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Algebraic Geometry — *Canonical vector heights on K3 surfaces—A nonexistence result*, by SHU KAWAGUCHI, communicated on 14 December 2012*.

ABSTRACT. — A. Baragar introduced a canonical vector height on a $K3$ surface X defined over a number field, and showed its existence if X has Picard rank two with infinite automorphism group. In another paper, A. Baragar and R. van Lujik performed numerical computation on certain $K3$ surfaces with Picard rank three, which strongly suggests that, in general, a canonical vector height does not exist. In this note, we prove this last assertion. We compare the set of periodic points of one automorphism with another on certain $K3$ surfaces.

KEY WORDS: Canonical height, $K3$ surface, automorphism.

MATHEMATICS SUBJECT CLASSIFICATION: 11G50, 14J50, 37P30, 37P35.

1. Introduction and the statement of the main results

Let K be a number field with a fixed algebraic closure $\overline{K}$. Let X be a $K3$ surface defined over K. We put $X_{\overline{K}} := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$. Let $\mathrm{Pic}(X_{\overline{K}})$ be the Picard group of $X_{\overline{K}}$. Since $X_{\overline{K}}$ is a $K3$ surface, $\mathrm{Pic}(X_{\overline{K}})$ is isomorphic to the Néron-Severi lattice $\mathrm{NS}(X_{\overline{K}})$ $(\subseteq H^2(X_{\overline{K}}, \mathbb{Z}))$ of $X_{\overline{K}}$, and $\mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite dimensional $\mathbb{R}$-vector space. Following Baragar [2], a *vector height* on $X(\overline{K})$ is a function

$$\mathbf{h} : X(\overline{K}) \to \mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$$

with the following two properties:

(i) For any $\sigma \in \mathrm{Aut}(X_{\overline{K}})$ and $P \in X(\overline{K})$, we have

$$\mathbf{h}(\sigma(P)) = \sigma_* \mathbf{h}(P) + \mathbf{O}(1),$$

where $\sigma_* := (\sigma^{-1})^*$ and $\mathbf{O}(1)$ denotes a bounded vector-valued function on $X(\overline{K})$.

(ii) For any divisor D of $X_{\overline{K}}$ and any Weil height h_D associated to D, we have

$$h_D(P) = \mathbf{h}(P) \cdot D + O(1),$$

where $O(1)$ denotes a bounded function on $X(\overline{K})$, and the dot $\cdot$ denotes the intersection paring on $\mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$.

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*Proposed by U. Zannier.

There always exist vector heights on X. Indeed, let $\{D_1, \dots, D_r\}$ be an $\mathbb{R}$-basis of $\mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\{D_1^*, \dots, D_r^*\}$ be the dual basis of $\{D_1, \dots, D_r\}$ with respect to the intersection paring of $\mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then the assignment of $P \in X(\overline{K})$ to $\mathbf{h}(P) := \sum_{i=1}^{r} h_{D_i^*}(P) D_i$ gives a vector height on X.

A vector height $\hat{\mathbf{h}} : X(\overline{K}) \to \mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$ is called a *canonical vector height* (over $\overline{K}$) if (i) is replaced by the stronger condition

$$\hat{\mathbf{h}}(\sigma(P)) = \sigma_* \hat{\mathbf{h}}(P) \tag{1.1}$$

for any $\sigma \in \mathrm{Aut}(X_{\overline{K}})$ and $P \in X(\overline{K})$. We say that a vector height $\hat{\mathbf{h}}$ is a *canonical vector height over* K if it satisfies (1.1) for any $\sigma \in \mathrm{Aut}(X)$ and $P \in X(\overline{K})$.

In [2], Baragar showed that, if X is a $K3$ surface with Picard rank 2 and $\mathrm{Aut}(X_{\overline{K}})$ is infinite, then a canonical vector height exists on $X(\overline{K})$. In [4] (see also [3]), Baragar and van Lujik performed numerical computation which strongly suggests that, in general, a canonical vector height does not exist. To be precise, Baragar and van Lujik considered a $K3$ surface Y over $\mathbb{Q}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by an explicit equation of multi-degree $(2, 2, 2)$, and showed that $Y_{\overline{\mathbb{Q}}}$ has Picard rank 3. Then for a suitable $P \in Y(\mathbb{Q})$, they computed of $\hat{\mathbf{h}}(P)$ in two ways and obtained a result which strongly suggests that a canonical vector height over $\mathbb{Q}$ does not exist on Y.

To the author's knowledge, there has been no proven $K3$ surface on which there does not exist a canonical vector height. The purpose of this note is to give such a $K3$ surface.

Let X be a $K3$ surface given by a $(2, 2, 2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over a number field K. For $1 \le i < j \le 3$, let $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the (i, j)-th factor. Since p_{ij} is generically a double cover, p_{ij} induces an involution $\sigma_k : X \to X$, where $\{i, j, k\} = \{1, 2, 3\}$. We put $\rho = \sigma_3 \circ \sigma_2 \circ \sigma_1$, and let $\mathrm{Per}(\rho)$ be the set of $\overline{K}$-valued periodic points of ρ. Let $\mathscr{A}$ be the subgroup of $\mathrm{Aut}(X_{\overline{K}})$ generated by σ_1, σ_2 and σ_3:

$$\mathscr{A} = \langle \sigma_1, \sigma_2, \sigma_3 \rangle. \tag{1.2}$$

We put

$$\mathscr{Q} = \{P \in X(\overline{K}) \mid \{\alpha(P) \mid \alpha \in \mathscr{A}\} \text{ is a finite set}\}. \tag{1.3}$$

Note that we have $\mathscr{Q} \subseteq \mathrm{Per}(\rho)$.

THEOREM 1.1. *Let X be $K3$ surface given by a $(2, 2, 2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over a number field K. Assume that p_{ij} is a finite morphism for any $i < j$ and $\mathscr{Q} \neq \mathrm{Per}(\rho)$. Then there does not exist a canonical vector height on X.*

We note that it is plausible that the condition $\mathscr{Q} \neq \mathrm{Per}(\rho)$ is always satisfied. (Indeed, $\mathrm{Per}(\rho)$ is an infinite set by the holomorphic Lefschetz fixed point formula. On the other hand, it might be hoped that $\mathscr{Q}$ is a finite set. See Remark 3.2 and Question 3.3.) For an explicitly given equation, one can often check that this assumption is satisfied as follows.

Corollary 1.2. *Let X be a $K3$ surface over $\mathbb{Q}(\sqrt{-2}, \sqrt{3})$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation*

$$x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 + 2x^2 + 2y^2 + 2z^2 + (2\sqrt{3}+2)x + \frac{5}{2}\sqrt{-2}y + 3z - 2\sqrt{3} - 4 = 0 \tag{1.4}$$

in the affine part $\mathbb{A}^3$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $P = (1, 0, 0)$ belongs to $\mathrm{Per}(\rho) \backslash \mathcal{Q}$, and there does not exist a canonical vector height on X. More strongly, there does not exist a canonical vector height over $\mathbb{Q}(\sqrt{-2}, \sqrt{3})$ on X.

