Rend. Lincei Mat. Appl. 24 (2013), 229–301
DOI 10.4171/RLM/654

Partial Differential Equations — *The energy graph of the non-linear Schrödinger equation*, by M. Procesi*, C. Procesi** and B. Van Nguyen***, communicated on 8 February 2013.

Abstract. — We discuss the stability of a class of normal forms of the completely resonant non-linear Schrödinger equation on a torus described in [12]. The discussion is essentially combinatorial and algebraic in nature.†

Key words: Normal form, NLS equation, Cayley graphs, stability.

Mathematics Subject Classification: 35Q55, 37K55, 05C31.

Contents

* Università di Roma, La Sapienza, supported by ERC grant HamPDEs under FP7, ** *** Università di Roma, La Sapienza.

† The results of this paper are related to the Conference "Dynamics of PDEs", which took place at the Accademia dei Lincei on October 14, 2011.

1. Introduction

In this paper we study the completely resonant cubic Nonlinear Schrödinger equation (NLS):

$$\mathrm{i}u_t - \Delta u = |u|^2 u \tag{1}$$

on the n dimensional torus $\mathbb{T}^n$. More precisely we analize the quadratic Normal form Hamiltonian, introduced in [12], of the NLS equation (1), with the purpose of proving *non-degeneracy* and stability results for its dynamics. Our dynamical results are summarized in Propositions 1.2 and 1.3 which in turn follow from our main Theorem 1. This theorem, whose lenghtly proof occupies most of the paper, is of algebraic, combinatorial and geometric nature, and can in principle be formulated with no previous knowledge of the NLS. In the first ten pages we recall, for convenience of the reader, the results on the NLS normal form proved in [12], and we show how to deduce our dynamical results from Theorem 1. Let us briefly- and somewhat naïvely- recall the theory of *Poincaré-Birkhoff Normal Form*. The Birkhoff normal form reduction was developed in order to study the long-time behaviour of the solutions of a dynamical system close to an equilibrium and represents a non-linear analog to the *canonical form* for matrices. For a classical introduction see [1], [5], [10], [8]; for the application to PDEs see for instance [4].

At a purely formal level, consider a non-linear Hamitonian dynamical system with an elliptic fixed point:

$$H(p,q) = \sum_{j \in I} \lambda_j (p_j^2 + q_j^2) + H^{>2}(p,q), \quad \lambda_j \in \mathbb{R}$$

here the index set I is finite or possibly denumberable while $H^{>2}(p,q)$ is some polynomial with minimal degree > 2. By definition the *normal form reduction* at order N is a symplectic change of variables Ψ_N which reduces H to its resonant terms:

$$H(p,q) \circ \Psi_N = \sum_j \lambda_j (p_j^2 + q_j^2) + H_{Res}^{>2}(p,q) + H^N(p,q)$$

where $H_{Res}^{>2}$ Poisson commutes with $\sum_j \lambda_j (p_j^2 + q_j^2)$ while $H^N(p,q)$ is a formal power series of minimal degree $> N+1$.

There are two classes of problems in this scheme:

(i) Even though H^N is of minimal order $N+1$ its norm may diverge as $N \to \infty$, due to the presence of small divisors.

(ii) If I is an infinite set it is not trivial, even when $N = 1$, to show that Ψ_N is an analytic change of variables.

Note that if the λ_j are rationally independent then the normal form $H_{Birk} = \sum_j \lambda_j (p_j^2 + q_j^2) + H_{Res}^{>2}(p,q)$ is integrable, a feature which is used in proving for instance long time stability results.

If the λ_j are resonant then H_{Birk} may not be integrable but it is possible that its dynamics is simpler than the one of the original Hamiltonian.

In particular in many examples, including the NLS, one can see that H_{Birk} has invariant tori of the form

$$p_i^2 + q_i^2 = \xi_i, \quad i \in S \subset I; \quad p_j = q_j = 0, \quad j \in S^c := I \backslash S \tag{2}$$

on which the dynamics is of the form $\psi \to \psi + \omega(\xi)t$ with $\omega(\xi)$ a diffeomorphism.

