|).\end{aligned}$$

The proposition follows for μ small enough if $|\delta| \ll \mu$.

The final statement follows easily since $\lambda_1 > 0$. □

REMARK. *We will also use the notation $C(\delta) = B(\delta) - B(0)$. Notice that*

$$C(\delta) = \sum_{i-1}^{n} C_i(\delta) \tag{4.33}$$

since $\sum_{i=1}^{n} x_i^2 = 1$ on the unit sphere.

5. Proof of the main theorem

In this section we prove Theorem 1.2.

By assumption we have

$$\lim_{r_j\to 0}\frac{u(r_jx)}{\|u(r_jx)\|_{L^2(B_1)}}=\frac{x_{n-1}^2-x_n^2}{\|x_{n-1}^2-x_n^2\|_{L^2(B_1)}}$$

for some sequence $r_j\to 0$. Therefore

$$\lim_{r_j\to 0}\frac{\Pi(u,r_j,0)}{\sup_{B_1}|\Pi(u,r_j,)\|_{L^2(B_1)}}=x_{n-1}^2-x_n^2. \tag{5.34}$$

For any $r>0$ we can define a $\delta(r)$ according to

$$\frac{\Pi(u,r,0)}{\sup_{B_1}|\Pi(u,r,0)\|_{L^2(B_1)}}=p_{\delta(r)}(x).$$

With this notation (5.34) implies that (see 4.11)

$$|\delta(r_j)|\to 0$$

so we may, by choosing j large enough, assume that $\delta(r_j)$ is as small as we need.

Also, from (3.6) and (3.5) we may deduce that

$$\sup_{B_1(0)}|\Pi(u,r_j,0)|\geq c|\ln(r_j)|$$

for j large enough.

If we denote $\sup_{B_1(0)}|\Pi(u,s,0)|=\tau_s\approx c|\ln(s)|$ for s small enough and $\tau_{2^{-j}s}$ is increasing in j (Lemma 3.3). Then Proposition 3.4 implies that

$$\Pi(u,r_j/2,0)=\Pi(u,r_j,0)+\Pi(Z_{p_\delta},1/2,0)+O(\tau_{r_j}^{-\alpha}). \tag{5.35}$$

The main step in our uniqueness proof for blow-up limits is

Lemma 5.1. *Let u be a solution to* (1.1) *and assume that* $\frac{\Pi(u,r,0)}{\sup_{B_1(0)}|\Pi(u,r,0)|}=p_{\delta(r)}$ *for some $\delta(r)$ satisfying $|\delta(r)|<\kappa_0$ for some universal κ_0.*

We also assume that

$$\sum_{i=1}^{n}C_i(\delta(r))<0. \tag{5.36}$$

Then for each $\gamma<1/8$ there exist a constant C_γ such that if

$$\max(\delta_1(r),\delta_2(r),\ldots,\delta_{n-2}(r))>C_\gamma\tau_r^{-\gamma} \tag{5.37}$$

then

$$\frac{\max(\delta_1(r/2),\delta_2(r/2),\ldots,\delta_{n-2}(r/2))}{1-\tilde{\delta}(r/2)} > \frac{\max(\delta_1(r),\delta_2(r),\ldots,\delta_{n-2}(r))}{1-\tilde{\delta}(r)}. \tag{5.38}$$

Moreover, if $\delta_j < 0$ *and*

$$\delta_j \le \min(\delta_1(r/2),\delta_2(r/2),\ldots,\delta_{n-2}(r/2))$$

then it follows that

$$\frac{\delta_j(r/2)}{1-\tilde{\delta}(r/2)} < \frac{\delta_j(r)}{1-\tilde{\delta}(r)}, \tag{5.39}$$

provided that (5.37) *holds.*

REMARK. *If* $\sum_{i=1}^{n} C_i(\delta(r)) > 0$ *a similar result holds and the proof goes through with trivial changes.*

PROOF. From (5.35) and (4.14) we can conclude that the coefficient of the x_j^2-term in $\Pi(u,r/2,0)$ is

$$\tau_r\delta_j(r) + K_0(n^2B_j(\delta(r)) - nB(\delta)) + O(\tau_r^{-2\gamma}). \tag{5.40}$$

Next we make the following claim

CLAIM. *For* $j = 1,\ldots,n-2$ *we have* $n^2B_j(0) - nB(0) = 0$.

