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Mathematical Analysis — *Restriction estimates for the full Laplacian on Métivier groups*, by Valentina Casarino and Paolo Ciatti, communicated on 9 November 2012.

Abstract. — In this paper we prove a restriction theorem for the full Laplacian on a group of Métivier type. In particular, we compute the spectral resolution of this operator and estimate the norm of the spectral projections between Lebesgue spaces.

Key words: Restriction theorem, two-step nilpotent groups, Métivier groups, full Laplacian, sublaplacian, twisted Laplacian.

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1. Introduction

In 1980 G. Métivier, motivated by the study of analytic hypoellipticity, introduced a class of two-step nilpotent Lie groups, whose quotients with respect to the codimension one subspaces of the centre are Heisenberg groups. The groups of H-type, introduced by A. Kaplan [K], are examples of groups satisfying the Métivier property, but there are Métivier groups which are not of H-type (for an example we refer the reader to [MuS]).

In this paper we focus our attention on the mapping properties of operators arising in the spectral decomposition of some left invariant differential operators on the class of groups of Métivier type.

On a group G of Métivier type, let $\Delta_{\mathfrak{z}}$ be the Laplacian on the centre and L a sublaplacian. We call full Laplacian their difference $\Delta_G = L - \Delta_{\mathfrak{z}}$. The operators L and Δ_G may be simultaneously diagonalized with positive spectrum. Thus it is possible to work out a joint spectral decomposition and then define the operators $\mathscr{P}_\mu^D$, $D = L$ or $D = \Delta_{\mathfrak{z}}$, as Dirac deltas $\delta_\mu(D)$ at a point $\mu > 0$. The family $\{\mathscr{P}_\mu^D\}_{\mu \in \mathbb{R}_+}$ is the spectral resolution of D.

There exists a vast literature concerning mapping properties between Lebesgue spaces of analogous operators in various settings. Hence, we can only mention the results that are most significant for this work.

The main example, of course, is that of the usual Laplacian Δ acting on the Euclidean space $\mathbb{R}^d$. As it is well known, the spectral resolution of Δ essentially consists of the family of the convolution operators with the Fourier transform $\widehat{\sigma}_r$

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of the spherical measures $d\sigma_r$. Indeed, we have $\Delta f * \widehat{\sigma_r} = -r^2 f * \widehat{\sigma_r}$. These operators appear in the restriction theory of the Fourier transform to the spheres and their mapping properties are described by the celebrated Stein-Tomas Theorem [St, Ch. 9].

THEOREM 1.1. *Suppose that* $1 \le p \le 2\frac{d+1}{d+3}$ *and let* $\frac{1}{p} + \frac{1}{p'} = 1$. *Then the estimate*

$$\|f * \widehat{\sigma_r}\|_{p'} \le C_r \|f\|_p$$

holds for all Schwartz functions on $\mathbb{R}^d$.

Since, according to the Knapp example [St], the above estimates fail if $2\frac{d+1}{d+3} < p \le 2$, this theorem yields the sharp result in this setting.

Analogously, in the framework of compact Riemannian manifolds, C. Sogge proved $L^p - L^2$ bounds, $1 \le p \le 2$, for the spectral projections associated to second order elliptic operators [So1], [So2].

Analogous estimates have been studied also for hypoelliptic, but not elliptic, laplacians. In this context, the most relevant result is due to D. Müller, who proved an analogue of the Stein-Tomas Theorem for the sublaplacian acting on the Heisenberg group [Mu]. First of all, he showed that the only available estimate between L^p spaces on $\mathbb{H}^n$ is the trivial $L^1 - L^\infty$ one. Thus he introduced the mixed Lebesgue norms

$$\|f\|_{L_t^r L_{x,y}^p} = \Big(\int_{\mathbb{R}^{2n}} \Big(\int_{-\infty}^{\infty} |f(x,y,t)|^r \, dt \Big)^{p/r} dx\, dy \Big)^{1/p}, \quad 1 \le p, r < \infty, \tag{1.1}$$

(with the obvious modifications when p or r are equal to ∞), proving that

$$\|\mathscr{P}_\mu^L f\|_{L_t^\infty L_{x,y}^{p'}} \le C_\mu \|f\|_{L_t^1 L_{x,y}^p}, \tag{1.2}$$

for all $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, and for all Schwartz functions on $\mathbb{H}^n$, and that there are no estimates between mixed spaces when the pair of exponents associated to the central variable is different from $(1, \infty)$.

Since the operators $\mathscr{P}_\mu^L$ operate on the t variable through the Fourier transform, that does not admit nontrivial restriction estimates on the real line, Müller suggested that on groups with higher dimensional centre better bounds could be available. Indeed, S. Thangavelu proved in [Th1] that the inequality

$$\|\mathscr{P}_\mu^L f\|_{L^{p'}(G)} \le C \|f\|_{L^p(G)}$$

holds for $1 \le p \le 2\frac{n+1}{n+3}$ on the direct product G of n copies of the three dimensional Heisenberg group $\mathbb{H}_1$.

In [CCi2] we extend the theorem of Müller to the sublaplacian on a group of Métivier type. Since these groups have in general centre of dimension bigger than one (actually, in the Métivier class only the Heisenberg groups have a one dimensional centre), we incorporate the Stein-Tomas Theorem in the estimate concerning the central variables. We also improve on (1.2) by replacing on the left-hand side p' with an exponent $q < p'$.

