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Algebraic Geometry — *Chern classes of rank two globally generated vector bundles on* $\mathbb{P}^2$, by PHILIPPE ELLIA, communicated on 9 November 2012*.

ABSTRACT. — We determine the Chern classes of globally generated rank two vector bundles on the projective plane.

KEY WORDS: Chern classes, rank two vector bundles, globally generated, projective plane.

MATHEMATICS SUBJECT CLASSIFICATION: 14F99, 14J99.

INTRODUCTION

Vector bundles generated by global sections come up in a variety of problems in projective algebraic geometry. In this paper we consider the following question: *which are the possible Chern classes of rank two globally generated vector bundles on* $\mathbb{P}^2$*?* (Here $\mathbb{P}^2 = \mathbb{P}^2_k$ with k algebraically closed, of charateristic zero.)

Clearly these Chern classes have to be positive. Naively one may think that this is the only restriction. A closer inspection shows that this is not true: since we are on $\mathbb{P}^2$, the construction of rank two vector bundles starting from codimension two, locally complete intersection subschemes is subject to the Cayley-Bacharach condition (see Section 2). So if we have an exact sequence $0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$, with F a rank two vector bundle and $Y \subset \mathbb{P}^2$ of codimension two, then Y satisfies Cayley-Bacharach for $c - 3$.

Now F is globally generated if and only if $\mathcal{I}_Y(c)$ is. If Y is contained in a *smooth* curve, T, of degree d, we have $0 \to \mathcal{O}(-d + c) \to \mathcal{I}_Y(c) \to \mathcal{I}_{Y,T}(c) \to 0$ and we see that, if $c \geq d$, $\mathcal{I}_Y(c)$ is globally generated if and only if the line bundle $\mathcal{L} = \mathcal{I}_{Y,T}(c)$ on T is globally generated. But there are gaps in the degrees of globally generated line bundles on a smooth plane curve of degree d (it is classically known that no such bundle exists if $d \geq 3$ and $1 \leq \deg \mathcal{L} \leq d - 2$). A remarkable theorem due to Greco-Raciti and Coppens ([5], [2] and Section 3) gives the exact list of gaps.

This is another obstruction, at least if Y lies on a *smooth* curve, T, of low degree with respect with $c = c_1(F)$ (in this case F tends to be not stable). The problem then is to have such a curve for *every* vector bundle with fixed Chern classes and then, to treat the case where T is not smooth. The first problem is solved in the *necessarily unstable* range ($\Delta(F) = c_1^2 - 4c_2 > 0$) (see Section 3). In the stable range there are no obstructions, this was already known to Le

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* Proposed by F. Catanese.

Potier (see [9]). For the second problem we use the following remark: if a line bundle $\mathcal{O}_T(Z)$ on a smooth plane curve of degree d is globally generated, then Z satisfies Cayley-Bacharach for $d-3$. Working with the minimal section of F we are able to have a similar statement even if T is singular (see 4.9). Finally with a slight modification of Theorem 3.1 in [5] we are able to show the existence of gaps.

To state our result we need some notations. Let $c > 0$ be an integer. Let's say that (c, y) is effective if there exists a globally generated rank two vector bundle on $\mathbb{P}^2$, F, with $c_1(F) = c$, $c_2(F) = y$. It is easy to see (cf Section 1) that it must be $0 \le y \le c^2$ and that (c, y) is effective if and only if $(c, c^2 - y)$ is. So we may assume $y \le c^2/2$. For every integer t, $2 \le t \le c/2$, let $G_t(0) = [c(t-1)+1, t(c-t)-1]$ (we use the convention that if $b < a$, then $[a, b] = \emptyset$). For every integer t, $4 \le t \le c/2$, denote by t_0 the integral part of $\sqrt{t-3}$, then for every integer a such that $1 \le a \le t_0$ define $G_t(a) = [(t-1)(c-a)+a^2+1, (t-1)(c-a+1)-1]$. Finally let

$$G_t = \bigcup_{a=0}^{t_0} G_t(a) \quad \text{and} \quad G = \bigcup_{t=2}^{c/2} G_t.$$

Then we have:

THEOREM 0.1. *Let $c > 0$ be an integer. There exists a globally generated rank two vector bundle on $\mathbb{P}^2$ with Chern classes $c_1 = c$, $c_2 = y$ if and only if one of the following occurs:*

(1) $y = 0$ *or* $c - 1 \le y < c^2/4$ *and* $y \notin G$
(2) $c^2/4 \le y \le 3c^2/4$
(3) $3c^2/4 < y \le c^2 - c + 1$ *and* $c^2 - y \notin G$ *or* $y = c^2$.

Although quite awful to state, this result is quite natural (see Section 3). As a by-product we get (Section 6) all the possible "bi-degrees" for generically injective morphisms from $\mathbb{P}^2$ to the Grassmannian $G(1, 3)$ (or more generally to a Grassmannian of lines). To conclude let's mention that some partial results on this problem can be found in [4].

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1. General facts and a result of Le Potier for stable bundles

Let F be a rank two globally generated vector bundle on $\mathbb{P}^2$ with Chern classes $c_1(F) =: c$, $c_2(F) =: y$. Since the restriction F_L to a line is globally generated, we get $c \ge 0$. A general section of F yields:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$$

where $Y \subset \mathbb{P}^2$ is a smooth set of y distinct points (cf [8], 1.4) or is empty. In the first case $y > 0$, in the second case $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$ and $y = 0$. In any case the Chern classes of a globally generated rank two vector bundle are positive.

Also observe ($Y \neq \emptyset$) that $\mathcal{I}_Y(c)$ is globally generated (in fact F globally generated $\Leftrightarrow \mathcal{I}_Y(c)$ is globally generated). This implies by Bertini's theorem that a general curve of degree c containing Y is smooth (hence irreducible).

Since $rk(F) + \dim(\mathbb{P}^2) = 4$, F can be generated by $V \subset H^0(F)$ with $\dim V = 4$ and we get:

$$0 \to E^* \to V \otimes \mathcal{O} \to F \to 0.$$

It follows that E is a rank two, globally generated vector bundle with Chern classes: $c_1(E) = c$, $c_2(E) = c^2 - y$. We will say that E is the *G-dual bundle* of F. Since a globally generated rank two bundle has positive Chern classes we get: $0 \le y \le c^2$, $c \ge 0$.

Definition 1.1. *We will say that (c, y) is effective if there exist a globally generated rank two vector bundle on $\mathbb{P}^2$ with $c_1 = c$ and $c_2 = y$. A non effective (c, y) will also be called a gap.*

Remark 1.2. *By considering G-dual bundles we see that (c, y) is effective if and only if $(c, c^2 - y)$ is effective. Hence it is enough to consider the range $0 \le y \le c^2/2$.*

If $c = 0$, then $F \simeq 2.\mathcal{O}$ and $y = 0$.