One wishes to obtain information on solutions of the complete Hamiltonian close to these tori. As is well known in order to obtain results one needs to study the Hamilton equations of H linearized at this family of invariant tori. That is one needs to study the dynamics induced on the normal bundle to these tori. This is described by a family of linear operators (between normal spaces) parametrized by the family and the points on the tori.

In terms of equations this is described by a quadratic Hamiltonian with coefficients depending on the parameters ξ and on the angle variables of the tori. The matrix obtained by linearizing H_{Birk} at the solutions (2) is referred to as the normal form matrix (or normal form). One of the main results of [12] exhibits, for the NLS and for generic choices of S, a symplectic change of variables which removes the dependence from the angles, this decouples the dynamics into the one on the tori and one on the normal space. Moreover in our infinite dimensional case the matrices of the normal form are block diagonal with blocks uniformly bounded. Thus one has a reduction to an infinite list of decoupled linear equations (depending on the parameters ξ).

In order to perform perturbation theory algorithms, to obtain informations on the solutions of H, one generally uses *non-degeneracy* conditions. One of the strongest requirements is that the matrix of the normal form has non-zero and distinct eigenvalues. This property is an instance of *structural stability*. In this paper we prove that this condition is satisfied for the normal forms of the NLS previously introduced provided the parameters ξ are taken outside a countable union of real hypersurfaces.

1.1. Structural stability. Structural stability, for an orbit of a dynamical system or a solution of a differential equation is a basic, and delicate, question both for theoretical and practical reasons. It essentially means that the qualitative behavior of the trajectories, close to the given solutions, is unaffected by small perturbations both of the initial data and of the system itself.

In the simplest case of the class of linear differential equation $\dot{x} = Ax$, where A is a real $n \times n$ matrix, the nature of the orbits depends upon the Jordan canonical form of A. In particular the discriminant of A is an hypersurface (in the space of all matrices) which contains all special normal forms; its complement is the set of matrices with distinct eigenvalues which decomposes into connected components. On each such component the number of real eigenvalues is constant, thus these regions are the regions of structural stability. Of course if the matrix A is subject to some restrictions (as being symmetric, symplectic etc.) the normal forms are further constrained [2].

1.1.1. Stability for the NLS. The normal form of the NLS is described by an infinite dimensional Hamiltonian which determines a linear operator $ad(N)$, depending on a finite number of parameters ξ_i (the actions of certain excited frequencies), and acting on a certain infinite dimensional vector space $F^{(0,1)}$ (see 2.6.1) of functions.

Stability for this infinite dimensional operator will be interpreted in the same way as it appears for finite dimensional linear systems, that is the property that the linear operator is semisimple with distinct eigenvalues.

This will be shown to be true outside a zero measure set of parameters, further on a smaller set of positive measure we shall show that the dynamic is elliptic. This condition in a more precise quantitative form (which will be discussed elsewhere) in the Theory of dynamical systems is referred to as the *second Melnikov condition*. We shall apply this in [11] in order to prove, by a KAM algorithm, the existence and stability of quasi-periodic solutions for the NLS (not just the normal form).

The fact that this non-degeneracy condition makes at all sense depends on the fact that the normal form matrix decomposes into an infinite direct sum of finite dimensional blocks. Furthermore, these finite dimensional blocks are described by translating, with suitable scalars, a finite number of combinatorially defined matrices, constructed from certain combinatorial objects called *marked colored graphs* (cf. Definition 2.8 and Remark 2.10). Thus the matrices appearing as blocks of the normal form matrix can be combinatorially classified and, in principle, computed. Indeed given a specific graph computing the associated matrix block is quite simple, so that the question is essentially that of classifying the possible graphs which describe blocks of the normal form.

The characteristic polynomials $\det(t - ad(N)_\Gamma)$ of the normal form operator $ad(N)$ restricted to the infinitely many blocks Γ are all polynomials in the variables ξ_i and t with integer coefficients. The issue is thus to prove that a rather complicated infinite list of polynomials in a variable t, of degree increasing with the space dimension, and with coefficients polynomials in the parameters ξ_i have distinct roots for generic values of the parameters.