The theorem of Müller corresponds in the realm of the classical Fourier analysis to the Stein-Thomas Theorem, which provides an $L^p(\mathbb{R}^d) - L^2(S^{d-1})$ or equivalently an $L^p(\mathbb{R}^d) - L^{p'}(\mathbb{R}^d)$ estimate. Our improvement, showing that the operators $\mathscr{P}_\mu^L$ are bounded from $L_t^1 L_{x,y}^p$, $1 \le p \le 2$, to $L_t^\infty L_{x,y}^q$ for some $q < p'$, is analogous to the enhancement of the Stein-Thomas Theorem, that would follow from the proof of the restriction conjecture.

In this paper we consider the same problem for the full Laplacian on a group of Métivier type. We work out its spectral resolution and find estimates for the norm of the spectral projectors as operators between Lebesgue spaces. In the proof we consider more general operators, that are defined as functions of the sublaplacian and the full Laplacian, and study the mapping properties of the operators arising in their spectral resolutions.

The main point here is the possibility to obtain bounds, analogous to (1.2), with q less than p' on the left; actually, we get $q = 2$.

To describe the idea, we consider the simpler case of the Heisenberg group. An essential rôle in our analysis will be played by the spectral projectors Λ_k^μ of the twisted Laplacian, that is a second order elliptic differential operator on $\mathbb{R}^{2n}$ with point spectrum. Indeed, the operators $\mathscr{P}_\mu^D$ are essentially given by the composition of two operations: the Fourier transform in the central variable t followed by the action in (x, y) of Λ_k^μ.

Sharp $L^p - L^2$ bounds for Λ_k^μ have been recently attained by H. Koch and F. Ricci [KoRi] (see also [CCi1] for a different proof). By means of these estimates, we obtain a result, that, when $D = L$, improves on that of Müller even on the Heisenberg group. In fact, we prove that

$$\|\mathscr{P}_\mu^D f\|_{L_t^\infty L_{x,y}^2} \le C_\mu \|f\|_{L_t^1 L_{x,y}^p},$$

for all Schwartz functions f on $\mathbb{H}^n$ and for all $1 \le p \le 2$.

The schema of the paper is the following. In the next section we recall some well known facts about the Heisenberg group. We mainly point the attention on the spectral resolution of the sublaplacian and on the Koch-Ricci estimates.

In the third section we introduce the groups of Métivier type, discussing some of their features. Then we briefly describe the spectral resolution of operators defined as functions of the sublaplacian and of the full Laplacian on such groups (we refer the reader to [CCi2] for a more detailed discussion).

In the fourth section we prove a restriction theorem for the full Laplacian on the Métivier groups. We first prove the estimates for a class of functions of L and Δ_G, obtaining a conditional statement, based on the assumption that the spectral projections of the twisted Laplacian are bounded between Lebesgue spaces. Then we obtain from this result the main theorem regarding the sublaplacian and the full Laplacian.

2. Preliminaries on the Heisenberg group

For the convenience of the reader, we collect in this section some results on the Heisenberg group $\mathbb{H}^n$. For more details we refer the reader to the book [Th2].

The Heisenberg group $\mathbb{H}_n$ is the space $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ equipped with the following non commutative multiplication law

$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)\right),$$

for x, x', y, y' in $\mathbb{R}^n$ and t, t' in $\mathbb{R}$. This product turns $\mathbb{H}_n$ into a two step nilpotent Lie group with centre given by $\{(0, 0, t) : t \in \mathbb{R}\}$. The bi-invariant Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure $dx\,dy\,dt$ on $\mathbb{R}^{2n+1}$.

The left invariant vector fields on $\mathbb{H}_n$

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

$j = 1, \dots, n$, span its Lie algebra, $\mathfrak{h}_n$.

In terms of these vector fields we define on $\mathbb{H}_n$ the sublaplacian

$$L = -\sum_{j=1}^{n} (X_j^2 + Y_j^2),$$

which is hypoelliptic, since the set $\{X_1 \dots, Y_n\}$ generates $\mathfrak{h}_n$ as a Lie algebra, and the full Laplacian

$$\Delta_{\mathbb{H}} = -\sum_{j=1}^{n} (X_j^2 + Y_j^2) - T^2 = L - T^2.$$

The operators L and $-iT$, or equivalently the operators L and $\Delta_{\mathbb{H}}$, extend to a pair of strongly commuting self-adjoint operators on $L^2(\mathbb{H}^n)$.

If f is a Schwartz function on $\mathbb{H}^n$, then its spectral decomposition with respect to the sublaplacian is

$$f(x, y, t) = \int_0^\infty \mathscr{P}_\mu f(x, y, t)\, d\mu, \quad \forall (x, y, t) \in \mathbb{H}^n, \tag{2.1}$$

where

$$\mathscr{P}_\mu f(x, y, t) = \frac{\mu^n}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \frac{1}{(2k+n)^{n+1}} \left(e^{-i\mu_k t} \Lambda_k^{\mu_k} f^{(\mu_k)}(x, y) + e^{i\mu_k t} \Lambda_k^{-\mu_k} f^{(-\mu_k)}(x, y)\right).$$

In this formula μ_k is a short for $\mu/(2k+n)$ and $f^{(\mu)}(x, y)$ denotes the Fourier transform of f in the central variable t, that is

$$f^{(\mu)}(x, y) = \int_{-\infty}^{\infty} f(x, y, t) e^{i\mu t}\, dt.$$

Finally, the operators Λ_k^μ are the spectral projections of a differential operator on $\mathbb{R}^{2n}$ called μ-twisted Laplacian. This is a second order elliptic differential operator on $\mathbb{R}^{2n}$ with point spectrum. We shall often write Λ_k in place of Λ_k^1.