In fact, we will consider a family of projective surfaces defined by

$$x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 + 2x^2 + 2y^2 + 2z^2 + ax + by + cz + d = 0$$

with $a, b, c, d \in \overline{\mathbb{Q}}$ and $abc \neq 0$. By suitably varying a, b, c, d, we can produce similar examples as Corollary 1.2.

The organization of this paper is as follows. In Sect. 2, we briefly review some geometric and arithmetical properties of X. In Sect. 3, we consider two automorphisms and show that, under the assumption of Theorem 1.1, the sets of periodic points of these automorphisms do not coincide with each other. Then we prove Theorem 1.1. We also give a criterion that $P \in X(\overline{K})$ does not belong to $\mathcal{Q}$ (Proposition 3.5 and Proposition 4.1), which might be of interest in itself. In Sect. 4, we prove Corollary 1.2. For the proof of Theorem 1.1, we use properties of canonical height functions associated to automorphisms of positive topological entropy (cf. [14, 1, 5, 11]). For the proof of Corollary 1.2, we use properties of canonical height functions associated to several morphisms (cf. [10]).

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2. Preliminaries

Let X be $K3$ surface given by a $(2, 2, 2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In this section, we briefly review some geometric and arithmetical properties of X. We refer the reader to [3, 5, 13, 15] for details.

Geometry over $\mathbb{C}$. Let X be $K3$ surface given by a $(2, 2, 2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over $\mathbb{C}$. For $1 \le i < j \le 3$, let $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projection to the (i, j)-th factor. Since p_{ij} is generically a double cover, p_{ij} induces an involution $\sigma_k : X \to X$, where $\{i, j, k\} = \{1, 2, 3\}$. We note that, if X is sufficiently general, then $\mathrm{Aut}(X) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ (cf. [16, p. 81]).

For $1 \le k \le 3$, let $p_k : X \to \mathbb{P}^1$ be the projection to the k-th factor. We define the divisor D_k on X by $D_k = p_k^*\{\infty\}$. The intersection paring $D_i \cdot D_j$ is given by

$$(D_i \cdot D_j)_{1 \le i,j \le 3} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Assume that p_{ij} are finite morphisms for all $i < j$, i.e., $p_{ij}^{-1}(P)$ is a finite set for any $P \in X(\mathbb{C})$. Then, for $P \in X(\mathbb{C})$, as a zero-cycle, we have the equality $p_{ij}^*(p_{ij}P) = (P) + (\sigma_k P)$ for $\{i, j, k\} = \{1, 2, 3\}$. By [5, Proposition 1.5 and its proof], we then have in $\mathrm{Pic}(X_{\bar{K}})$

$$\sigma_k^* D_i = D_i, \tag{2.1}$$

$$\sigma_k^* D_j = D_j, \tag{2.2}$$

$$\sigma_k^* D_k = 2D_i + 2D_j - D_k. \tag{2.3}$$

Automorphisms over $\mathbb{C}$. Let $\sigma : X \to X$ be an automorphism over $\mathbb{C}$. Then σ induces an action $\sigma^* : \mathrm{NS}(X) \to \mathrm{NS}(X)$, and then $\sigma_{\mathbb{C}}^* : \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$.

Let λ be the spectral radius of $\sigma_{\mathbb{C}}^*$, i.e.,

$$\lambda := \max\{|\gamma| \mid \gamma \text{ is an eigenvalue of } \sigma_{\mathbb{C}}^* : \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}\}.$$

The Néron-Severi lattice $\mathrm{NS}(X)$ ($\subseteq H^2(X, \mathbb{Z})$) equipped with the intersection form has signature $(1, \operatorname{rank} \mathrm{NS}(X) - 1)$. Thus, if $\lambda > 1$, then $\sigma_{\mathbb{C}}^* : \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ has a unique eigenvalue γ with $|\gamma| > 1$, and $\gamma = \lambda$ is a real number with multiplicity 1. Then, starting from any general ample divisor A of X, $E := \lim_{n \to \infty} \frac{1}{\lambda^n} \sigma_{\mathbb{C}}^{*n}(A) \neq 0$ exists in $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Since the limit of ample divisors are nef, E is nef. The *topological entropy* of σ is equal to $\log \lambda$. To sum up, we have the following.

PROPOSITION 2.1 ([7], Theorem 2). *Let $\sigma : X \to X$ be an automorphism over $\mathbb{C}$ with positive topological entropy $\log \lambda > 0$. Then, up to a multiple of positive numbers, there exists a unique nonzero nef $\mathbb{R}$-divisor E such that $\sigma^* E = \lambda E$.*

EXAMPLE 2.2 (cf. [15], Lemma 2.1). We give an example which will be used later. We set $\rho = \sigma_3 \circ \sigma_2 \circ \sigma_1$ as in the introduction. We put

$$E^+ := \frac{1 - \sqrt{5}}{2} D_1 + D_2 + \frac{1 + \sqrt{5}}{2} D_3, \tag{2.4}$$

$$E^- := \frac{1 + \sqrt{5}}{2} D_1 + D_2 + \frac{1 - \sqrt{5}}{2} D_3. \tag{2.5}$$

Then E^+, E^- are nef. Assume that p_{ij}'s are finite. Then using (2.1)–(2.3), we have $\rho^* E^+ = (9 + 4\sqrt{5}) E^+$ and $(\rho^{-1})^* E^- = (9 + 4\sqrt{5}) E^-$. Thus ρ and ρ^{-1} are both automorphisms with positive topological entropy $\log(9 + 4\sqrt{5})$.

Similarly, we set $\tau = \sigma_1 \circ \sigma_3 \circ \sigma_2$. We put

$$F^+ := \frac{1+\sqrt{5}}{2} D_1 + \frac{1-\sqrt{5}}{2} D_2 + D_3, \tag{2.6}$$

$$F^- := \frac{1-\sqrt{5}}{2} D_1 + \frac{1+\sqrt{5}}{2} D_2 + D_3. \tag{2.7}$$

Then F^+, F^- are nef with $\tau^* F^+ = (9+4\sqrt{5})F^+$ and $(\tau^{-1})^* F^- = (9+4\sqrt{5})F^-$. Thus τ and τ^{-1} are both automorphisms with positive topological entropy $\log(9+4\sqrt{5})$.

Canonical height functions. Let X be a $K3$ surface given by a $(2,2,2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over a number field K. We fix an embedding $K \subset \mathbb{C}$, and let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. For an automorphism $\sigma : X_{\overline{K}} \to X_{\overline{K}}$, the topological entropy of σ is defined by that of $\sigma_{\mathbb{C}} : X_{\overline{K}} \times_{\mathrm{Spec}(\overline{K})} \mathrm{Spec}(\mathbb{C}) \to X_{\overline{K}} \times_{\mathrm{Spec}(\overline{K})} \mathrm{Spec}(\mathbb{C})$.