If $y = c^2$ then $c_2(E) = 0$, hence $E \simeq \mathcal{O} \oplus \mathcal{O}(c)$ and:

$$0 \to \mathcal{O}(-c) \to 3.\mathcal{O} \to F \to 0.$$

Such bundles exists for any $c \ge 0$. If $y = 0$, $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$.

Definition 1.3. *If F is a rank two vector bundles on $\mathbb{P}^2$ we denote by F_{norm} the unique twist of F such that $-1 \le c_1(F_{norm}) \le 0$. The bundle F is* stable *if $h^0(F_{norm}) = 0$.*

By a result of Schwarzenberger, if F is stable with $c_1(F) = c$, $c_2(F) = y$, then $\Delta(F) := c^2 - 4y < 0$ (and $\Delta(F) \neq -4$). Moreover there exist a stable rank two vector bundle with Chern classes (c, y) if and only if $\Delta := c^2 - 4y < 0$, $\Delta \neq -4$.

Concerning stable bundles we have the following result of Le Potier [9]:

Proposition 1.4 (Le Potier). *Let $\mathcal{M}(c_1, c_2)$ denote the moduli space of stable rank two bundles with Chern classes c_1, c_2 on $\mathbb{P}^2$. There exists a non empty open subset of $\mathcal{M}(c_1, c_2)$ corresponding to globally generated bundles if and only if one of the following holds:*

(1) *$c_1 > 0$ and $\chi(c_1, c_2) \ge 4$ ($\chi(c_1, c_2) = 2 + \frac{c_1(c_1+3)}{2} - c_2$)*
(2) *$(c_1, c_2) = (1, 1)$ or $(2, 4)$.*

Using this proposition we get:

Corollary 1.5. *If* $c > 0$ *and*

$$\frac{c^2}{4} \le y \le \frac{3c^2}{4}$$

then (c, y) *is effective.*

Proof. The existence condition $(\Delta < 0, \Delta \neq -4)$ translates as: $y > c^2/4$, $y \neq c^2/4 + 1$. Condition (1) of 1.4 gives: $\frac{c(c+3)}{2} - 2 \ge y$, hence if $\frac{c(c+3)}{2} - 2 \ge y > \frac{c^2}{4}$ and $y \neq \frac{c^2}{4} + 1$, (c, y) is effective.

Let's show that $\left(c, \frac{c^2}{4}\right)$ is effective for every $c \ge 2$ (c even). Consider:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(2) \to 0$$

where Y is one point. Then F is globally generated with Chern classes $(2, 1)$. For every $m \ge 0$, $F(m)$ is globally generated with $c_1^2 = 4c_2$.

In the same way let's show that $\left(c, \frac{c^2}{4} + 1\right)$ is effective for every $c \ge 2$ (c even). This time consider:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(2) \to 0$$

where Y is a set of two points; F is globally generated with Chern classes $(2, 2)$. For every $m \ge 0$, $F(m)$ is globally generated with the desired Chern classes.

We conclude that if $\frac{c(c+3)}{2} - 2 \ge y \ge \frac{c^2}{4}$, then (c, y) is effective. By *duality*, (c, y) is effective if $\frac{3c^2}{4} \ge y \ge \frac{c(c-3)}{2} + 2$. Putting every thing together we get the result. □

Remark 1.6. *Since* $3c^2/4 > c^2/2$, *we may, by duality, concentrate on the range* $y < c^2/4$, *i.e. on not stable bundles with* $\Delta > 0$, *that's what we are going to do in the next section.*

2. Cayley-Bacharach

Definition 2.1. *Let* $Y \subset \mathbb{P}^2$ *be a locally complete intersection (l.c.i.) zero-dimensional subscheme. Let* $n \ge 1$ *be an integer. We say that* Y *satisfies Cayley-Bacharach for curves of degree* n *(CB(n)), if any curve of degree* n *containing a subscheme* $Y' \subset Y$ *of colength one (i.e. of degree* $\deg Y - 1$*), contains* Y.

Remark 2.2. *Since* Y *is l.c.i. for any* $p \in Supp(Y)$ *there exists a unique subscheme* $Y' \subset Y$ *of colength one (locally) linked to* p *in* Y. *So Def.* 2.1 *makes sense even if* Y *is non reduced.*

Let's recall the following ([6]):

PROPOSITION 2.3. *Let $Y \subset \mathbb{P}^2$ be a zero-dimensional l.c.i. subscheme. There exists an exact sequence:*

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$$

with F a rank two vector bundle if and only if Y satisfies $CB(c-3)$.

See [6] (under the assumption that Y is reduced) and [1] for the general case. The proposition gives conditions on the Chern classes of bundles having a section, in our case:

LEMMA 2.4. *Let F be a globally generated rank two vector bundle on $\mathbb{P}^2$ with $c_1(F) = c$, $c_2(F) = y$, then:*

$$c - 1 \leq y \leq c^2 - c + 1 \quad or \quad y = c^2 \quad or \quad y = 0.$$

PROOF. Since F is globally generated a general section vanishes in codimension two or doesn't vanish at all. In the second case $F \simeq 2.\mathcal{O}$ and $y = 0$. Let's assume, from now on, that a general section vanishes in codimension two. We have an exact sequence:

$$0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$$

where Y is a zero-dimensional subscheme (we may assume Y smooth) which satisfies Cayley-Bacharach condition for $c-3$.

If $c - 3 \geq y - 1$, $\forall p \in Y$ there exists a curve of degree $c-3$ containing $Y_p := Y \backslash \{p\}$ and not containing Y (consider a suitable union of lines). Since Y must satisfy the Cayley-Bacharach condition, it must be $y \geq c-1$.

Let F be a globally generated rank two vector bundle with $c_1(F) = c$, $c_2(F) = y$. Consider the *G-dual* bundle:

$$0 \to F^* \to 4.\mathcal{O} \to E \to 0$$

then E is a rank two, globally generated, vector bundle with $c_1(E) = c$, $c_2(E) = c^2 - y$. By the previous part: $c_2(E) = 0$ (i.e. $y = c^2$) or $c^2 - y = c_2(E) \geq c_1(E) - 1 = c - 1$. So $c^2 - c + 1 \geq y$. □

REMARK 2.5. *It is easy to check that for $1 \leq c \leq 3$, every value of y, $c - 1 \leq y \leq c^2 - c + 1$ is effective (take $Y \subset \mathbb{P}^2$ of maximal rank with $c - 1 \leq y \leq c^2/2$ and use Castelnuovo-Mumford's lemma to show that $\mathcal{I}_Y(c)$ is globally generated). In fact gaps occur only for $c \geq 6$. In the sequel we will assume that $c \geq 4$.*

3. THE STATEMENT

From now on we may restrict our attention to the range: $c - 1 \leq y < c^2/4$ (1.6, 2.4) for $c \geq 4$ (2.5). In this range $\Delta(F) = c^2 - 4y > 0$, hence F is *necessarily unstable* (i.e. not semi-stable). In particular, if c is even: $h^0\big(F\big(-\frac{c}{2}\big)\big) = h^0\big(\mathcal{I}_Y\big(\frac{c}{2}\big)\big) \neq 0$