The estimates of the norms of the spectral projections Λ_k^μ as operators acting on Lebesgue spaces will be an important tool in the proof. The sharp bounds for the norms $\|\Lambda_k\|_{L^p(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})}$ have been recently obtained by H. Koch and F. Ricci. By improving some earlier results in [RRaTh], [SteZ], they showed that

$$\|\Lambda_k\|_{L^p(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})} \le C(2k+n)^{\gamma(1/p)}, \quad 1 \le p \le 2, \tag{2.2}$$

where γ is the piecewise affine function on $\left[\frac{1}{2}, 1\right]$ defined by

$$\gamma\Big(\frac{1}{p}\Big) := \begin{cases} n\big(\frac{1}{p}-\frac{1}{2}\big)-\frac{1}{2} & \text{if } 1 \le p \le p_*, \\ \frac{1}{2}\big(\frac{1}{2}-\frac{1}{p}\big) & \text{if } p_* \le p \le 2, \end{cases}$$

with critical point $p_* = p_*(2n)$, given by $p_*(2n) := 2\frac{2n+1}{2n+3}$. Observe that $p_*(m)$ is the critical exponent found by Stein and Thomas in the restriction theorem in dimension m.

3. Spectral resolution of the sublaplacian on Métivier groups

Let G be a connected, simply connected, two-step nilpotent Lie group and assume that its Lie algebra $\mathfrak{g}$ is endowed with an inner product $\langle\cdot,\cdot\rangle$. We denote the centre of $\mathfrak{g}$ by $\mathfrak{z}$, its orthogonal complement by $\mathfrak{v}$, and set $\dim\mathfrak{z} = d$ and $\dim\mathfrak{v} = k$.

Let $|\cdot|$ denote the norm induced by $\langle\cdot,\cdot\rangle$ on $\mathfrak{z}^*$, the dual of $\mathfrak{z}$. We introduce the unit sphere S in $\mathfrak{z}^*$, that is,

$$S := \{\omega \in \mathfrak{z}^* : |\omega| = 1\}.$$

For any fixed $\omega \in S$, there is an element $Z_\omega \in \mathfrak{z}$ such that $\omega(Z_\omega) = 1$ and $|Z_\omega| = 1$.

The centre of the Lie algebra decomposes into the sum

$$\mathfrak{z} = \mathbb{R}Z_\omega \oplus \ker\omega. \tag{3.1}$$

Observe that for every $Z \in \ker\omega$ we have $\langle Z_\omega, Z\rangle = 0$. Abusing notation, we shall identify the quotient $\mathfrak{z}/\ker\omega$ with $\mathbb{R}Z_\omega$. Then, setting

$$\mathfrak{g}_\omega := \mathbb{R}Z_\omega \oplus \mathfrak{v},$$

we have $\mathfrak{g}_\omega \simeq \mathfrak{g}/\ker\omega$. Since $\ker\omega$, being a subspace of the centre, is an ideal in $\mathfrak{g}$, $\mathfrak{g}_\omega$ is a Lie algebra. The connected simply connected subgroup of G with Lie algebra $\mathfrak{g}_\omega$ will be denoted by G_ω.

We assume that G satisfies a non-degeneracy condition, that may be expressed in terms of the bilinear application $B_\omega(X, Y) = \omega([X, Y])$, with X, Y in $\mathfrak{v}$ and ω

in S. Recall that B_ω is *non-degenerate* if the space $\{V \in \mathfrak{v} : \omega([V, U]) = 0$ for all U in $\mathfrak{v}\}$ is trivial.

Definition 3.1 [M]. *We say that G is a Métivier group if B_ω is non-degenerate for all ω in S.*

This nondegeneracy property implies that G_ω is isomorphic to the Heisenberg group for all $\omega \in S$, and that $\mathfrak{v}$ generates $\mathfrak{g}$ as a Lie algebra. In particular, $\mathfrak{v}$ has even dimension, say $\dim \mathfrak{v} = 2n$, so that G_ω is isomorphic to $\mathbb{H}_n$ with Lie algebra $\mathfrak{h}_n = \mathbb{R} \oplus \mathfrak{v}_n$, $\mathfrak{v}_n = \mathbb{R}^{2n}$.

Since $\mathfrak{g}$ is nilpotent, the exponential map, $\exp : \mathfrak{g} \to G$, is surjective. Thus we may parametrize G by $\mathfrak{v} \oplus \mathfrak{z}$, endowing it with the exponential coordinates. More precisely, we fix a basis $\{Z_1, \ldots, Z_d, V_1, \ldots, V_{2n}\}$ of $\mathfrak{g}$, with $\{Z_1, \ldots, Z_d\}$ a basis of $\mathfrak{z}$ and $\{V_1, \ldots, V_{2n}\}$ a basis of $\mathfrak{v}$, and identify a point g of G with the point (V, Z) in $\mathbb{R}^{2n} \times \mathbb{R}^d$, such that

$$g = \exp(V, Z) = \exp\Big(\sum_{j=1}^{2n} v_j V_j + \sum_{a=1}^{d} z_a Z_a\Big).$$

In these coordinates the product law is given by the Baker-Campbell-Hausdorff formula

$$(V, Z)(V', Z') = \Big(V + V', Z + Z' + \frac{1}{2}[V, V']\Big),$$

for all $V, V' \in \mathfrak{v}$ and $Z, Z' \in \mathfrak{z}$.