For each $\mathbb{R}$-divisor D, one can associate an absolute logarithmic Weil height function $h_D : X(\overline{K}) \to \mathbb{R}$. For a given D, h_D is determined only up to a bounded function $O(1)$ on $X(\overline{K})$. Height functions are $\mathbb{R}$-linear with respect to D, and have the functoriality $h_{\sigma^*(D)} = h_D \circ \sigma + O(1)$ for any automorphism $\sigma : X_{\overline{K}} \to X_{\overline{K}}$. We refer the reader, for example, to [8, 6] for more details on height functions.

Theorem 2.3 ([11], Theorem C). *Let $\sigma : X_{\overline{K}} \to X_{\overline{K}}$ be an automorphism of positive topological entropy* $\log \lambda > 0$. *Then we have the following.*

(1) *There exists nef $\mathbb{R}$-divisors $E^+, E^- \in \mathrm{Pic}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\sigma^*(E^+) = \lambda E^+$ and $(\sigma^{-1})^*(E^-) = \lambda E^-$.*
(2) *There exists a unique height function $\hat{h}_{\sigma,E^+}$ (resp. $\hat{h}_{\sigma^{-1},E^-}$) associated to E^+ (resp. E^-) such that $\hat{h}_{\sigma,E^+}(\sigma(P)) = \lambda \hat{h}_{\sigma,E^+}(P)$ (resp. $\hat{h}_{\sigma,E^-}(\sigma(P)) = \lambda \hat{h}_{\sigma,E^-}(P)$) for all $P \in X(\overline{K})$.*
(3) *The functions $\hat{h}_{\sigma,E^+}$, $\hat{h}_{\sigma^{-1},E^-}$ are non-negative, i.e., for any $P \in X(\overline{K})$, we have $\hat{h}_{\sigma,E^+}(P) \geq 0$ and $\hat{h}_{\sigma^{-1},E^-}(P) \geq 0$.*
(4) *Assume that $E^+ + E^-$ is $\mathbb{R}$-ample. Then*

$$\mathrm{Per}(\sigma) = \{P \in X(\overline{K}) \mid \hat{h}_{\sigma,E^+}(P) = 0\} = \{P \in X(\overline{K}) \mid \hat{h}_{\sigma^{-1},E^-}(P) = 0\}.$$

Further, for any finite extension field L of K, the set $\mathrm{Per}(\sigma) \cap X(L)$ *is a finite set.*

Remark 2.4. The height functions $\hat{h}_{\sigma,E^+}$, $\hat{h}_{\sigma^{-1},E^-}$ are called canonical height functions, and defined by

$$\hat{h}_{\sigma,E^+}(P) := \lim_{n\to\infty} \frac{1}{\lambda^n} h_{E^+}(\sigma^n(P)), \quad \hat{h}_{\sigma^{-1},E^-}(P) := \lim_{n\to\infty} \frac{1}{\lambda^n} h_{E^-}(\sigma^{-n}(P)),$$

where h_{E^+} and h_{E^+} are any Weil heights associated to E^+ and E^-. The construction is due to Silverman [14], who defined canonical height functions on Wehler $K3$ surfaces with the composition of two involutions. In [11], we have treated automorphisms with positive topological entropy on smooth projective surfaces, where $E^+ + E^-$ may not be $\mathbb{R}$-ample in general.

Example 2.5. Let $\rho = \sigma_3 \circ \sigma_2 \circ \sigma_1 : X \to X$ be the automorphism in Example 2.2. Then we can take $E^+, E^- \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ as in (2.4) and (2.5). Since $E^+ + E^- = D_1 + 2D_2 + D_3$ is ample, Theorem 2.3(4) holds for $\hat{h}_{\rho, E^+}$ and $\hat{h}_{\rho^{-1}, E^-}$. We have similar statements for $\tau = \sigma_1 \circ \sigma_3 \circ \sigma_2 : X \to X$. The canonical height functions $\hat{h}_{\rho, E^+}$ and $\hat{h}_{\rho^{-1}, E^-}$ are studied in [1, 5, 15].

3. Proof of Theorem 1.1

Let X be a $K3$ surface given by a $(2,2,2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over a number field K. In this section, we prove Theorem 1.1.

Similarly to ρ and τ in Example 2.2, we put $\upsilon = \sigma_2 \circ \sigma_1 \circ \sigma_3$. Then ρ, τ, υ are all automorphisms of X of positive topological entropy $\log(9 + 4\sqrt{5})$.

First we compare the sets of periodic points $\mathrm{Per}(\rho)$, $\mathrm{Per}(\tau)$, $\mathrm{Per}(\upsilon)$ of ρ, τ, υ in $X(\overline{K})$. Note that, since ρ, τ, υ has positive topological entropy, it follows from the holomorphic Lefschetz fixed point formula that $\mathrm{Per}(\rho)$, $\mathrm{Per}(\tau)$, $\mathrm{Per}(\upsilon)$ are infinite sets. Recall that $\mathcal{Q}$ is a subset of $X(\overline{K})$ defined by (1.3).

Lemma 3.1. *Assume that* $\mathcal{Q} \neq \mathrm{Per}(\rho)$. *Then we have either* $\mathrm{Per}(\rho) \neq \mathrm{Per}(\tau)$, $\mathrm{Per}(\rho) \neq \mathrm{Per}(\upsilon)$, *or* $\mathrm{Per}(\tau) \neq \mathrm{Per}(\upsilon)$.

Proof. We assume that $\mathcal{P} := \mathrm{Per}(\rho) = \mathrm{Per}(\tau) = \mathrm{Per}(\upsilon)$, and derive a contradiction. Since $\sigma_1 \rho \sigma_1^{-1} = \tau$, we have $\sigma_1(\mathrm{Per}(\rho)) = \mathrm{Per}(\tau)$. Thus $\sigma_1 \mathcal{P} = \mathcal{P}$. We similarly have $\sigma_2 \mathcal{P} = \mathcal{P}$ and $\sigma_3 \mathcal{P} = \mathcal{P}$.

Let P be any element in $\mathrm{Per}(\rho) = \mathcal{P}$. Then $\{\sigma(P) \mid \sigma \in \mathcal{A}\} \subseteq \mathcal{P} = \mathrm{Per}(\rho) = \{Q \in X(\overline{K}) \mid h_{\rho, E^+}(Q) = 0\}$. We take a finite extension field L of K such that $P \in X(L)$. Then $\{\sigma(P) \mid \sigma \in \mathcal{A}\} \subseteq \{Q \in X(L) \mid h_{\rho, E^+}(Q) = 0\}$. By Theorem 2.3(4), the right-hand side of the above equation is a finite set, and so $\{\sigma(P) \mid \sigma \in \mathcal{A}\}$ is a finite set, i.e., $P \in \mathcal{Q}$. This contradicts with the assumption $\mathcal{Q} \neq \mathrm{Per}(\rho) = \mathcal{P}$. □

Remark 3.2. Lemma 3.1 is related to so-called unlikely intersections. Let $f, g : \mathbb{P}^N \to \mathbb{P}^N$ be noninvertible morphisms defined over a number field K. Let $\mathrm{PrePer}(f)$ and $\mathrm{PrePer}(g)$ denote the sets of preperiodic points for f and g in $\mathbb{P}^N(\overline{K})$. Let $\hat{h}_f$ and $\hat{h}_g$ denote canonical height functions.