PROOF OF THE CLAIM. This is easy to verify since we can calculate Z_{p_0}, and thus $B_i(0)$ explicitly (cf. [2, Lemma 4.4]):

Define $v : (0,+\infty)\times[0,+\infty) \to \mathbb{R}$ by

$$v(x_{n-1},x_n) := -4x_{n-1}x_n\log(x_{n-1}^2+x_n^2) + 2(x_{n-1}^2-x_n^2)\Big(\frac{\pi}{2} - 2\arctan\Big(\frac{x_n}{x_{n-1}}\Big)\Big) - \pi(x_{n-1}^2+x_n^2).$$

Moreover, let

$$w(x_{n-1},x_n) := \begin{cases} v(x_{n-1},x_n), & x_{n-1}x_n \ge 0,\ x_{n-1}\ne 0,\\ -v(-x_{n-1},x_n), & x_{n-1}<0,\ x_n\ge 0,\\ -v(x_{n-1},x_n), & x_{n-1}>0,\ x_n\le 0,\end{cases}$$

and define

$$\tilde{Z}_{x_{n-1}x_n}(x_{n-1},x_n) := \frac{w(x_{n-1},x_n) - \pi(x_{n-1}^2+x_n^2) + 8x_{n-1}x_n}{8\pi}.$$

In particular, $\tilde{Z}_{x_{n-1}x_n}(x_{n-1}, x_n)$ is a rotation of Z_{p_0}. It is clear that

$$\Pi(\tilde{Z}_{x_{n-1}x_n}, 1/2, 0) = \frac{\ln(2)}{\pi} x_{n-1}x_n,$$

or equivalently

$$\Pi(Z_{p_0}, 1/2, 0) = \frac{\ln(2)}{2\pi}(x_{n-1}^2 - x_n^2).$$

It follows that $n^2 B_j(0) - nB(0) = 0$ for $j = 1, \ldots, n-2$. This proves the claim.

By the definition of $C_j(\delta)$ we may thus write, for $j = 1, \ldots, n-2$, the coefficient of the x_j^2-term in $\Pi(u, r/2, 0)$ (that is equation (5.40))

$$\tau_r \delta_j(r) - K_0(n^2 C_j(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma}).$$

Similarly we can express the x_{n-1}^2 coefficient of $\Pi(u, r/2, 0)$ according to

$$\tau_r(1 - \tilde{\delta}(r)) + \frac{\ln(2)}{2\pi} - K_0(n^2 C_{n-1}(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma}).$$

The quotient of the x_j^2 and the x_{n-1}^2 coefficients of $\Pi(u, r/2, 0)$ is thus equal to

$$\frac{\tau_r \delta_j(r) - K_0(n^2 C_j(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})}{\tau_r(1 - \tilde{\delta}(r)) + \frac{\ln(2)}{2\pi} - K_0(n^2 C_{n-1}(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})}.$$

Let us first prove the Lemma under the assumption

$$\delta_j(r) = \max(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r)). \tag{5.41}$$

Then the claim of the Lemma is

$$\frac{\tau_r \delta_j(r) - K_0(n^2 C_j(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})}{\tau_r(1 - \tilde{\delta}(r)) + \frac{\ln(2)}{2\pi} - K_0(n^2 C_{n-1}(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})} > \frac{\delta_j(r)}{1 - \tilde{\delta}(r)}. \tag{5.42}$$

The inequality (5.42) hold if

$$\begin{aligned} -K_0(1 - \tilde{\delta}(r))n^2 C_j(\delta(r)) + K_0 n(1 - \tilde{\delta}(r) - \delta_j(r))C(\delta) \\ + O(|\delta|\delta_j + \tau_r^{-2\gamma}) > 0. \end{aligned} \tag{5.43}$$

From (5.41) and Proposition 4.3 we have

$$(n-1)C_j(\delta) \le \sum_{i=1}^{n-2} C_i(\delta) + O(|\delta|) = \sum_{i=1}^{n} C_i(\delta) = C(\delta)$$

where we used Lemma 4.2 in the first equality and (4.33) in the last equality. Using this and $\delta_j > 0$ in (5.43) we can deduce that the Lemma holds if

$$-K_0(1-\tilde{\delta})C_j(\delta) > O(|\delta|\delta_j + \tau_r^{-2\gamma}),$$

or equivalently if

$$-C_j(\delta) > O(\tau_r^{-2\gamma}),$$

where we used that $|C_j(\delta)| \approx |\delta|\,|\ln(|\delta|)|$.