(resp. $h^0\big(F\big(-\frac{(c+1)}{2}\big)\big) = h^0\big(\mathscr{I}_Y\big(\frac{c-1}{2}\big)\big) \neq 0$, if c is odd). So Y is forced to lie on a curve of relatively low degree. In fact something more precise can be said, for this we need the following elementary remark:

Lemma 3.1 (The trick). *Let F be a rank two vector bundle on $\mathbb{P}^2$ with $h^0(F) \neq 0$. If $c_2(F) < 0$, then $h^0(F(-1)) \neq 0$.*

Proof. A non-zero section of F cannot vanish in codimension two (we would have $c_2 > 0$), nor can the section be nowhere non-zero (F would split as $F \simeq \mathcal{O} \oplus \mathcal{O}(c)$, hence $c_2(F) = 0$). It follows that any section vanishes along a divisor. By dividing by the equation of this divisor we get $h^0(F(-1)) \neq 0$. □

Actually this works also on $\mathbb{P}^n$, $n \geq 2$.

For $2 \leq t \leq c/2$ ($c \geq 4$) we define:

$$\bar{A}_t := [(t-1)(c-t+1), t(c-t)] = [(t-1)c - (t-1)^2, (t-1)c - (t^2 - c)].$$

The ranges $\bar{A}_t$ cover $\big[c-1, \frac{c^2}{4}\big[$, the interval we are interested in. From our point of view we may concentrate on the interior points of $\bar{A}_t$. Indeed if $y = ab$, with $a + b = c$, we may take $F \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$. So we define:

$$A_t = \,](t-1)(c-t+1), t(c-t)[, \quad 2 \leq t \leq c/2.$$

Lemma 3.2. *If $y \in A_t$, and if Y is the zero-locus of a section of F, a rank two vector bundle with Chern classes (c, y), then $h^0(\mathscr{I}_Y(t-1)) \neq 0$.*

Proof. We have an exact sequence $0 \to \mathcal{O} \to F \to \mathscr{I}_Y(c) \to 0$. Now $c_2(F(-(c-t)) = (-c+t)t + y$. By our assumptions, $y < t(c-t)$, hence $c_2(F(-c+t)) < 0$. Looking at the graph of $c_2(F(x)) = x^2 + cx + y$, we see that $c_2(F(x)) < 0$ for $(-c - \sqrt{\Delta(F)})/2 < x \leq -c/2$. Since $c_2(F(-c+t)) < 0$, $-c + t < -c/2$ and $h^0(F(-c/2)) \neq 0$, by induction, using Lemma 3.1, we conclude that $h^0(F(-c+t-1)) = h^0(\mathscr{I}_Y(t-1)) \neq 0$. □

So if $y \in A_t$, Y is forced to lie on a degree $(t-1)$ curve (but not on a curve of degree $t-2$). If general principles are respected we may think that if $y \in A_t$, $Y \subset T$, where T is a *smooth* curve of degree $t-1$ and that $h^0(\mathscr{I}_Y(t-2)) = 0$. If this is the case we have an exact sequence:

$$0 \to \mathcal{O}(-t+1) \to \mathscr{I}_Y \to \mathscr{I}_{Y,T} \to 0$$

twisting by $\mathcal{O}_T(c)$:

$$0 \to \mathcal{O}(c-t+1) \to \mathscr{I}_Y(c) \to \mathcal{O}_T(c-Y) \to 0.$$

Since $c - t + 1 > 0$ (because $c \geq 2t$), we see that: $\mathscr{I}_Y(c)$ is globally generated if and only if $\mathcal{O}_T(c-Y)$ is globally generated. The line bundle $\mathscr{L} = \mathcal{O}_T(c-Y)$

has degree $l := c(t-1) - y$. So the question is: for which l does there exists a degree l line bundle on T generated by global sections? This is, by its own, a quite natural problem which, strangely enough, has been solved only recently ([5], [2]). First a definition:

Definition 3.3. *Let C be a smooth irreducible curve. The Lűroth semi-group of C, $LS(C)$, is the semi-group of nonnegative integers which are degrees of rational functions on C. In other words: $LS(C) = \{n \in \mathbb{N} \mid \exists \mathscr{L}$, of degree n, such that $\mathscr{L}$ is globally generated$\}$.*

Then we have:

Theorem 3.4 (Greco-Raciti-Coppens). *If C is a smooth plane curve of degree $d \geq 3$, then*

$$LS(C) = LS(d) := \mathbb{N} \Big\backslash \bigcup_{a=1}^{n_0} [(a-1)d + 1, a(d-a) - 1]$$

where n_0 is the integral part of $\sqrt{d-2}$.

Of course $LS(1) = LS(2) = \mathbb{N}$. We observe that $LS(C)$ doesn't depend on C but only on its degree.

Going back to our problem we see that if $c(t-1) - y \notin LS(t-1)$, then $\mathscr{L} = \mathcal{O}_T(c - Y)$ can't be globally generated and the same happens to $\mathscr{I}_Y(c)$. In conclusion if $c(t-1) - y \in \bigcup_{a=1}^{n_0} [(a-1)(t-1) + 1, a(t-1-a) - 1]$, or if $c(t-1) - y < 0$, under our assumptions, (c, y) is not effective. The assumption is that the *unique* curve of degree $t-1$ containing Y is *smooth*. (Observe that $\deg \mathcal{O}_T(t-1-Y) < 0$, hence $h^0(\mathscr{I}_Y(t-1)) = 1$.)

Our theorem says that general principles are indeed respected. In order to have a more manageable statement let's introduce some notations:

Definition 3.5. *Fix an integer $c \geq 4$. An integer $y \in A_t$ for some $2 \leq t \leq c/2$, will be said to be admissible if $c(t-1) - y \in LS(t-1)$. If $c(t-1) - y \notin LS(t-1)$, y will be said to be* non-admissible.

Observe that $y \in A_t$ is non-admissible if and only if: $y \in G_t(0) = [c(t-1)+1, t(c-t)-1]$ (this corresponds to $c(t-1) - y < 0$), or $y \in G_t(a) = [(t-1)(c-a) + a^2 + 1, (t-1)(c-a+1) - 1]$ for some $a \geq 1$ such that $a^2 + 2 \leq t-1$ (i.e. $a \leq t_0$).

In order to prove Theorem 0.1 it remains to show:

Theorem 3.6. *For any $c \geq 4$ and for any $y \in A_t$ for some $2 \leq t \leq c/2$, (c, y) is effective if and only if y is admissible.*

The proof splits into two parts:

(1) (*Gaps*) If $c(t-1) - y \notin LS(t-1)$, one has to prove that (c, y) is not effective. This is clear if Y lies on a smooth curve, T, of degree $t-1$, but there is no reason for this to be true and the problem is when T is singular.
(2) (*Existence*) If $c(t-1) - y \in LS(t-1)$, one knows that there exists $\mathscr{L}$ globally generated, of degree $c(t-1) - y$ on a smooth curve, T, of degree $t-1$. The problem is to find such an $\mathscr{L}$ such that $\mathscr{M} := \mathcal{O}_T(c) \otimes \mathscr{L}^*$ has a section vanishing along a Y satisfying the Cayley-Bacharach condition for $(c-3)$.