If we denote by dV and dZ the Lebesgue measures on $\mathfrak{v}$ and $\mathfrak{z}$ respectively, then the product measure $dV\,dZ$ is a left-invariant Haar measure on G. We shall denote by $L^p(G)$ the corresponding Lebesgue spaces.

Finally, we call $\mathscr{S}(G)$ the Schwartz space on G, that is, the space of functions f on G such that $f \circ \exp$ belongs to the usual Schwartz space on the Euclidean space $\mathfrak{g}$.

The vectors fields

$$\tilde{V}_j = \frac{\partial}{\partial v_j} + \frac{1}{2}\sum_{a=1}^{d} \langle Z_a, [V, V_j]\rangle \frac{\partial}{\partial z_a}, \qquad \tilde{T}_a = \frac{\partial}{\partial z_a},$$

where $j = 1, \ldots, 2n$ and $a = 1, \ldots d$, yield a basis of $\mathfrak{g}$. In terms of these vectors we define the sublaplacian

$$L = -\tilde{V}_1^2 - \cdots - \tilde{V}_{2n}^2,$$

the Laplacian on the centre

$$\Delta_{\mathfrak{z}} = \tilde{T}_1^2 + \cdots + \tilde{T}_d^2,$$

and the full Laplacian

$$\Delta_G = L - \Delta_{\mathfrak{z}}.$$

These operators are positive and essentially self-adjoint on $L^2(G)$. Moreover, L and Δ_G are hypoelliptic, since the set of vector fields $\{\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_{2n}\}$ generates $\mathfrak{g}$ as a Lie algebra.

Following a well known procedure ([Str], [T]), by taking the Radon transform in the central variables and using the Métivier property, we reduce the computation of the spectral decompositon of a function on G to the spectral decomposition of its Radon transform on a Heisenberg group.

Given a function f in $\mathscr{S}(G)$, we denote the partial Fourier transform of f in the central variables by

$$\hat{f}(\eta; V) := \hat{f}^{(\eta)}(V) := \int_{\mathfrak{z}} e^{i\eta(Z)} f(V, Z)\, dZ,$$

for all $\eta \in \mathfrak{z}^*$, $t \in \mathbb{R}$, $V \in \mathfrak{v}$, where $\mathfrak{z}^*$ is the dual of $\mathfrak{z}$. The partial Radon transform in the central variables of f is

$$R_\omega f(V, t) := \int_{\{Z' \in \ker \omega\}} f(V, tZ_\omega + Z')\, dZ',$$

for all $\omega \in S$, $t \in \mathbb{R}$, $V \in \mathfrak{v}$, where dZ' denotes the Lebesgue measure on the hyperplane $\ker \omega$ in $\mathfrak{z}$. We recall that the Radon transform factorizes the Fourier transform

$$f^{(\rho\omega)}(V) = \int_{-\infty}^{\infty} e^{i\rho t} R_\omega f(V, t)\, dt = (R_\omega f)^{(\rho)}(V), \tag{3.2}$$

for all $\omega \in S$, $\rho \geq 0$, $V \in \mathfrak{v}$.

For each fixed ω, $R_\omega f$ is a function on the subgroup G_ω, which is isomorphic to $\mathbb{H}_n$. The family of functions $\{R_\omega f\}_{\omega \in S}$ determines f. In [CCi2] we obtained the spectral decomposition of f with respect to L from the spectral decomposition of the functions $R_\omega f$ with respect to the sublaplacian L_ω, that is defined by

$$R_\omega(Lf) = L_\omega(R_\omega f).$$

This sublaplacian is conjugated via a linear map on $\mathfrak{v}$, A_ω, to the canonical sublaplacian, $L_{\mathbb{H}}$ on $\mathbb{H}_n$, so that the decomposition of $R_\omega f$ with respect to L_ω may be deduced from that of $(R_\omega f) \circ A_\omega^{-1}$ with respect to $L_{\mathbb{H}}$. By (2.1), we hence obtain

$$((R_\omega f) \circ A_\omega^{-1})(V, t) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda t} \Pi_k^{\lambda,\omega} (R_\omega f)^{(\lambda)}(A_\omega^{-1} V) |\lambda|^n\, d\lambda \tag{3.3}$$

where

$$\Pi_k^{\lambda,\omega} g(V) = \Lambda_k^{\lambda}(g \circ A_\omega^{-1})(A_\omega V). \tag{3.4}$$

Since $R_\omega(f \circ A_\omega^{-1}) = (R_\omega f) \circ A_\omega^{-1}$, from (3.3) it follows that

$$(R_\omega f)^{(\lambda)}(V) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \Pi_k^{\lambda,\omega} f^{(\lambda)}(V)|\lambda|^n,$$

whence we deduce

$$(R_\omega f)^{(\lambda)}(V) = \frac{1}{(2\pi)^{n+1}} \sum_{k=0}^{\infty} \Pi_k^{\lambda,\omega} f^{(\lambda\omega)}(V)|\lambda|^n.$$