Yuan–Zhang [17] showed that $\mathrm{PrePer}(f) = \mathrm{PrePer}(g)$ if and only if $\mathrm{PrePer}(f) \cap \mathrm{PrePer}(g)$ is Zariski dense in $\mathbb{P}^N$ if and only if $\hat{h}_f = \hat{h}_g$. It is known that the identical canonical heights give quite strict restrictions on f and g. For example, if f and g are polynomials in one variable, then $\hat{h}_f = \hat{h}_g$ implies that f and g commute, except for the power maps and Chebychev maps (cf. [12]).

Let us pose a question related to this remark. If the answer to Question 3.3 is affirmative, then we will always have $\mathcal{Q} \neq \mathrm{Per}(\rho)$. One might even hope that $\mathcal{Q} \subset \mathrm{Per}(\rho) \cap \mathrm{Per}(\tau) \cap \mathrm{Per}(\upsilon)$ is a finite set.

Question 3.3.

(1) Let X be smooth projective surface over $\mathbb{C}$, and let σ and σ' be automorphisms of X with positive topological entropy. Then is it true that $\mathrm{Per}(\sigma) \cap \mathrm{Per}(\sigma')$ is Zariski dense in X if and only if $\mathrm{Per}(\sigma) = \mathrm{Per}(\sigma')$?
(2) Let X be $K3$ surface given by a $(2,2,2)$ hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over $\mathbb{C}$. let σ and σ' be automorphisms of X with positive topological entropy $\log\lambda$ and $\log\lambda'$. Assume that the one-dimensional λ-eigenspace of σ is not equal to the one-dimensional λ'-eigenspace of σ' nor the one-dimensional λ'-eigenspace of σ'^{-1}. Then is it true that $\mathrm{Per}(\sigma) \cap \mathrm{Per}(\sigma')$ is not Zariski dense in X?

We now assume that there exists a vector height $\hat{\mathbf{h}}$ on X. We show some properties of $\hat{\mathbf{h}}$.

Proposition 3.4 [3]. *Let $\sigma : X \to X$ be an automorphism of positive topological entropy* $\log\lambda > 0$. *We take a nonzero nef $\mathbb{R}$-divisor $E \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ with $\sigma^* E = \lambda E$. Then we have $\hat{\mathbf{h}}(P) \cdot E = \hat{h}_{\sigma,E}(P)$ for any $P \in X(\overline{K})$.*

Proof. This is shown in [3, §1]. For the reader's convenience, we give a quick proof using Theorem 2.3(2). We regard $\hat{\mathbf{h}} \cdot E$ as a function on $X(\overline{K})$ which maps $P \in X(\overline{K})$ to $\hat{\mathbf{h}}(P) \cdot E$. Then $\hat{\mathbf{h}} \cdot E$ is a height function associated to E, and satisfies

$$\hat{\mathbf{h}}(\sigma(P)) \cdot E = \sigma_* \hat{\mathbf{h}}(P) \cdot E = (\sigma^{-1})^* \hat{\mathbf{h}}(P) \cdot E = \hat{\mathbf{h}}(P) \cdot \sigma^* E = \lambda(\hat{\mathbf{h}}(P) \cdot E).$$

By uniqueness of $\hat{h}_{\sigma,E}$ in Theorem 2.3(2), we have $\hat{\mathbf{h}} \cdot E = \hat{h}_{\sigma,E}$. □

Proof of Theorem 1.1. Let us begin the proof of the main theorem. We suppose that there exists a canonical vector height $\hat{\mathbf{h}}$ on X, and we will derive a contradiction. By Lemma 3.1, at least two of the sets $\mathrm{Per}(\rho)$, $\mathrm{Per}(\tau)$, $\mathrm{Per}(\upsilon)$ are different. We treat the case $\mathrm{Per}(\rho) \neq \mathrm{Per}(\tau)$. The other cases can be treated similarly via the symmetry of $\sigma_1, \sigma_2, \sigma_3$.

Let $E^{\pm}, F^{\pm} \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be as in (2.4)–(2.7). Then we have

$$\frac{3+\sqrt{5}}{2} E^+ + E^- = F^+ + \frac{3+\sqrt{5}}{2} F^-.$$

Thus we get

$$\frac{3+\sqrt{5}}{2} \hat{\mathbf{h}} \cdot E^+ + \hat{\mathbf{h}} \cdot E^- = \hat{\mathbf{h}} \cdot F^+ + \frac{3+\sqrt{5}}{2} \hat{\mathbf{h}} \cdot F^-.$$

By Proposition 3.4, we have

$$\frac{3+\sqrt{5}}{2}\hat{h}_{\rho,E^+} + \hat{h}_{\rho^{-1},E^-} = \hat{h}_{\tau,F^+} + \frac{3+\sqrt{5}}{2}\hat{h}_{\tau^{-1},F^-}.$$

Since $\mathrm{Per}(\rho) \neq \mathrm{Per}(\tau)$, there exists either $P_0 \in \mathrm{Per}(\rho)\backslash\mathrm{Per}(\tau)$ or $Q_0 \in \mathrm{Per}(\tau)\backslash\mathrm{Per}(\rho)$. In the former case, it follows from Theorem 2.3(4) that $\hat{h}_{\rho,E^+}(P_0) = \hat{h}_{\rho^{-1},E^-}(P_0) = 0$, and that $\hat{h}_{\tau,F^+}(P_0) > 0$ and $\hat{h}_{\tau^{-1},F^-}(P_0) > 0$. We have

$$\begin{aligned} 0 &= \frac{3+\sqrt{5}}{2}\hat{h}_{\rho,E^+}(P_0) + \hat{h}_{\rho^{-1},E^-}(P_0) \\ &= \hat{h}_{\tau,F^+}(P_0) + \frac{3+\sqrt{5}}{2}\hat{h}_{\tau^{-1},F^-}(P_0) > 0, \end{aligned}$$

which is a contradiction. In the latter case, we similarly have the contradiction

$$\begin{aligned} 0 &< \frac{3+\sqrt{5}}{2}\hat{h}_{\rho,E^+}(Q_0) + \hat{h}_{\rho^{-1},E^-}(Q_0) \\ &= \hat{h}_{\tau,F^+}(Q_0) + \frac{3+\sqrt{5}}{2}\hat{h}_{\tau^{-1},F^-}(Q_0) = 0. \end{aligned}$$