4. The proof (gaps)

In this section we fix an integer $c \geq 4$ and prove that non-admissible $y \in A_t$, $2 \leq t \leq c/2$ are gaps. For this we will assume that such a y is effective and will derive a contradiction. From 3.2 we know that $h^0(\mathscr{I}_Y(t-1)) \neq 0$. The first task is to show that under our assumption (y not-admissible), $h^0(\mathscr{I}_Y(t-2)) = 0$ (see 4.3); this will imply that $F(-c+t-1)$ has a section vanishing in codimension two.

To begin with let's observe that non-admissible $y \in A_t$ may occur only when t is small with respect to c.

Lemma 4.1. *Assume $c \geq 4$. If $t > \frac{2\sqrt{3}}{3}\sqrt{c-2}$, then every $y \in A_t$ is admissible.*

Proof. Recall (see 3.5) that $y \in A_t$, $2 \leq t \leq c/2$, is non admissible if and only if $y \in G_t(a)$ for some a, $0 \leq a \leq t_0$.

We have $G_t(0) \neq \emptyset \Leftrightarrow t(c-t) - 1 \geq c(t-1) + 1 \Leftrightarrow t \leq \sqrt{c-2}$.

For $a \geq 1$, $G_t(a) \cap A_t \neq \emptyset \Rightarrow (t-1)(c-a) + a^2 + 1 < t(c-t)$. This is equivalent to: $a^2 - at + t^2 - c + a + 1 < 0$ $(*)$. The discriminant of this equation in a is $\Delta = -3t^2 + 4(c-a-1)$ and we must have $\Delta \geq 0$, i.e. $\frac{2\sqrt{3}}{3}\sqrt{c-2} \geq t$. □

Let's get rid of the $y's$ in $G_t(0)$:

Lemma 4.2. *If $y \in A_t$ is non-admissible and effective, then $y \in G_t(a)$ for some a, $1 \leq a \leq \sqrt{t-3}$.*

Proof. We have to show that if $c(t-1) - y < 0$ and $y \in A_t$, then y is not effective. By 3.2 $h^0(\mathscr{I}_Y(t-1)) \neq 0$. If y is effective then $\mathscr{I}_Y(c)$ is globally generated and Y is contained in a complete intersection of type $(t-1, c)$, hence $\deg Y = y \leq c(t-1)$: contradiction. □

Now we show that if y is non-admissible and effective, then $h^0(\mathscr{I}_Y(t-1)) = 1$:

Lemma 4.3. *Let $c \geq 4$ and assume $y \in A_t$ for some t, $2 \leq t \leq c/2$. Assume furthermore that y is non-admissible and effective i.e.:*

$$y = (t-1)(c-a) + \alpha, \quad a^2 + 1 \leq \alpha \leq t-2$$

for a given a such that $t-1 \geq a^2 + 2$. Under these assumptions, $h^0(\mathscr{I}_Y(t-1)) = 1$.

PROOF. If $h^0(\mathcal{I}_Y(t-2)) \neq 0$, then $y \leq c(t-2)$ (the general $F_c \in H^0(\mathcal{I}_Y(c))$ is integral since $\mathcal{I}_Y(c)$ is globally generated. Moreover $t-1 < c$ so $F_c \neq T$). It follows that:

$$y = (t-1)(c-a) + \alpha \leq c(t-2) = c(t-1) - c.$$

This yields $a(t-1) \geq c + \alpha$. We have $c + \alpha \geq c + a^2 + 1$, hence:

$$0 \geq a^2 - a(t-1) + c + 1 \quad (*).$$

The discriminant of $(*)$ (viewed as an equation in a) is: $\Delta = (t-1)^2 - 4(c+1)$. If $\Delta < 0$, $(*)$ is never satisfied and $h^0(\mathcal{I}_Y(t-2)) = 0$. Now $\Delta < 0 \Leftrightarrow (t-1)^2 < 4(c+1)$. In our context $\Delta < 0 \Leftrightarrow t < 1 + 2\sqrt{c+1}$. In conclusion if $t < 1 + 2\sqrt{c+1}$ and if y is non-admissible, then $h^0(\mathcal{I}_Y(t-2)) = 0$.

Now by 4.1 if y is non-admissible, we have: $t \leq \frac{2\sqrt{3}}{3}\sqrt{c-2}$. Since $\frac{2\sqrt{3}}{3}\sqrt{c-2} < 1 + 2\sqrt{c+1}$, $\forall c > 0$, we are done.

Since $h^0(\mathcal{I}_Y(t-1)) \neq 0$, $F(-c+t-1)$ has a non-zero section, since $h^0(\mathcal{I}_Y(t-2)) = 0$ the section vanishes in codimension two. Hence we have:

$$0 \to \mathcal{O} \to F(-c+t-1) \to \mathcal{I}_W(-c+2t-2) \to 0$$

where $\deg W = y - (t-1)(c-t+1)$. Since $-c+2t-2 < 0$ (because $c \geq 2t$), we get $h^0(F(-c+t-1)) = 1 = h^0(\mathcal{I}_Y(t-1))$. □

NOTATIONS 4.4. *Let F be a globally generated rank two vector bundle with Chern classes (c, y). A section $s \in H^0(F)$ defines $Y_s = (s)_0$. If $y \in A_t$, $h^0(\mathcal{I}_{Y_s}(t-1)) \neq 0$, moreover if y is non-admissible $h^0(\mathcal{I}_{Y_s}(t-1)) = 1$ and there is a unique $T_s \in H^0(\mathcal{I}_{Y_s}(t-1))$. It follows that $F(-c+t-1)$ has a unique section (hence vanishing in codimension two): $0 \to \mathcal{O} \xrightarrow{u} F(-c+t-1) \to \mathcal{I}_W(-c+2t-2) \to 0$.*

LEMMA 4.5. *If $y \in A_t$ is non-admissible and effective, with notations as in 4.4:*

(1) *Y_s and W are bilinked on T_s*
(2) *The curves T_s are precisely the elements of $H^0(\mathcal{I}_W(t-1))$*
(3) *$\mathcal{I}_W(t-1)$ is globally generated, in particular for $s \in H^0(F)$ general, T_s is reduced.*

PROOF. (1) (2) We have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O} & = & \mathcal{O} & & \\
 & & & & \downarrow u & & \downarrow T_s & & \\
0 & \to & \mathcal{O}(-c+t-1) & \xrightarrow{s} & F(-c+t-1) & \to & \mathcal{I}_Y(t-1) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}(-c+t-1) & \xrightarrow{s} & \mathcal{I}_W(-c+2t-2) & \to & \mathcal{I}_{Y,T_s}(t-1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

This diagram is obtained as follows: the section T_s lifts to, u, the unique section of $F(-c+t-1)$, hence $Coker(u) \simeq \mathscr{I}_W(-c+2t-2)$, then take the first horizontal line corresponding to s and complete the full diagram.