In particular if $|\delta|$ is small and (5.41) holds then (5.38) holds if $\delta_j \geq C_\gamma \tau^{-\gamma}$. This is exactly what we wanted to prove.

Next we chose any $\delta_j < 0$ in order to prove (5.39).

Then the claim of the Lemma is

$$\frac{\tau_r\delta_j(r) - K_0(n^2C_j(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})}{\tau_r(1-\tilde{\delta}(r)) + \frac{\ln(2)}{2\pi} - K_0(n^2C_{n-1}(\delta(r)) - nC(\delta)) + O(\tau_r^{-2\gamma})} < \frac{\delta_j(r)}{1-\tilde{\delta}(r)}. \tag{5.44}$$

The inequality (5.44) hold if

$$\begin{aligned} -K_0(1-\tilde{\delta}(r))n^2C_j(\delta(r)) + K_0n(1-\tilde{\delta}(r)-\delta_j(r))C(\delta) \\ + O(|\delta|\delta_j + \tau_r^{-2\gamma}) < 0. \end{aligned} \tag{5.45}$$

We either have that

$$C(\delta) < -\tilde{C}\tau_r^{-\gamma} \tag{5.46}$$

or

$$C_j(\delta) > \tilde{C}\tau_r^{-\gamma} \tag{5.47}$$

for some universal $\tilde{C}$. This since if $\delta_k = \max(\delta_1(r/2), \delta_2(r/2), \ldots, \delta_{n-2}(r/2)) \geq C_\gamma\tau_r^{-\gamma}$ then $C_k(\delta) < -cC_\gamma\tau_r^{-\gamma}|\ln(\tau_r)|$ so if $C(\delta) \geq -C\tau_r^{-\gamma}$ then at least one of $C_l(\delta)$, for $l = 1, \ldots, n-2$, must satisfy $C_l(\delta) > cC_\gamma\tau_r^{-\gamma}|\ln(\tau_r)| \gg C_\gamma\tau_r^{-\gamma}$ since $|\delta| \ll 1$. By the monotonicity of $C_l(\delta)$ it follows that $C_j(\delta) > \tilde{C}\tau_r^{-\gamma}$.

In either case (5.46) or (5.47) it follows that (5.45) holds true. The Lemma follows. □

We may now proceed with our proof of the main Theorem. From Lemma 5.1 and (1.2) it follows that

$$|\delta(r)| \leq C\tau_r^{-\gamma}. \tag{5.48}$$

If not then we have by Lemma 5.1 that

$$\max(\delta_1(r/2), \delta_2(r/2), \ldots, \delta_{n-2}(r/2)) > \max(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r))$$

if

$$\max(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r)) > 0$$

and

$$\min(\delta_1(r/2), \delta_2(r/2), \ldots, \delta_{n-2}(r/2)) < \min(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r))$$

if

$$\min(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r)) < 0.$$

Since $\tau_{r/2^k} > \tau_{r/2^l}$ for $k > l$ we may iterate this and conclude that if (5.48) is not true then

$$\lim_{k\to\infty} \max(\delta_1(r/2^k), \delta_2(r/2^k), \ldots, \delta_{n-2}(r/2^k)) \geq \max(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r))$$

and/or

$$\lim_{k\to\infty} \min(\delta_1(r/2^k), \delta_2(r/2^k), \ldots, \delta_{n-2}(r/2^k)) \leq \min(\delta_1(r), \delta_2(r), \ldots, \delta_{n-2}(r)).$$

This would contradict (1.2).