By (3.2), $(R_\omega f)^{(\rho)}$, the Fourier transform in t of $R_\omega f$, coincides with the Fourier transform in Z of f at $\rho\omega$. Hence, by the Fourier inversion formula in polar coordinates, we obtain

$$\begin{aligned} f(V,Z) &= \frac{1}{(2\pi)^d} \int_0^\infty \int_S e^{-i\rho\omega(Z)} f^{(\rho\omega)}(V)\, d\omega \rho^{d-1}\, d\rho \\ &= \frac{1}{(2\pi)^{d+n+1}} \sum_{k=0}^{\infty} \int_0^\infty \int_S e^{-i\rho\omega(Z)} \Pi_k^{\rho,\omega} f^{(\rho\omega)}(V)\, d\omega \rho^{d+n-1}\, d\rho. \end{aligned} \tag{3.5}$$

Formula (3.5) may be interpreted as the decomposition of f in terms of the joint eigendistributions of $\Delta_\mathfrak{z}$ and L,

$$\mathcal{Q}_{\rho,k} f(V,Z) = \int_S e^{-i\rho\omega(Z)} \Pi_k^{\rho,\omega} f^{(\rho\omega)}(V)\, d\omega,$$

corresponding respectively to the eigenvalues ρ and $\rho(2k+n)$.

Given a function $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$, we define the operator $\Phi(L, \Delta_\mathfrak{z})$ by

$$\begin{aligned} \Phi(L,\Delta_\mathfrak{z}) f(V,Z) = \sum_{k=0}^{\infty} \int_0^\infty & \phi(\rho(2k+n),\rho) \\ & \times \int_S e^{-i\rho\omega(Z)} \Pi_k^{\rho,\omega} f^{(\rho\omega)}(V)\, d\omega \rho^{d+n-1}\, d\rho, \end{aligned} \tag{3.6}$$

where we make the assumption on ϕ that the expression on the right hand side is a well defined distribution for all Schwartz functions f. We also suppose that $\phi(\rho(2k+n),\rho)$ is a differentiable function of ρ with strictly positive derivative on $\mathbb{R}_+$, satisfying $\lim_{\rho\to 0+} \phi(\rho(2k+n),\rho) = 0$ and $\lim_{\rho\to+\infty} \phi(\rho(2k+n),\rho) = +\infty$.

The equation $\phi(\rho(2k+n),\rho) = \mu$ may be solved for each positive μ and each k to give $\rho = \rho_k(\mu)$. Replacing in (3.6) ρ, which is the eigenvalue of $\Delta_\mathfrak{z}$, with the eigenvalue μ of $\Phi(L, \Delta_\mathfrak{z})$, we obtain the spectral decomposition of the latter operator,

$$\Phi(L, \Delta_\mathfrak{z})f(V, Z) = \int_0^\infty \Big(\sum_{k=0}^\infty \mu \rho_k(\mu)^{d+n-1} \rho_k'(\mu) \times \int_S e^{-i\rho_k \omega(Z)} \Pi_k^{\rho_k(\mu), \omega} f^{(\rho_k(\mu))}(V)\, d\omega \Big)\, d\mu,$$

here ρ_k' denotes the derivative of ρ_k. Similarly, replacing ρ with μ in (3.5), we obtain the spectral decomposition of a Schwartz function f,

$$f(V, Z) = \int_0^\infty \Big(\sum_{k=0}^\infty \rho_k(\mu)^{d+n-1} \rho_k'(\mu) \times \int_S e^{-i\rho_k \omega(Z)} \Pi_k^{\rho_k(\mu), \omega} f^{(\rho_k(\mu))}(V)\, d\omega \Big)\, d\mu. \tag{3.7}$$

We use (3.7) to introduce the operators $\delta_\mu(\Phi(L, \Delta_\mathfrak{z}))$ for $\mu > 0$, which are defined by

$$\mathscr{P}_\mu^\Phi f = \delta_\mu(\Phi(L, \Delta_\mathfrak{z}))f = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \chi_{(\mu-\varepsilon, \mu+\varepsilon)}(\Phi(L, \Delta_\mathfrak{z}))f,$$

where f is a Schwartz function and $\chi_{(\mu-\varepsilon, \mu+\varepsilon)}$ is the characteristic function of the interval $(\mu - \varepsilon, \mu + \varepsilon)$. Since the function between parentheses in (3.7) is continuous, we find

$$\begin{aligned} \mathscr{P}_\mu^\Phi f(V, Z) &= \sum_{k=0}^\infty \rho_k(\mu)^{d+n-1} \rho_k'(\mu) \Big(\int_S e^{-i\rho_k \omega(Z)} \hat{h}_{\rho_k(\mu)}(\omega) \Pi_k^{\rho_k(\mu), \omega} g(V)\, d\omega \Big) \\ &= \sum_{k=0}^\infty \rho_k(\mu)^{d+n-1} \rho_k'(\mu) \mathscr{Q}_{\rho_k(\mu), k} f(V, Z), \end{aligned} \tag{3.8}$$

where, to make the formulas more readable, we assume that f is a tensor function, that is $f(V, Z) = (g \otimes h)(V, Z) = g(V)h(Z)$, and use the notation $\hat{h}(\rho\omega) = \hat{h}_\rho(\omega)$ for the Fourier transform of h. With this notation (3.7) becomes

$$f(V, Z) = \int_0^\infty \mathscr{P}_\mu^\Phi f(V, Z)\, d\mu.$$

There are two main examples in the theory, the sublaplacian and the full Laplacian. For the sublaplacian $\phi(\rho, \lambda) = \rho$, we have $\mu = \rho(2k + n)$, which yields

$$\rho_k(\mu) = \frac{\mu}{2k + n}. \tag{3.9}$$

By (3.8), we obtain

$$\mathscr{P}_\mu^L f(V,Z) = \mu^{n+d-1} \sum_{k=0}^{\infty} (2k+n)^{-n-d} \mathscr{Q}_{\mu/(2k+n),k} f(V,Z). \tag{3.10}$$