We see that s corresponds to an element of $H^0(\mathscr{I}_W(t-1))$ and the quotient $\mathcal{O}(-t+1) \xrightarrow{s} \mathscr{I}_W$ has support on T_s and is isomorphic to $\mathscr{I}_{Y,T_s}(-t+c+1)$, hence $\mathscr{I}_{W,T_s}(-h) \simeq \mathscr{I}_{Y,T_s}$ where $h = c-t+1$. This shows that W and Y_s are bilinked on T_s. Indeed by composing with the inclusion $\mathscr{I}_{Y,T_s} \to \mathcal{O}_{T_s}$ we get an injective morphism $\varphi : \mathscr{I}_{W,T_s}(-h) \to \mathcal{O}_{T_s}$, now take a curve $C \in H^0(\mathscr{I}_W(k))$, without any irreducible component in common with T_s and take C' such that $\varphi(C) = C'$ in $H^0_*(\mathcal{O}_{T_s})$, then W is bilinked to Y by $C \cap T_s$ and $C' \cap T_s$.

This can be seen in another way: take $C' \in H^0(\mathscr{I}_Y(k))$ with no irreducible common component with T_s, the complete intersection $T_s \cap C'$ links Y to a subscheme Z. By mapping cone:

$$0 \to F^*(h-k) \to \mathcal{O}(-t+1) \oplus \mathcal{O}(-k) \oplus \mathcal{O}(-(k-h)) \xrightarrow{(T_s, C', C)} \mathscr{I}_Z \to 0.$$

Observe that C and T_s do not share any common component. Indeed if $T_s = AT'$ and $C = A\tilde{C}$, then $(C' \cap A) \subset Z$ (as schemes), because Z is the schematic intersection of T_s, C' and C. This is impossible because $C' \cap A$ contains points of Y (otherwise $Y \subset T'$ but $h^0(\mathscr{I}_Y(t-2)) = 0$). The complete intersection $T_s \cap C$ links Z to a subscheme W' and by mapping cone, we get that W' is a section of $F(-h)$. By uniqueness it follows that $W' = W$. The same argument starting from W and $T \in H^0(\mathscr{I}_W(t-1))$, instead of Y and T_s, works even better and shows that W is bilinked on T to a section Y_s of F. In conclusion the curves T_s are given by $s \wedge u$, where u is the unique section of $F(-h)$ and where $s \in H^0(F)$ vanishes in codimension two.

(3) The exact sequence $0 \to \mathcal{O}(c-t+1) \to F \to \mathscr{I}_W(t-1) \to 0$ shows that $\mathscr{I}_W(t-1)$ is globally generated, hence the general element in $H^0(\mathscr{I}_W(t-1))$ is reduced. □

Since W could well be non reduced with embedding dimension two, concerning T, this is the best we can hope. However, *and this is the point*, we may reverse the construction and start from W.

Lemma 4.6. *Let $W \subset \mathbb{P}^2$ be a zero-dimensional, locally complete intersection (l.c.i.) subscheme. Assume $\mathscr{I}_W(n)$ is globally generated, then if $T, T' \in H^0(\mathscr{I}_W(n))$ are sufficiently general, the complete intersection $T \cap T'$ links W to a smooth subscheme Z such that $W \cap Z = \emptyset$.*

Proof. If $p \in Supp(W)$, denote by W_p the subscheme of W supported at p. Since W is l.c.i, $\mathscr{I}_{W,p} = (f,g) \subset \mathcal{O}_p$. By assumption the map $H^0(\mathscr{I}_W(n)) \otimes \mathcal{O}_p \xrightarrow{ev} \mathscr{I}_{W,p}$ which takes $T \in H^0(\mathscr{I}_W(n))$ to its germ, T_p, at p, is surjective. Hence there exists T such that $T_p = f$ (resp. T' such that $T'_p = g$). It follows that in a neighbourhood of p: $T \cap T' = W_p$. If G is the Grassmannian of lines of $H^0(\mathscr{I}_W(n))$ for $\langle T, T' \rangle \in G$ the property $T \cap T' = W_p$ (in a neighbourhood of p) is open (it means that the local degree at p of $T \cap T'$ is minimum). We conclude that there exists a dense open subset, $U_p \subset G$, such that for $\langle T, T' \rangle \in U_p$, $T \cap T' = W_p$ (locally at p). If $Supp(W) = \{p_1, \dots, p_r\}$ there exists a dense open subset $U \subset U_1 \cap \cdots \cap U_r$ such that if $\langle T, T' \rangle \in U$, then $T \cap T'$ links W to Z and $W \cap Z = \emptyset$.

By Bertini's theorem the general curve $T \in H^0(\mathcal{I}_W(n))$ is smooth out of W. If $C \subset T$ is an irreducible component, the curves of $H^0(\mathcal{I}_W(n))$ cut on C, residually to $W \cap C$, a base point free linear system. By the previous part the general member, Z_C, of this linear system doesn't meet $Sing(C)$ (because $Z_C \cap W = \emptyset$), it follows, by Bertini's theorem, that Z_C is smooth. So for general $T, T' \in H^0(\mathcal{I}_W(n))$, $T \cap T'$ links W to a smooth subscheme, Z, such that $W \cap Z = \emptyset$. □

Corollary 4.7. *Let $y \in A_t$ be non-admissible. If y is effective, with notations as in* 4.4, *if $T, T' \in H^0(\mathcal{I}_W(t-1))$ are sufficiently general, then $T \cap T'$ links W to a smooth subscheme, Z, such that $W \cap Z = \emptyset$. Furthermore $\mathcal{I}_Z(c)$ is globally generated and if $S_c \in H^0(\mathcal{I}_Z(c))$ is sufficiently general, then $T \cap S_c$ links Z to a smooth subscheme Y, where Y is the zero locus of a section of F and where $Z \cap Y = \emptyset$.*

Proof. The first statement follows from 4.6. From the exact sequence

$$0 \to \mathcal{O}(c - 2t + 2) \to F(-t+1) \to \mathcal{I}_W \to 0$$

we get by mapping cone:

$$0 \to F^*(-t+1) \to \mathcal{O}(-c) \oplus 2.\mathcal{O}(-t+1) \to \mathcal{I}_Z \to 0 \quad (*)$$

which shows that $\mathcal{I}_Z(c)$ is globally generated. Since Z is smooth and contained in the smooth locus of T and since $\mathcal{I}_Z(c)$ is globally generated, if C is an irreducible component of T, the curves of $H^0(\mathcal{I}_Z(c))$ cut on C, residually to $C \cap Z$, a base point free linear system. In particular the general member, D, of this linear system doesn't meet $Sing(C)$. By Bertini's theorem we may assume D smooth. It follows that if $S_c \in H^0(\mathcal{I}_Z(c))$ is sufficiently general, $S_c \cap T$ links Z to a smooth Y such that $Z \cap Y = \emptyset$. By mapping cone, we see from $(*)$ that Y is the zero-locus of a section of F. □

The previous lemmas will allow us to apply the following (classical, I think) result:

Lemma 4.8. *Let $Y, Z \subset \mathbb{P}^2$ be two zero-dimensional subschemes linked by a complete intersection, X, of type (a, b). Assume:*

(1) $Y \cap Z = \emptyset$
(2) *$\mathcal{I}_Y(a)$ globally generated.*

Then Z satisfies Cayley-Bacharach for $(b-3)$.