This completes the proof of Theorem 1.1. □

In the rest of this section, we give a criterion that a point $P \in X(\overline{K})$ does not belong to $\mathcal{Q}$, which will be used in the next section. Let $h : \mathbb{P}^1(\overline{K}) \to \mathbb{R}$ denote the usual Weil height function. Since $D_k = p_k^*\{\infty\}$ for $k = 1, 2, 3$, we have $h_{D_k} = h \circ p_k + O(1)$ on $X(\overline{K})$. Note that h_{D_k} is determined only up to a bounded function. We choose a representative of h_{D_k} by setting

$$h_{D_k} := h \circ p_k. \tag{3.1}$$

By slight abuse of the notation, we write h_{D_k} for this representative. We assume that p_{ij} is a finite morphism for any $i < j$. By (2.3), there exists constants $c_k > 0$ and $c_k' > 0$ such that

$$2h_{D_i} + 2h_{D_j} + c_k \geq h_{D_k} \circ \sigma_k + h_{D_k} \geq 2h_{D_i} + 2h_{D_j} - c_k' \tag{3.2}$$

for $k = 1, 2, 3$ and $\{i, j, k\} = \{1, 2, 3\}$.

Proposition 3.5. *Let the notation and assumption be as above. Let $P_0 \in X(\overline{K})$. Then, if*

$$h_{D_1}(P_0) + h_{D_2}(P_0) + h_{D_3}(P_0) > \frac{1}{2}(c_1' + c_2' + c_3'),$$

then $P_0 \notin \mathcal{Q}$.

PROOF. We use [10]. We set $D := D_1 + D_2 + D_3$, and we set $h_D := h_{D_1} + h_{D_2} + h_{D_3}$. Recall that $\mathscr{A}$ is defined in (1.2). We put $\mathscr{A}_0 := \{\mathrm{id}\} \subseteq \mathscr{A}$, and for each $n \geq 1$

$$\mathscr{A}_n := \{\sigma_{\ell_1} \circ \sigma_{\ell_2} \circ \cdots \circ \sigma_{\ell_n} \in \mathscr{A} \mid \ell_1, \ell_2, \dots, \ell_n \in \{1, 2, 3\}\}.$$

Since $\sigma_1^* D + \sigma_2^* D + \sigma_3^* D = 5D$ in $\mathrm{Pic}(X)$, [10, Theorem 1.2.1] shows that, for any $P \in X(\overline{K})$, the limit

$$\hat{h}_{D,\mathscr{A}}(P) := \lim_{n\to\infty} \left(\frac{1}{5}\right)^n \sum_{\sigma \in \mathscr{A}_n} h_D(\sigma(P))$$

exists. In fact, by (3.2), we have

$$h_D(\sigma_1(P)) + h_D(\sigma_2(P)) + h_D(\sigma_3(P)) \geq 5h_D(P) - (c_1' + c_2' + c_3')$$

for any $P \in X(\overline{K})$. Then

$$\begin{aligned}\left(\frac{1}{5}\right)^{n+1} \sum_{\sigma \in \mathscr{A}_{n+1}} h_D(\sigma(P)) &= \left(\frac{1}{5}\right)^{n+1} \sum_{\sigma \in \mathscr{A}_n} \sum_{i=1}^{3} h_D(\sigma_i(\sigma(P))) \\ &\geq \left(\frac{1}{5}\right)^{n} \sum_{\sigma \in \mathscr{A}_n} h_D(\sigma(P)) - \frac{3^n}{5^{n+1}}(c_1' + c_2' + c_3') \\ &\geq \cdots \geq h_D(P) - \sum_{i=0}^{n} \frac{3^i}{5^{i+1}}(c_1' + c_2' + c_3').\end{aligned}$$

Since $\sum_{i=0}^{\infty} \frac{3^i}{5^{i+1}}(c_1' + c_2' + c_3') = \frac{1}{2}(c_1' + c_2' + c_3')$, letting n to the infinity, we have

$$\hat{h}_{D,\mathscr{A}}(P) \geq h_D(P) - \frac{1}{2}(c_1' + c_2' + c_3'). \tag{3.3}$$

Further, it follows from [10, Theorem 1.2.1] that, for $P \in X(\overline{K})$, we have $\hat{h}_{D,\mathscr{A}}(P) = 0$ if and only if $P \in \mathscr{Q}$. Thus, if $P \in \mathscr{Q}$, then $h_D(P) \leq \frac{1}{2}(c_1' + c_2' + c_3')$. This completes the proof. □

REMARK 3.6. In the following section, we consider a particular X and estimate c_1', c_2', c_3' in (3.2) to show that a given point Q does not belong to $\mathrm{Per}(\rho)$. The referee points out that it may be possible to show $Q \notin \mathrm{Per}(\rho)$ by reducing modulo p for various primes and use [9].

4. PROOF OF COROLLARY 1.2

In this section, we prove Corollary 1.2. We first give an explicit estimate of c_k' in (3.2). One way to give such an estimate would be to use an effective Hilbert

Nullstellensatz, but for automorphisms of X, the computation would be complicated and seems not to yield a nice estimate. For $K3$ surfaces including the one in Corollary 1.2, we can give the following good estimate.

PROPOSITION 4.1. *Let K be a number field, and let $a, b, c, d \in K$ with $abc \neq 0$. Let X be a projective surface over K in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation*

$$x^2y^2z^2 + x^2y^2 + x^2z^2 + y^2z^2 + 2x^2 + 2y^2 + 2z^2 + ax + by + cz + d = 0 \tag{4.1}$$

in the affine part $\mathbb{A}^3$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We assume that X is smooth. For $k = 1, 2, 3$, let h_{D_k} be the height function defined in (3.1). Then $p_{ij} : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is finite for any $i < j$, and we have

$$\begin{aligned} h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \\ \geq 2h_{D_2}(P) + 2h_{D_3}(P) - 2h((a : b : c : d : 1)) - \log 162, \end{aligned} \tag{4.2}$$

$$\begin{aligned} h_{D_2}(P) + h_{D_2}(\sigma_2(P)) \\ \geq 2h_{D_1}(P) + 2h_{D_3}(P) - 2h((a : b : c : d : 1)) - \log 162, \end{aligned} \tag{4.3}$$

$$\begin{aligned} h_{D_3}(P) + h_{D_3}(\sigma_3(P)) \\ \geq 2h_{D_1}(P) + 2h_{D_2}(P) - 2h((a : b : c : d : 1)) - \log 162 \end{aligned} \tag{4.4}$$

for any $P \in X(\overline{K})$.