In the case of the full Laplacian $\phi(\rho,\lambda) = \rho + \lambda^2$. Hence, if $\mu = \rho(2k+n) + \rho^2$, then

$$\rho_k(\mu) = \frac{1}{2}\sqrt{4\mu + (2k+n)^2} - \frac{2k+n}{2}. \tag{3.11}$$

Therefore, (3.8) yields

$$\mathscr{P}_\mu^{\Delta_G} f(V,Z) = \frac{1}{2^{n+d-1}} \sum_{k=0}^{\infty} \frac{(\sqrt{4\mu + (2k+n)^2} - 2k - n)^{n+d-1}}{\sqrt{4\mu + (2k+n)^2}} \times \mathscr{Q}_{\rho_k(\mu),k} f(V,Z). \tag{3.12}$$

4. Restriction estimates for Métivier groups

In this section we find a bound on the norms of the operators $\mathscr{P}_\mu^\Phi$. The first result we prove is a conditional one, based on the assumption that the projections Λ_k of the twisted Laplacian are bounded from $L^p(\mathfrak{v}_n)$ to $L^q(\mathfrak{v}_n)$.

Theorem 4.1. *Assume that $1 \le r \le 2\frac{d+1}{d+3}$. If the projections Λ_k are bounded from $L^p(\mathfrak{v}_n)$ to $L^q(\mathfrak{v}_n)$, with $1 \le p \le 2 \le q \le \infty$, then the following inequality holds*

$$\begin{aligned} \|\mathscr{P}_\mu^\Phi f\|_{L^q(\mathfrak{v})L^{r'}(\mathfrak{z})} &\le C \|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})} \\ &\times \Big(\sum_{k=0}^{\infty} \rho_k(\mu)^{d(1/r-1/r')+n(1/p-1/q)-1} \rho_k'(\mu) \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \Big). \end{aligned} \tag{4.1}$$

The proof of this theorem depends on the following lemma, which is proved in [CCi2].

Lemma 4.2. *Fix ω in S. Suppose that $\Lambda_k : L^p(\mathfrak{v}_n) \to L^q(\mathfrak{v}_n)$ for some p, q. The following inequality holds*

$$\|\Pi_k^{\rho\omega} g\|_{L^q(\mathfrak{v})} \le C\rho^{n(1/p-1/q-1)} \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \|g\|_{L^p(\mathfrak{v})}$$

for all g in $\mathscr{S}(\mathfrak{v})$ and all $\rho > 0$.

PROOF OF THEOREM 4.1. In order to simplify the notation, we write $f(V,Z) = h(Z)g(V)$, with h and g Schwartz functions. However, in the proof we will never use this fact. We take $\alpha : \mathfrak{v} \to \mathbb{C}$ and $\beta : \mathfrak{z} \to \mathbb{C}$, $\alpha \in \mathscr{S}(\mathfrak{v})$, $\beta \in \mathscr{S}(\mathfrak{z})$. Then

$$\begin{aligned}\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}} &= \int_{\mathfrak{v}} \int_{\mathfrak{z}} \overline{\alpha(V)\beta(Z)} \sum_{k=0}^{\infty} \rho_k^{d+n-1} \rho_k' \\ &\quad \times \Big(\int_S e^{i\rho_k \omega(Z)} \widehat{h_{\rho_k}}(\omega) (\Pi_k^{\rho_k,\omega} g)(V)\, d\sigma(\omega) \Big)\, dZ\, dV \\ &= \sum_{k=0}^{\infty} \rho_k^{d+n-1} \rho_k' \Big(\int_S \int_{\mathfrak{v}} \widehat{h_{\rho_k}}(\omega)(\Pi_k^{\rho_k,\omega} g)(V) \\ &\quad \times \int_{\mathfrak{z}} e^{i\rho_k\omega(Z)} \overline{\alpha(V)\beta(Z)}\, dZ\, dV\, d\sigma(\omega) \Big) \\ &= \sum_{k=0}^{\infty} \rho_k^{d+n-1} \rho_k' \Big(\int_S \langle \widehat{h_{\rho_k}}(\omega)(\Pi_k^{\rho_k,\omega} g), \widehat{\beta_{\rho_k}}(\omega)\alpha \rangle_{\mathfrak{v}}\, d\sigma(\omega) \Big).\end{aligned}$$

Applying the Hölder's inequality to the inner integral we deduce that

$$\begin{aligned}|\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}}| &\le \sum_{k=0}^{\infty} \rho_k^{d+n-1} \rho_k' \\ &\quad \times \Big(\int_S \|\widehat{h_{\rho_k}}(\omega)(\Pi_k^{\rho_k,\omega} g)\|_{L^q(\mathfrak{v})} \|\widehat{\beta_{\rho_k}}(\omega)\alpha\|_{L^{q'}(\mathfrak{v})}\, d\sigma(\omega) \Big).\end{aligned}$$