Proof. Notice that Z and Y are l.c.i. Now let P be a curve of degree $b-3$ containing $Z' \subset Z$ of colength one. We have to show that P contains Z. Since $\mathcal{I}_Y(a)$

is globally generated and since $Y \cap Z = \emptyset$, there exists $F \in H^0(\mathcal{I}_Y(a))$ not passing through p. Now PF is a degree $a+b-3$ curve containing $X \setminus \{p\}$. Since complete intersections (a,b) verify Cayley-Bacharach for $a+b-3$ (the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$ exists!), PF passes through p. This implies that P contains Z. □

Gathering everything together:

Corollary 4.9. *Let $y \in A_t$ be non-admissible. If y is effective, then there exists a smooth zero-dimensional subscheme Z such that:*

(1) *Z lies on a pencil $\langle T, T' \rangle$ of curves of degree $t-1$, the base locus of this pencil is zero-dimensional.*
(2) $\deg Z = c(t-1) - y$
(3) *Z satisfies Cayley-Bacharach for $t-4$*

Proof. By 4.7 there is a Y zero-locus of a section of F which is linked by a complete intersection of type $(c, t-1)$ to a Z such that $Y \cap Z = \emptyset$. Since $\mathcal{I}_Y(c)$ is globally generated, by 4.8, Z satisfies Cayley-Bacharach for $t-4$. □

Now we conclude with:

Proposition 4.10. *Let $Z \subset \mathbb{P}^2$ be a smooth zero-dimensional subscheme contained in a curve of degree d. Let $a \geq 1$ be an integer such that $d \geq a^2 + 2$. Assume $h^0(\mathcal{I}_Z(a-1)) = 0$. If $(a-1)d + 1 \leq \deg Z \leq a(d-a) - 1$, then Z doesn't verify Cayley-Bacharach for $d-3$.*

Remark 4.11. *This proposition is Theorem* 3.1 *in* [5] *with a slight modification: we make no assumption on the degree d curve* (*which can be singular, even non reduced*)*, but we assume $h^0(\mathcal{I}_Z(a-1)) = 0$* (*which follows from Bezout if the degree d curve is integral*).

Since this proposition is a key point, and for convenience of the reader, we will prove it. We insist on the fact that the proof given is essentially the proof of Theorem 3.1 *in* [5].

Notations 4.12. *We recall that if $Z \subset \mathbb{P}^2$, the numerical character of Z, $\chi = (n_0, \dots, n_{\sigma-1})$ is a sequence of integers which encodes the Hilbert function of Z* (*see* [7])*:*

(1) *$n_0 \geq \dots \geq n_{\sigma-1} \geq \sigma$ where σ is the minimal degree of a curve containing Z*
(2) $h^1(\mathcal{I}_Z(n)) = \sum_{i=0}^{\sigma-1} [n_i - n - 1]_+ - [i - n - 1]_+$ $([x]_+ = max\{0, x\})$.
(3) *In particular* $\deg Z = \sum_{i=0}^{\sigma-1} (n_i - i)$.

The numerical character is said to be connected if $n_i \leq n_{i+1} + 1$, for all $0 \leq i < \sigma - 1$. For those more comfortable with the Hilbert function, $H(Z, -)$ and its first difference function, $\Delta(Z, i) = H(Z, i) - H(Z, i-1)$, we recall that $\Delta(i) = i + 1$

for $i < \sigma$ while $\Delta(i) = \#\{l \mid n_l \geq i+1\}$. It follows that the condition $n_{r-1} > n_r + 1$ is equivalent to $\Delta(n_r + 1) = \Delta(n_r)$. Also recall that for $0 \leq i < \sigma$, $n_i = min\{t \geq i \mid \Delta(t) \leq i\}$.

Lemma 4.13. *Let $Z \subset \mathbb{P}^2$ be a smooth zero-dimensional subscheme. Let $\chi = (n_0, \dots, n_{\sigma-1})$ be the numerical character of Z. If $n_{r-1} > n_r + 1$, then Z doesn't verify Cayley-Bacharach for every $i \geq n_r - 1$.*

Proof. It is enough to show that Z doesn't verify $CB(n_r - 1)$. By [3] there exists a curve, R, of degree r such that $R \cap Z = E'$ where $\chi(E') = (n_0, \dots, n_{r-1})$. Moreover if E'' is the residual of Z with respect to the divisor R, $\chi(E'') = (m_0, \dots, m_{\sigma-1-r})$, with $m_i = n_{r+i} - r$. It follows that $h^1(\mathcal{I}_{E''}(n_r - r - 1)) = 0$. This implies that given $X \subset E''$ of colength one, there exists a curve, P, of degree $n_r - r - 1$ passing through X but not containing E''. The curve RP has degree $n_r - 1$, passes through $Z' := E' \cup X$ but doesn't contain Z (because $R \cap Z = E'$). □

Proof of Proposition 4.10. Observe that the assumptions imply $d \geq 3$, moreover if $d = 3$, then $a = \deg Z = 1$ and the statement is clear; so we may assume $d \geq 4$.

Assume to the contrary that Z satisfies $CB(d-3)$. This implies $h^1(\mathcal{I}_Z(d-3)) \neq 0$. If $a = 1$, then $\deg Z \leq d-2$ and necessarily $h^1(\mathcal{I}_Z(d-3)) = 0$, so we may assume $a \geq 2$. Now if $h^1(\mathcal{I}_Z(d-3)) \neq 0$, then $n_0 \geq d-1$, where $\chi(Z) = (n_0, \dots, n_{\sigma-1})$ is the numerical character of Z. Since $\sigma \geq a$, $n_{a-1} \in \chi(Z)$.

We claim that $n_{a-1} < d-2$. Indeed otherwise $n_0 \geq d-1$ and $n_0 \geq \dots \geq n_{a-1} \geq d-2$ implies

$$\deg Z = \sum_{i=0}^{\sigma-1}(n_i - i) \geq \sum_{i=0}^{a-1}(n_i - i) \geq 1 + \sum_{i=0}^{a-1}(d-2-i)$$
$$= 1 + a(d-2) - \frac{a(a-1)}{2}.$$

If $a \geq 1$, then $1 + a(d-2) - \frac{a(a-1)}{2} > a(d-a) - 1 \geq \deg Z$: contradiction.