PROOF. We first check that p_{ij} is a finite morphism. It suffices to check for p_{12}. If $x = (\alpha : 1)$, $y = (\beta : 1)$ are in the affine part, then

$$\begin{aligned} p_{12}^{-1}(((\alpha : 1), (\beta : 1))) = \{((\alpha : 1), (\beta : 1), (z_0 : z_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \mid z_0^2(\alpha^2\beta^2 + \alpha^2 + \beta^2 + 2) + cz_0z_1 \\ + (\alpha^2\beta^2 + 2\alpha^2 + 2\beta^2 + a\alpha + b\beta + d)z_1^2 = 0\}. \end{aligned}$$

Since $c \neq 0$, we see that $p_{12}^{-1}(((\alpha : 1), (\beta : 1)))$ is a finite set. If $x = (1 : 0)$ is the point at infinity, but $y = (\beta : 1)$ is in the affine part, then

$$\begin{aligned} p_{12}^{-1}&(((1 : 0), (\beta : 1))) \\ &= \{((1 : 0), (\beta : 1), (z_0 : z_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid (\beta^2 + 1)z_0^2 + (\beta^2 + 2)z_1^2 = 0\}. \end{aligned}$$

Since $\beta^2 + 1 \neq \beta^2 + 2$, we see that $p_{12}^{-1}(((1 : 0), (\beta : 1)))$ is a finite set. The case, where $x = (\alpha : 1)$ is in the affine part, but $y = (1 : 0)$ is the point at infinity, is similar. Finally, if $x = (1 : 0)$ and $y = (1 : 0)$ are the points at infinity, we have

$$p_{12}^{-1}(((1 : 0), (1 : 0))) = \{((1 : 0), (1 : 0), (z_0 : z_1)) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \mid z_0^2 + z_1^2 = 0\}.$$

Thus $p_{12}^{-1}(((1 : 0), (1 : 0))) = \{(\pm\sqrt{-1} : 1)\}$ is a finite set.

We will show (4.2). The other inequalities are proven similarly.

(Step 1). We treat the case where P and $\sigma_1(P)$ lie in the affine part $\mathbb{A}^3$. Let $P = (x, y, z)$ in the affine coordinate. We take a finite extension field L of K such that $P \in \mathbb{A}^3(L) \subseteq \mathbb{P}^3(L)$. Let M_L denote the set of places of L. For a non-Archimedean place $v \in M_L$ which lies over a prime $p \in \mathbb{Z}$, we set $n_v = [L_v : \mathbb{Q}_p]$. For such v, let $|\cdot|_v$ be the v-adic norm on L normalized as $|p|_v = p^{-1}$. If v is Archimedean, we set $n_v = 1$ if v is real, and $n_v = 2$ if v is complex. For an Archimedean place $v \in M_L$, let $|\cdot|_v$ denote the usual absolute value.

We write $\sigma_1(P) = (x', y, z) \in \mathbb{A}^3(L) \subseteq \mathbb{P}^3(L)$. If v is non-Archimedean, then $|x + x'|_v \le \max\{|x|_v, |x'|_v\}$, so that

$$\max\{1, |x|_v, |x'|_v, |xx'|_v\} \ge \max\{1, |x + x'|_v, |xx'|_v\}.$$

If v is Archimedean, then $|x + x'|_v \le 2\max\{|x|_v, |x'|_v\}$, so that

$$\max\{1, |x|_v, |x'|_v, |xx'|_v\} \ge \frac{1}{2}\max\{1, |x + x'|_v, |xx'|_v\}.$$

Then we compute

$$\begin{aligned} h_{D_1}(P) + h_{D_1}(\sigma_1(P)) &= h((x : 1)) + h((x' : 1)) \\ &= \frac{1}{[L : \mathbb{Q}]}\sum_{v \in M_L} n_v(\log\max\{1, |x|_v\} + \log\max\{1, |x'|_v\}) \\ &\ge \frac{1}{[L : \mathbb{Q}]}\sum_{v \in M_L} n_v \log\max\{1, |x + x'|_v, |xx'|_v\} - \log 2 \\ &= h((x + x' : xx' : 1)) - \log 2. \end{aligned}$$

Here, we have

$$x + x' = \frac{-a}{y^2z^2 + y^2 + z^2 + 2}, \quad xx' = \frac{y^2z^2 + 2y^2 + 2z^2 + by + cz + d}{y^2z^2 + y^2 + z^2 + 2}.$$

We thus obtain

$$\begin{aligned} (4.5) \quad & h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \\ &\ge h\Big(\Big(\frac{-a}{y^2z^2 + y^2 + z^2 + 2} : \frac{y^2z^2 + 2y^2 + 2z^2 + by + cz + d}{y^2z^2 + y^2 + z^2 + 2} : 1\Big)\Big) \\ &\quad - \log 2 \\ &= h((-a : y^2z^2 + 2y^2 + 2z^2 + by + cz + d : y^2z^2 + y^2 + z^2 + 2)) \\ &\quad - \log 2. \end{aligned}$$

On the other hand, we have

$$h_{D_2}(P) = h((y:1)), \quad h_{D_3}(P) = h((z:1)).$$

Let us compare $h((-a : y^2z^2 + 2y^2 + 2z^2 + by + cz + d : y^2z^2 + y^2 + z^2 + 2))$ and $2h((y:1)) + 2h((z:1))$. To this end, for each $v \in M_L$, we will compare

$$\max\{|a|_v, |y^2z^2 + 2y^2 + 2z^2 + by + cz + d|_v, |y^2z^2 + y^2 + z^2 + 2|_v\} \tag{4.6}$$

and

$$\max\{1, |y|_v\}^2 \cdot \max\{1, |z|_v\}^2 = \max\{1, |y^2|_v, |z^2|_v, |y^2z^2|_v\}. \tag{4.7}$$

We set

$$C_v := \max\{|a|_v, |b|_v, |c|_v, |d|_v, 1\}.$$

Case 1. Let $v \in M_L$ be a non-Archimedean place.

Subcase 1. Assume that $|y|_v > 1$ and $|z|_v > 1$. In this case,

$$(4.6) \geq |y^2z^2 + y^2 + z^2 + 2|_v = |y^2z^2|_v = (4.7).$$

Subcase 2. Assume that $|y|_v > C_v$ and $|z|_v \leq 1$. We further divide this subcase into two.

First suppose that $|z^2 + 1|_v = 1$. Then

$$(4.6) \geq |y^2z^2 + y^2 + z^2 + 2|_v = |y^2(z^2 + 1) + z^2 + 2|_v = |y^2|_v = (4.7).$$

Next suppose that $|z^2 + 1|_v < 1$. Then $|z^2 + 2|_v = \max\{|z^2 + 1|_v, |1|_v\} = 1$. Since $|y|_v > C_v$, we have $|y^2|_v > |b|_v|y|_v$, $|y^2|_v > |c|_v \geq |c|_v|z|_v$ and $|y^2|_v > |d|_v$. Thus

$$\begin{aligned}(4.6) &\geq |y^2z^2 + 2y^2 + 2z^2 + by + cz + d|_v \\ &= |y^2(z^2 + 2) + 2z^2 + by + cz + d|_v = |y^2|_v = (4.7).\end{aligned}$$

Subcase 3. Assume that $|y|_v \leq 1$ and $|z|_v > C_v$. Similar to Subcase 2, we have $(4.6) \geq (4.7)$.