Using Lemma 4.2, we then obtain

$$\begin{aligned}|\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}}| &\le C \sum_{k=0}^{\infty} \rho_k^{d+n(1/p-1/q)-1} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \\ &\quad \times \Big(\int_S \|\widehat{h_{\rho_k}}(\omega) g\|_{L^p(\mathfrak{v})} \|\widehat{\beta_{\rho_k}}(\omega)\alpha\|_{L^{q'}(\mathfrak{v})}\, d\sigma(\omega) \Big).\end{aligned}$$

Then the Cauchy-Schwarz inequality implies

$$\begin{aligned}|\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}}| &\le C \sum_{k=0}^{\infty} \rho_k^{d+n(1/p-1/q)-1} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \\ &\quad \times \Big(\int_S \|\widehat{h_{\rho_k}}(\omega) g\|_{L^p(\mathfrak{v})}^2\, d\sigma(\omega) \Big)^{1/2} \\ &\quad \times \Big(\int_S \|\widehat{\beta_{\rho_k}}(\omega)\alpha\|_{L^{q'}(\mathfrak{v})}^2\, d\sigma(\omega) \Big)^{1/2}.\end{aligned}$$

Since $p \le 2 \le q$, it follows that $\frac{2}{p} \ge 1$ and $\frac{2}{q'} \ge 1$. Therefore we can apply to the integrals on the right hand side the Minkowski integral inequality, to attain

$$|\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}}| \le C \sum_{k=0}^{\infty} \rho_k^{d+n(1/p-1/q)-1} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}$$
$$\times \Big(\int_S \Big(\int_{\mathfrak{v}} |\widehat{h_{\rho_k}}(\omega) g(V)|^p \, dV \Big)^{2/p} d\sigma(\omega) \Big)^{1/2}$$
$$\times \Big(\int_S \Big(\int_{\mathfrak{v}} |\widehat{\beta_{\rho_k}}(\omega) \alpha(V)|^{q'} \, dV \Big)^{2/q'} d\sigma(\omega) \Big)^{1/2}$$
$$\le C \sum_{k=0}^{\infty} \rho_k^{d+n(1/p-1/q)-1} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}$$
$$\times \Big(\int_{\mathfrak{v}} \Big(\int_S |\widehat{h_{\rho_k}}(\omega) g(V)|^2 \, d\sigma(\omega) \Big)^{p/2} dV \Big)^{1/p}$$
$$\times \Big(\int_{\mathfrak{v}} \Big(\int_S |\widehat{\beta_{\rho_k}}(\omega) \alpha(V)|^2 \, d\sigma(\omega) \Big)^{q'/2} dV \Big)^{1/q'}.$$

The Stein-Tomas theorem, that for $1 \le r \le r_*(d) = 2\frac{d+1}{d+3}$ yields the bound

$$\|\widehat{h_\rho}\|_{L^2(S)} \le C \|h_\rho\|_{L^r(\mathfrak{z})} = C\rho^{-d/r'} \|h\|_{L^r(\mathfrak{z})},$$

may now be pressed into service to give

$$|\langle \mathscr{P}_\mu^M f, \alpha \otimes \beta \rangle_{\mathfrak{v}\oplus\mathfrak{z}}| \le C \Big(\sum_{k=0}^{\infty} \rho_k^{d+n(1/p-1/q)-1-2(d/r')} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \Big)$$
$$\times \Big(\int_{\mathfrak{v}} \Big(\int_{\mathfrak{z}} |h(Z) g(V)|^r \, dZ \Big)^{p/r} dV \Big)^{1/p}$$
$$\times \Big(\int_{\mathfrak{v}} \Big(\int_{\mathfrak{z}} |\beta(Z) \alpha(V)|^r \, dZ \Big)^{q'/r} dV \Big)^{1/q'},$$

whence it follows that

$$\|\mathscr{P}_\mu^M f\|_{L^q(\mathfrak{v}) L^{r'}(\mathfrak{z})} \le C \|f\|_{L^p(\mathfrak{v}) L^r(\mathfrak{z})}$$
$$\times \Big(\sum_{k=0}^{\infty} \rho_k^{d(1/r-1/r')+n(1/p-1/q)-1} \rho_k' \|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)} \Big),$$

proving the assertion. □

Now we implement Theorem 4.1 with the Koch-Ricci estimates for $\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^q(\mathfrak{v}_n)}$, and then we apply it to the operators arising in the the spectral resolutions of L and Δ_G, that are respectively given by (3.10) and (3.12). For the sake of simplicity, we state only the estimates for $q = 2$. The estimates for $q > 2$ may be obtained by interpolation.

THEOREM 4.3. *Suppose that* $1 \le r \le r_*(d) = 2\frac{d+1}{d+3}$. *Then for all* p *satisfying* $1 \le p \le 2$ *and for all Schwartz functions* f, *we have*

$$\|\mathscr{P}^L_\rho f\|_{L^2(\mathfrak{v})L^{r'}(\mathfrak{z})} \le C\rho^{d(2/r-1)+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}. \tag{4.2}$$

PROOF. The proof reduces to the study of the convergence of the series in (4.1), which for $\Phi = L$, according to (3.9), takes the form

$$\sum_{k=0}^{\infty}(2k+n)^{-d(1/r-1/r')-n(1/p-1/2)}\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^2(\mathfrak{v}_n)}.$$

We omit the details, which are given in [CCi2]. □

Finally, we consider the full Laplacian on G.