Let's show that $n_{a-1} \geq d-a$. Assume to the contrary $n_{a-1} < d-a$. Then there exists k, $1 \leq k \leq a-1$ such that $n_k \leq d-2$ and $n_{k-1} \geq d-1$ (indeed $n_0 \geq d-1$ and $n_{a-1} < d-a \leq d-2$). If $n_{k-1} \geq n_k \geq \dots \geq n_{a-1}$ is connected, then $n_{k-1} < d-a+r$ where $a = k+r$. Hence $d-a+r > n_{k-1} \geq d-1$, which implies $r \geq a$ which is impossible since $k \geq 1$. It follows that there is a gap in $n_{k-1} \geq n_k \geq \dots \geq n_{a-1}$, i.e. there exists r, $k \leq r \leq a-1$, such that $n_{r-1} > n_r + 1$. Since $d-2 \geq n_k \geq n_r$, we conclude by 4.13 that Z doesn't satisfy $CB(d-3)$: contradiction.

So far we have $d-a \leq n_{a-1} < d-2$ and $n_0 \geq d-1$. Set $n_{a-1} = d-a+r$ ($r \geq 0$). We claim that there exists k such that $n_k \geq d-1$ and $n_k \geq \dots \geq n_{a-1} = d-a+r$ is connected. Since $n_0 \geq d-1$, this follows from 4.13, otherwise Z doesn't verify $CB(d-3)$.

We have $\chi(Z) = (n_0, \dots, n_k, \dots, n_{a-1}, \dots, n_{\sigma-1})$ with $n_k \geq d-1$, $n_{a-1} = d-a+r$. Since $(n_k, \dots, n_{a-1})$ is connected and $n_k \geq d-1$, we have $n_i \geq d-1+k-i$ for $k \leq i \leq a-1$. Since $n_{a-1} = d-a+r \geq d-1+k-(a-1)$, we get $r \geq k$. It follows that:

$$\begin{aligned}\deg Z = \sum_{i=0}^{\sigma-1}(n_i - i) &= \sum_{i=0}^{k-1}(n_i - i) + \sum_{i=k}^{a-1}(n_i - i) + \sum_{i\geq a}(n_i - i)\\ &\geq \sum_{i=0}^{k-1}(d-1-i) + \sum_{i=k}^{a-1}(d-1-2i+k) + \sum_{i\geq a}(n_i - i)\\ &\geq \sum_{i=0}^{k-1}(d-1-i) + \sum_{i=k}^{a-1}(d-1-2i+k) = (+).\end{aligned}$$

We have:

$$\sum_{i=k}^{a-1}(d-1-2i+k) = (a-k)(d-a) \quad (*).$$

If $k=0$, we get $\deg Z \geq a(d-a)$, a contradiction since $\deg Z \leq a(d-a)-1$ by assumption. Assume $k>0$. Then:

$$\sum_{i=0}^{k-1}(d-1-i) = k(d-1) - \frac{k(k-1)}{2} = k\left(d-1-\frac{(k-1)}{2}\right).$$

From $(+)$ and $(*)$ we get:

$$\deg Z \geq (a-k)(d-a) + k\left(d-1-\frac{(k-1)}{2}\right) = a(d-a) + k\left(a-1-\frac{(k-1)}{2}\right)$$

and to conclude it is enough to check that $a-1 \geq (k-1)/2$. Since $r \geq k$, this will follow from $a-1 \geq (r-1)/2$. If $a < (r+1)/2$, then $n_{a-1} = d-a+r > d+a-1 \geq d$, in contradiction with $n_{a-1} < d-2$. The proof is over. □

We can now conclude and get the "gaps part" of 3.6:

Corollary 4.14. *For $c \geq 4$ let $y \in A_t$ for some t, $2 \leq t \leq c/2$. If y is non admissible, then y is a gap (i.e. (c, y) is not effective).*

Proof. Since y is non-admissible, $y \in G_t(a)$ for some $a \geq 1$ (see 4.2), or equivalently $\deg Z = c(t-1) - y \in [(a-1)(t-1)+1, a(t-1-a)-1]$ for some $a \geq 1$ such that $a^2+1 \leq t-1$. In view of 4.9 it is enough to show that Z cannot verify Cayley-Bacharach for $t-4$. For this we want to apply 4.10. The only thing we have to show is $h^0(\mathcal{I}_Z(a-1)) = 0$. Let P be a curve of degree $\sigma < a$ containing Z. If P doesn't have a common component with some curve of

$H^0(\mathcal{I}_Z(t-1))$, then $\deg Z \leq \sigma(t-1) \leq (a-1)(t-1)$. But this is impossible since $\deg Z \geq (a-1)(t-1)+1$. On the other hand Z is contained in a pencil $\langle T, T' \rangle$ of curves of degree $t-1$ and this pencil has a base locus of dimension zero (see 4.9). So we may always find a curve in $H^0(\mathcal{I}_Z(t-1))$ having no common component with P. □

5. The proof (existence)

In this section we assume that $y \in A_t$ is admissible and prove that y is indeed effective. Since y is admissible we know by [2] that there exists a smooth plane curve, T, of degree $t-1$ and a globally generated line bundle, $\mathcal{L}$, on T of degree $z := c(t-1) - y$.

Lemma 5.1. *Assume $y \in A_t$ is admissible. If T is a smooth plane curve of degree $t-1$ and if $\mathcal{L}$ is a globally generated line bundle on T with* $\deg \mathcal{L} = c(t-1) - y$, *then $\mathcal{L}^*(c)$ is non special and globally generated.*

Proof. We have $\deg \mathcal{L}^*(c) = y$. It is enough to check that $y \geq 2g_T + 1 = (t-2)(t-3)+1$. We have $y \geq (t-1)(c-t+1)+1$. Since $c \geq 2t$ it follows that $y \geq (t-1)(t+1)+1 = t^2$. □

Lemma 5.2. *Assume $y \in A_t$ is admissible. If there exists a smooth plane curve, T, of degree $t-1$, carrying a globally generated line bundle, $\mathcal{L}$, with* $\deg \mathcal{L} =$ *$c(t-1) - y$ and with $h^1(\mathcal{L}) \neq 0$, then y is effective.*

Proof. Let Z be a section of $\mathcal{L}$. If $h^1(\mathcal{L}) = h^0(\mathcal{L}^*(t-4)) \neq 0$, then Z lies on a curve, R, of degree $t-4$. Set $X = T \cap R$. By 5.1 $\mathcal{L}^*(c)$ is globally generated, so we may find a $s \in H^0(\mathcal{L}^*(c))$ such that $(s)_0 \cap X = \emptyset$. Set $Y = (s)_0$. We have $\mathcal{O}_T(c) \simeq \mathcal{O}_T(Z+Y)$ and $Y \cap Z = \emptyset$. So Y and Z are linked by a complete intersection $I = F_c \cap T$. Let's prove that Y satisfies $CB(c-3)$. First observe that there exists a degree $t-1$ curve, T', containing Z such that $T' \cap Y = \emptyset$: indeed since $Y \cap X = \emptyset$, we just take $T' = R \cup C$ where C is a suitable cubic. Now let $p \in Y$ and let P be a degree $c-3$ curve containing $Y' = Y \backslash \{p\}$. The curve $T'P$ contains $I \backslash \{p\}$ and has degree $c+t-4$. Since the complete intersection I satisfies $CB(c+t-4)$ and since $T' \cap Y = \emptyset$, $p \in P$.