Subcase 4. Assume that $|y|_v \leq C_v$ and $|z|_v \leq C_v$. In this case, $(4.7) \leq C_v^2$. We have

$$(4.6) \geq |a|_v \geq \frac{|a|_v}{C_v^2} \cdot (4.7).$$

Case 2. Let $v \in M_L$ be an Archimedean place.

Subcase 1. Assume that $|y|_v > 2$ and $|z|_v > 2$. In this case,

$$\begin{aligned}
(4.6) &\geq |y^2z^2 + y^2 + z^2 + 2|_v \\
&\geq |y^2z^2|_v - |y^2|_v - |z^2|_v - 2 \\
&= \frac{1}{4}(|y|_v^2 - 4)(|z|_v^2 - 4) + \frac{3}{8}(|y|_v^2|z|_v^2 - 16) + \frac{3}{8}|y|_v^2|z|_v^2 \\
&\geq \frac{3}{8}|y|_v^2|z|_v^2 = \frac{3}{8}\cdot(4.7).
\end{aligned}$$

Subcase 2. Assume that $|y|_v > 6C_v + 3$ and $|z|_v \leq 2$. We have either $|z^2 + 1|_v \geq \frac{1}{2}$ or $|z^2 + 2|_v \geq \frac{1}{2}$. We divide this case into two.

Suppose that $|z^2 + 1|_v \geq \frac{1}{2}$. Noting that $|y|_v > 9$, we get

$$\begin{aligned}
(4.6) &\geq |y^2z^2 + y^2 + z^2 + 2|_v = |y^2(z^2 + 1) + z^2 + 2|_v \\
&\geq \frac{1}{2}|y|_v^2 - (|z|_v^2 + 2) \geq \frac{1}{2}|y|_v^2 - 6 \geq \frac{1}{4}|y|_v^2 \geq \frac{1}{16}\cdot(4.7).
\end{aligned}$$

Next suppose that $|z^2 + 2|_v \geq \frac{1}{2}$. Then

$$\begin{aligned}
(4.6) &= |y^2z^2 + 2y^2 + 2z^2 + by + cz + d|_v \\
&= |y^2(z^2 + 2) + 2z^2 + by + cz + d|_v \\
&\geq \frac{1}{2}|y|_v^2 - C_v|y|_v - (8 + 3C_v) \\
&= \frac{1}{4}(|y|_v - 2C_v)^2 - (8 + 3C_v + C_v^2) + \frac{1}{4}|y|_v^2 \\
&\geq \frac{1}{4}(4C_v + 3)^2 - (8 + 3C_v + C_v^2) + \frac{1}{4}|y|_v^2 \\
&\geq \left(3C_v^2 + 3C_v - \frac{23}{4}\right) + \frac{1}{4}|y|_v^2 > \frac{1}{4}|y|_v^2 \geq \frac{1}{16}\cdot(4.7).
\end{aligned}$$

Subcase 3. Assume that $|y|_v \leq 2$ and $|z|_v > 6C_v + 3$. Then a similar argument as in Subcase 2 gives $(4.6) \geq \frac{1}{16}\cdot(4.7)$.

Subcase 4. Assume that $|y|_v \leq 6C_v + 3$ and $|z|_v \leq 6C_v + 3$. As in Case 1, we have $(4.7) \leq (6C_v + 3)^2$. Thus

$$(4.6) \geq |a|_v = \frac{|a|_v}{(6C_v + 3)^2}\cdot(4.7) \geq \frac{|a|_v}{(9C_v)^2}\cdot(4.7).$$

Since $\frac{|a|_v}{(9C_v)^2} \leq \frac{1}{9^2} \leq \frac{1}{16}$, we have $(4.6) \geq \frac{|a|_v}{(9C_v)^2}\cdot(4.7)$ in Case 2. Combining Cases 1 and 2, we get

$$\begin{aligned}
&h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \\
&\quad \geq \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_L} n_v \log\max\{|a|_v, |y^2z^2 + 2y^2 + 2z^2 + by + cz + d|_v, \\
&\qquad\qquad\qquad\qquad |y^2z^2 + y^2 + z^2 + 2|_v\} - \log 2 \\
&\quad \geq \frac{1}{[L:\mathbb{Q}]} \Bigg[\sum_{v \in M_L, v:\text{non-Arch}} n_v(\log|a|_v - 2\log C_v) \\
&\qquad + \log\max\{1, |y^2|_v, |z^2|_v, |y^2z^2|_v\} \\
&\qquad + \sum_{v \in M_L, v:\text{Arch}} n_v(\log|a|_v - 2\log(9C_v)) \\
&\qquad + \log\max\{1, |y^2|_v, |z^2|_v, |y^2z^2|_v\} \Bigg] - \log 2 \\
&\quad \geq \frac{1}{[L:\mathbb{Q}]} \Big(\sum_{v \in M_L} n_v \log|a|_v - \sum_{v \in M_L} n_v 2\log C_v - [L:\mathbb{Q}]4\log 3 \\
&\qquad + \sum_{v \in M_L} \log\max\{1, |y^2|_v, |z^2|_v, |y^2z^2|_v\} \Big) - \log 2 \\
&\quad = -2h((a:b:c:d:1)) - 4\log 3 + 2h_{D_2}(P) + 2h_{D_3}(P) - \log 2,
\end{aligned}$$

where we used the product formula $\prod_{v \in M_L} |a|_v^{n_v} = 1$ in the last equality. Hence we obtain (4.2) when P and $\sigma_1(P)$ are both in the affine part.

(Step 2). We treat the case where $P = (x, y, z)$ or $\sigma_1(P)$ does not lie in the affine part $\mathbb{A}^3$. We write $\sigma_1(P) = (x', y, z)$ with $x', y, z \in \mathbb{P}^1$.

Case 1. We consider the case where x or x' is the point at infinity, but $y = (y:1)$ and $z = (z:1)$ are in the affine part. In this case, we may assume that x' is the point at infinity. Then $y^2z^2 + y^2 + z^2 + 2 = 0$, and we have

$$x = -\frac{y^2z^2 + 2y^2 + 2z^2 + by + cz + d}{a}.$$

It follows that

$$\begin{aligned}
(4.8)\qquad & h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \\
&= h((x:1)) = h\Big(\Big(-\frac{y^2z^2 + 2y^2 + 2z^2 + by + cz + d}{a} : 1\Big)\Big) \\
&= h((y^2z^2 + 2y^2 + 2z^2 + by + cz + d : -a)) \\
&= h((-a : y^2z^2 + y^2 + z^2 + 2 : y^2z^2 + 2y^2 + 2z^2 + by + cz + d)) \\
&\quad > h((-a : y^2z^2 + y^2 + z^2 + 2 : y^2z^2 + 2y^2 + 2z^2 + by + cz + d)) \\
&\quad - \log 2,
\end{aligned}$$

where we use $y^2z^2 + y^2 + z^2 + 2 = 0$ in the fourth equality. The quantity (4.8) is equal to (4.5). Thus the argument in Step 1 shows that (4.2) holds in Case 1.