So (5.48) has to hold. This implies in particular that

$$\left|\frac{\Pi(u,r,0)}{\sup_{B_1}|\Pi(u,r,0)|} - \frac{\Pi(u,r/2,0)}{\sup_{B_1}|\Pi(u,r/2,0)|}\right| \leq C\frac{\tau_r^{-\gamma}}{\sup_{B_1}|\Pi(u,r,0)|} \leq C\tau_r^{-1-\gamma}.$$

We may iterate and conclude that

$$\left|\frac{\Pi(u,r,0)}{\sup_{B_1}|\Pi(u,r,0)|} - \frac{\Pi(u,r/2^k,0)}{\sup_{B_1}|\Pi(u,r/2^k,0)|}\right| \leq C\sum_{j=1}^{k} C\tau_{r/2^k}^{-1-\gamma} \tag{5.49}$$

$$\leq C\sum_{j=1}^{k}(k\ln(2) + \ln(1/r))^{-1-\gamma}$$

since $\tau_r > c|\ln(r)|$. Since $\gamma > 0$ it follows that (5.49) is convergent and we may directly conclude that

$$\lim_{r\to 0}\frac{u(rx)}{\|u(rx)\|_{L^2(B_1)}}$$

exists. The first claim (1.3) of Theorem 1.2 follows.

That

$$S \cap B_{r_0}(0) \cap \left\{x; \sum_{i=1}^{n-2} x_i^2 \leq \eta(x_{n_1}^2 + x_n^2)\right\}$$

consists of two C^1 manifolds intersection at right angles at the origin is now standard (see Corollary 9.2 or in [3]).

To prove that

$$S_{n-2} \cap B_{r_0}(0)$$

is contained in a C^1 manifold of dimension $(n-2)$ for some small r_0 we may proceed as in Theorem 12.2 in [3]. This proves Theorem 1.2.

References

[1] A. Ambrosetti - M. Struwe, *Existence of steady vortex rings in an ideal fluid*, Arch. Rational Mech. Anal., 108(2):97–109, 1989.

[2] J. Andersson - H. Shahgholian - S. G. Weiss, *Uniform regularity close to cross singularities in an unstable free boundary problem*, Comm. Math. Phys., 2010.

[3] J. Andersson - H. Shahgholian - S. G. Weiss, *On the singularities of a free boundary through fourier expansion*, Inventiones Mathematicae, 187:535–587, 2012. 10.007/s00222-011-0336-5.

[4] J. Andersson - S. G. Weiss, *Cross-shaped and degenerate singularities in an unstable elliptic free boundary problem*, J. Differential Equations, 228(2):633–640, 2006.

[5] I. Blank, *Eliminating mixed asymptotics in obstacle type free boundary problems*, Comm. Partial Differential Equations, 29(7–8):1167–1186, 2004.

[6] E. Bombieri - E. De Giorgi - E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math., 7:243–268, 1969.

[7] S. Chanillo - D. Grieser - M. Imai - K. Kurata - I. Ohnishi, *Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes*, Comm. Math. Phys., 214(2):315–337, 2000.

[8] S. Chanillo - D. Grieser - K. Kurata, *The free boundary problem in the optimization of composite membranes*, In Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999), volume 268 of Contemp. Math., pages 61–81. Amer. Math. Soc., Providence, RI, 2000.

[9] S. Chanillo - C. E. Kenig, *Weak uniqueness and partial regularity for the composite membrane problem*, J. Eur. Math. Soc. (JEMS), 10(3):705–737, 2008.

[10] S. Chanillo - C. E. Kenig - T. To, *Regularity of the minimizers in the composite membrane problem in* $\mathbb{R}^2$, J. Funct. Anal., 255(9):2299–2320, 2008.

[11] J. I. Diaz - S. Shmarev, *Lagrangian approach to the study of level sets: application to a free boundary problem in climatology*, Arch. Ration. Mech. Anal., 194(1):75–103, 2009.

[12] L. Karp - A. S. Margulis, *Newtonian potential theory for unbounded sources and applications to free boundary problems*, J. Anal. Math., 70:1–63, 1996.

[13] O. Kowalski - D. Preiss, *Besicovitch-type properties of measures and submanifolds*, J. Reine Angew. Math., 379:115–151, 1987.

[14] R. Monneau - G. S. Weiss, *An unstable elliptic free boundary problem arising in solid combustion*, Duke Math. J., 136(2):321–341, 2007.

[15] H. Shahgholian, *The singular set for the composite membrane problem*, Comm. Math. Phys., 271(1):93–101, 2007.

[16] L. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Based on lecture notes by Norbert Hungerbühler.

[17] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2), 88:62–105, 1968.

[18] B. White, *Tangent cones to two-dimensional area-minimizing integral currents are unique*, Duke Math. J., 50(1):143–160, 1983.

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