It follows that we have: $0 \to \mathcal{O} \to F \to \mathcal{I}_Y(c) \to 0$ where F is a rank two vector bundle with Chern classes (c, y). Since $\mathcal{I}_{Y,T}(c) \simeq \mathcal{L}$ is globally generated, $\mathcal{I}_Y(c)$ and therefore F are globally generated. □

We need a lemma:

Lemma 5.3. *For any integer r, $1 \leq r \leq h^0(\mathcal{O}(t-1)) - 3$, there exists a smooth zero-dimensional subscheme, R, of degree r such that $\mathcal{I}_R(t-1)$ is globally generated with $h^0(\mathcal{I}_R(t-1)) \geq 3$.*

Proof. Take R of degree r, of maximal rank. If $h^0(\mathcal{O}(t-2)) \geq r$, then $h^1(\mathcal{I}_R(t-2)) = 0$ and we conclude by Castelnuovo-Mumford's lemma. Assume

$h^0(\mathcal{O}(t-2)) < r$ and take R of maximal rank and minimally generated (i.e. all the maps $\sigma(m) : H^0(\mathcal{I}_R(m)) \otimes H^0(\mathcal{O}(1)) \to H^0(\mathcal{I}_R(m+1))$ are of maximal rank). If $\sigma(t-1)$ is surjective we are done, otherwise it is injective and the minimal free resolution looks like:

$$0 \to d.\mathcal{O}(-t-1) \to b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t+1) \to \mathcal{I}_R \to 0.$$

By assumption $a \geq 3$.

Since $\mathcal{H}om(d - \mathcal{O}(-t-1), b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t+1))$ is globally generated, if $\varphi \in Hom(d.\mathcal{O}(-t-1), b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t+1))$ is sufficiently general, then $Coker(u) \simeq \mathcal{I}_R$ with R smooth of codimension two. Furthermore since $b.\mathcal{O}(1)$ is globally generated, it can be generated by $b+2$ sections; it follows that the general morphism $f : d.\mathcal{O} \to b.\mathcal{O}(1)$ is surjective ($d = a+b-1 \geq b+2$). In conclusion the general morphism $\varphi = (f, g) : d.\mathcal{O}(-t-1) \to b.\mathcal{O}(-t) \oplus a.\mathcal{O}(-t+1)$ has $Coker(\varphi) \simeq \mathcal{I}_R$ with R smooth, with the induced morphism $a.\mathcal{O}(-t+1) \to \mathcal{I}_R$ surjective. □

Proposition 5.4. *Let $c \geq 4$ be an integer. For every $2 \leq t \leq c/2$, every admissible $y \in A_t$ is effective.*

Proof. By [2] there exists a globally generated line bundle, $\mathcal{L}$, of degree $l = c(t-1) - y$ on a smooth plane curve, T, of degree $t-1$. If $h^1(\mathcal{L}) \neq 0$ we conclude with 5.2. Assume $h^1(\mathcal{L}) = 0$. Then $h^0(\mathcal{L}) = l - g_T + 1 \geq 2$ (we may assume $\mathcal{L} \neq \mathcal{O}_T$, because if $y = c(t-1)$, we are done). So $l \geq \frac{(t-2)(t-3)}{2} + 1$. Since $(t-1)(c-t+1) + 1 \leq y \leq t(c-t) - 1$, we have:

$$(t-1)^2 - 1 \geq l \geq \frac{(t-2)(t-3)}{2} + 1 \quad (*).$$

It follows that:

$$l = (t-1)^2 - r, \quad 1 \leq r \leq \frac{t(t+1)}{2} - 3 = h^0(\mathcal{O}(t-1)) - 3 \quad (**).$$

For r, $1 \leq r \leq h^0(\mathcal{O}(t-1)) - 3$, let $R \subset \mathbb{P}^2$ be a general set of r points of maximal rank, with $h^0(\mathcal{I}_R(t-1)) \geq 3$ and $\mathcal{I}_R(t-1)$ globally generated (see 5.3). It follows that R is linked by a complete intersection $T \cap T'$ of two smooth curves of degree $t-1$, to a set, Z, of $(t-1)^2 - r = l$ points. Since $\mathcal{I}_R(t-1)$ is globally generated, $\mathcal{I}_{R,T}(t-1) \simeq \mathcal{O}_T(t-1-R)$ is globally generated. Since $\mathcal{O}_T(t-1) \simeq \mathcal{O}_T(R+Z)$, we see that $\mathcal{L} := \mathcal{O}_T(Z)$ is globally generated. Moreover, by construction, $h^0(\mathcal{I}_Z(t-1)) \geq 2$. By 5.1, $\mathcal{L}^*(c)$ is globally generated so there exists $s \in H^0(\mathcal{L}^*(c))$ such that: $Y := (s)_0$ satisfies $Y \cap (T \cap T') = \emptyset$. As in the proof of 5.2, we see that Y satisfies $CB(c-3)$: indeed T' is a degree $t-1$ curve containing Z such that $T' \cap Y = \emptyset$. Since $\mathcal{I}_{Y,T}(c) \simeq \mathcal{L}$ is globally generated, we conclude that $\mathcal{I}_Y(c)$ is globally generated. □

Proposition 5.4 and Corollary 4.14 (and Remark 2.5) prove Theorem 3.6. It follows that the proof of Theorem 0.1 is complete.

6. Morphisms from $\mathbb{P}^2$ to $G(1,3)$

It is well known that finite morphisms $\varphi : \mathbb{P}^2 \to G(1,3)$ are in bijective correspondence with exact sequences of vector bundles on $\mathbb{P}^2$:

$$0 \to E^* \to 4.\mathcal{O} \to F \to 0 \quad (*)$$

where F has rank two and is globally generated with $c_1(F) = c > 0$. If φ is generically injective, then $\varphi(\mathbb{P}^2) = S \subset G \subset \mathbb{P}^5$ (the last inclusion is given by the Plűcker embedding) has degree c^2 (as a surface of $\mathbb{P}^5$) and bidegree $(y, c^2 - y)$, $y = c_2(F)$ (i.e. there are y lines of S through a general point of $\mathbb{P}^3$ and $c^2 - y$ lines of S contained in a general plane of $\mathbb{P}^3$). Theorem 0.1 gives all the possible (c, y) (but it doesn't tell if φ exists). Finally, by [10], if φ is an embedding then $(c, y) \in \{(1,0), (1,1), (2,1), (2,3)\}$.

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