Case 2. We consider the case where $y = (1 : 0)$ is the point at infinity, $z = (z : 1)$ is in the affine part, and $x = (x_0 : x_1)$, $x' = (x'_0 : x'_1) \in \mathbb{P}^1$. Since $P = (x, y, z)$ is a point in X, it follows from the projectivized equation of (4.1) that

$$(z^2 + 1)x_0^2 + (z^2 + 2)x_1^2 = 0.$$

We divide this case into two subcases.

Subcase 1. Assume that $x = (x : 1)$ is in the affine part. Then $z \neq \pm\sqrt{-1}$. In this case, x and $x' = (x' : 1)$ satisfies

$$xx' = 0, \quad x + x' = \frac{z^2 + 2}{z^2 + 1}.$$

Then we have, as in (4.5),

$$\begin{aligned} h_{D_1}(P) + h_{D_1}(\sigma_1(P)) &= h((x + x' : xx' : 1)) - \log 2 \\ &= h\Big(\Big(\frac{z^2 + 2}{z^2 + 1} : 0 : 1\Big)\Big) - \log 2 \\ &= h((z^2 + 2 : z^2 + 1)) - \log 2. \end{aligned}$$

Since $h((z^2 + 2 : z^2 + 1)) \geq 2h((z : 1)) - 2\log 2$ and $h_{D_2}(P) = 0$, we obtain

$$h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \geq 2h_{D_2}(P) + 2h_{D_3}(P) - 3\log 2.$$

In particular, (4.2) holds.

Subcase 2. Assume that x is the point at infinity. Then we have $z = \pm\sqrt{-1}$. We have

$$h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \geq 02h_{D_2}(P) + h((\pm\sqrt{-1} : 0)) = 2h_{D_2}(P) + 2h_{D_3}(P).$$

In particular, (4.2) holds.

Case 3. We consider the case where $z = (1 : 0)$ is the point at infinity, $y = (y : 1)$ is in the affine part, and $x = (x_0 : x_1)$, $x' = (x'_0 : x'_1) \in \mathbb{P}^1$. Similar arguments as in Case 2 give that (4.2) holds also in Case 3.

Case 4. We consider the case where each of $y = (1 : 0)$ and $z = (1 : 0)$ is the point at infinity. Then

$$h_{D_1}(P) + h_{D_1}(\sigma_1(P)) \geq 0 = 2h_{D_2}(P) + 2h_{D_3}(P),$$

so that (4.2) holds in this case.

This completes the proof of Proposition 4.1. □

Remark 4.2. Let X be a $K3$ surface over a number field K in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation

$$x^2y^2z^2 + \alpha x^2y^2 + \beta x^2z^2 + \gamma y^2z^2 + \delta x^2 + \varepsilon y^2 + \lambda z^2 + ax + by + cz + d = 0$$

in the affine part $\mathbb{A}^3$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. If $\alpha \neq \delta$, $\beta \neq \varepsilon$, $\gamma \neq \lambda$, $abc \neq 0$, then the proof of Proposition 4.1 works and give an explicit estimate of c_1', c_2' and c_3'.

Using Proposition 3.5 and Proposition 4.1, let us prove Corollary 1.2.

Proof of Corollary 1.2. Let X be a surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by (1.4) in the affine part. One can check that X is smooth.

Let $P = (1, 0, 0)$. One can check that, in the affine coordinate,

$$\sigma_1(P) = (\sqrt{3}, 0, 0), \qquad \sigma_2 \circ \sigma_1(P) = \left(\sqrt{3}, \frac{\sqrt{-2}}{2}, 0\right),$$

$$\rho(P) = \sigma_3 \circ \sigma_2 \circ \sigma_1(P) = \left(\sqrt{3}, \frac{\sqrt{-2}}{2}, 1\right), \qquad \sigma_1 \circ \rho(P) = \left(1, \frac{\sqrt{-2}}{2}, 1\right),$$

$$\sigma_2 \circ \sigma_1 \circ \rho(P) = (1, 0, 1), \quad \rho^2(P) = \sigma_3 \circ \sigma_2 \circ \sigma_1 \circ \rho(P) = (1, 0, 0).$$

Thus P is a ρ-periodic point with exact period 2.

Next, we show that $P \notin \mathcal{Q}$. We use Proposition 3.5. By Proposition 4.1, we can take c_k' in (3.2) as

$$\begin{aligned} c_1' = c_2' = c_3' &= 2h\left(\left(2\sqrt{3} + 2 : \frac{5}{2}\sqrt{-2} : 3 : -2\sqrt{3} - 4 : 1\right)\right) + 4\log 3 + \log 2 \\ &= 2h((4(1 + \sqrt{3}) : 5\sqrt{-2} : 6 : -4(\sqrt{3} + 2) : 2)) + \log(2 \cdot 3^4) \\ &\le \log(2^{5/2} \cdot 3^4 \cdot 5 \cdot (2 + \sqrt{3})). \end{aligned}$$

On the other hand, in the affine coordinate, we compute

$$\begin{aligned} \sigma_3(P) &= (1, 0, 1), \\ \sigma_1 \circ \sigma_3(P) &= \left(\frac{-1 + 2\sqrt{3}}{3}, 0, 1\right), \\ \sigma_2 \circ \sigma_1 \circ \sigma_3(P) &= \left(\frac{-1 + 2\sqrt{3}}{3}, \frac{3^2 \cdot 5 \cdot (53 + 8\sqrt{3})\sqrt{-2}}{2 \cdot 2617}, 1\right), \\ \sigma_3 \circ \sigma_2 \circ \sigma_1 \circ \sigma_3(P) &= \left(\frac{-1 + 2\sqrt{3}}{3}, \frac{3^2 \cdot 5 \cdot (53 + 8\sqrt{3})\sqrt{-2}}{2 \cdot 2617}, \right. \\ &\qquad \left. \frac{3^3(2 \cdot 449 \cdot 86729 + 17064953 \cdot \sqrt{3})}{2 \cdot 11 \cdot 61 \cdot 71 \cdot 15913}\right), \end{aligned}$$

We set $P_0 := \sigma_3 \circ \sigma_2 \circ \sigma_1 \circ \sigma_3(P)$. Then we have

$$h_{D_1}(P_0) + h_{D_2}(P_0) + h_{D_3}(P_0) > \frac{3}{2} \log(2^{5/2} \cdot 3^4 \cdot 5 \cdot (2 + \sqrt{3})).$$

It follows from Proposition 3.5 that $P_0 \notin \mathcal{Q}$. Hence $P \notin \mathcal{Q}$. Since P and σ_1, σ_2, σ_3 are all defined over $\mathbb{Q}(\sqrt{-2}, \sqrt{3})$, proof of Theorem 1.1 shows that there does not exist a canonical vector height over $\mathbb{Q}(\sqrt{-2}, \sqrt{3})$ on X. This completes the proof of Corollary 1.2. □

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