THEOREM 4.4. *Suppose that* $1 \le r \le r_*(d) = 2\frac{d+1}{d+3}$. *For* $1 \le p \le p_*$ *we have*

$$\|\mathscr{P}^{\Delta_G}_\mu f\|_{L^2(\mathfrak{v})L^{r'}(\mathfrak{z})} \le C\mu^{(d/2)(1/r-1/r')+n(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})} \tag{4.3}$$

and for $p_* \le p \le 2$ *we have*

$$\|\mathscr{P}^{\Delta_G}_\mu f\|_{L^2(\mathfrak{v})L^{r'}(\mathfrak{z})} \le C\mu^{(d/2)(1/r-1/r')+((2n-1)/4)(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}. \tag{4.4}$$

PROOF. Plugging (3.11) in (4.1), we obtain

$$\begin{aligned}
&\|\mathscr{P}^{\Delta_G}_\mu f\|_{L^2(\mathfrak{v})L^{r'}(\mathfrak{z})} \le C\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\\
&\quad\times\Bigg(\sum_{k=0}^{\infty}\frac{(\sqrt{4\mu+(2k+n)^2}-2k+n)^{d(1/r-1/r')+n(1/p-1/2)-1}}{\sqrt{4\mu+(2k+n)^2}}\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^2(\mathfrak{v}_n)}\Bigg)\\
&\quad\le C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\\
&\quad\times\Bigg(\sum_{k=0}^{\infty}(\sqrt{4\mu+(2k+n)^2}+2k+n)^{1-d(1/r-1/r')-n(1/p-1/2)}\frac{\|\Lambda_k\|_{L^p(\mathfrak{v}_n)\to L^2(\mathfrak{v}_n)}}{\sqrt{4\mu+(2k+n)^2}}\Bigg).
\end{aligned}$$

We split the sum into the sum over those k such that $2k+n \le 2\sqrt{\mu}$ and those such that $2k+n > 2\sqrt{\mu}$. Then we control the first term, say I, by

$$\begin{aligned}
I &\le C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\mu^{-(d/2)(1/r-1/r')-(n/2)(1/p-1/2)}\\
&\quad\times\Bigg(\sum_{2k+n\le 2\sqrt{\mu}}\|\Lambda_k\|_{L^p(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})}\Bigg)\\
&\le C\mu^{(d/2)(1/r-1/r')+(n/2)(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\Bigg(\sum_{2k+n\le 2\sqrt{\mu}}\|\Lambda_k\|_{L^p(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})}\Bigg)
\end{aligned}$$

and the second, say II, by

$$II \leq C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\Big(\sum_{2k+n\geq 2\sqrt{\mu}}\frac{\|\Lambda_k\|_{L^p(\mathbb{R}^{2n})\to L^2(\mathbb{R}^{2n})}}{(2k+n)^{d(1/r-1/r')+n(1/p-1/2)}}\Big).$$

When $1 \leq p \leq p_*$ by (2.2) we have

$$\begin{aligned} I &\leq C\mu^{(d/2)(1/r-1/r')+(n/2)(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\Big(\sum_{2k+n\leq 2\sqrt{\mu}}(2k+n)^{n(1/p-1/2)-1/2}\Big) \\ &\leq C\mu^{(d/2)(1/r-1/r')+(n/2)(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\mu^{(n/2)(1/p-1/2)+1/4} \\ &\leq C\mu^{(d/2)(1/r-1/r')+n(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})} \end{aligned}$$

and

$$\begin{aligned} II &\leq C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\Big(\sum_{2k+n\geq 2\sqrt{\mu}}\frac{(2k+n)^{-1/2}}{(2k+n)^{d(1/r-1/r')}}\Big) \\ &\leq C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\mu^{-(d/2)(1/r-1/r')+1/4} \\ &\leq C\mu^{(d/2)(1/r-1/r')+n(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}. \end{aligned}$$

proving (4.3).

When $p_* \leq p \leq 2$ we have

$$\begin{aligned} I &\leq C\mu^{(d/2)(1/r-1/r')+(n/2)(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\Big(\sum_{2k+n\leq 2\sqrt{\mu}}(2k+n)^{(1/2)(1/2-1/p)-1/2}\Big) \\ &\leq C\mu^{(d/2)(1/r-1/r')+(n/2)(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\mu^{(1/4)(1/2-1/p)+1/4} \\ &\leq C\mu^{(d/2)(1/r-1/r')+((2n-1)/4)(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})} \end{aligned}$$

and

$$\begin{aligned} II &\leq C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})} \\ &\quad\times\Big(\sum_{2k+n\geq 2\sqrt{\mu}}(2k+n)^{-d(1/r-1/r')-((2n+1)/2)(1/p-1/2)-1/2}\Big) \\ &\leq C\mu^{d(1/r-1/r')+n(1/p-1/2)-1}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}\mu^{-(d/2)(1/r-1/r')-((2n+1)/4)(1/p-1/2)+1/4} \\ &\leq C\mu^{(d/2)(1/r-1/r')-((2n-1)/4)(1/p-1/2)-3/4}\|f\|_{L^p(\mathfrak{v})L^r(\mathfrak{z})}, \end{aligned}$$

proving (4.4). □

REMARK 4.5. In [CCi2] we prove that in the estimates for the sublaplacian the range of r in (4.2) is sharp. An analogous argument shows that the same is true for the estimates in the above theorem.

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