In general, in order to prove that a single polynomial has distinct roots, one has to prove the non-vanishing of its discriminant, for two polynomials to have different roots the condition is the non-vanishing of the resultant. In our case we can consider all the characteristic polynomials as having coefficients in the field of rational functions in the parameters ξ_i, its algebraic closure is a *field of algebraic functions*. Thus if the discriminant $D(\xi)$ of a given polynomial and the resultant $R(\xi)$ of two distinct polynomials in $\mathbb{Q}(\xi_1, \dots, \xi_m)[t]$ are non-zero as polynomials in the ξ we have that outside the real hypersurfaces $R(\xi) = 0$, $D(\xi) = 0$ the two polynomials have distinct roots. Although both the discriminant and the resultant can be computed by explicit formulas a proof of their non-vanishing for the infinite list of complicated polynomials appearing seems to be a hopeless task.

We thus followed a different approach. Remark that, if we have a list of different polynomials in one variable t, with coefficients in a field F of characteristic 0, a sufficient condition that all their roots (in the algebraic closure $\bar{F}$ of F) be distinct is that they are all *irreducible* (over F) and distinct. This follows immediately from the fact that an irreducible polynomial $f(t)$ is uniquely determined as the minimal polynomial of each of its roots (cf. [3]) and, in characteristic 0, its derivative $f'(t)$ is non-zero. By the irreducibility of $f(t)$ the greatest common divisor between $f(t)$, $f'(t)$ is 1 so all the roots of $f(t)$ are distinct.

Therefore by a rather complex induction (setting some variables ξ_i equal to zero) we prove:

Theorem 1 (Separation and Irreducibility Theorem). *The characteristic polynomials of the possible graphs giving blocks of the normal form of the NLS are all* distinct, *and* irreducible *as polynomials with integer coefficients, that is in* $\mathbb{Z}[\xi_1, \dots, \xi_m, t] \subset \mathbb{Q}(\xi_1, \dots, \xi_m)[t]$.

In general proving that a polynomial in several variables is irreducible is not an easy task, few general methods are available and none of these seems to apply to our case. For a given polynomial with integer coefficients there exist reasonable computer algebra algorithms to test irreducibility but this is not a practical method in our case where the polynomials are infinite and their degrees also tend to infinity. Fortunately the combinatorics comes to our help as follows. We start from one of the matrices describing the Hamiltonian for a block associated to a given graph Γ. If we set one of the parameters $\xi_i = 0$ it is easy to verify that the matrix specializes to a direct sum of smaller blocks of the same type for less parameters (cf. Corollary 8.3). This remark gives a powerful tool for induction. The characteristic polynomial specializes to the product of the characteristic polynomials of the blocks and, by induction, we may assume that these factors are irreducible. We thus obtain a factorization for the specialized polynomial.

We repeat the argument with a different variable obtaining a different specialization and a different factorization. It is possible that these two factorizations cannot arise from the same factorization of the given polynomial. If this happens we are sure that the polynomial we started with is irreducible. This is the method we follow in order to prove Theorem 1 and it is the content of Part 2.

Unfortunately this still requires a rather tedious and lengthy case analysis and a reduction to some basic cases which we treat by computer algebra algorithms.

The fact that the polynomials are distinct (cf. Lemma 9.2) is based by induction on the irreducibility theorem and it is relatively easy to prove.

There is another delicate point in this proof, in order for the induction to work we need to have a complete control on the graphs that may appear, which is not proved in [12] and which we do not know for $q > 1$. We need to know that the possible graphs satisfy a *geometric non-degeneracy* or *non resonance* restriction, given by Proposition 2.2. Precisely one of the presentations of our graphs is by describing the vertices as integral vectors (in $\mathbb{Z}^m$), then the non degeneracy condition is that these vectors are affinely independent. The possible graphs are obtained by associating to the combinatorial graphs a system of d linear and quadratic equations, in n variables, which depend on the tangential sites in a quadratic way, where $d+1$ is the number of vertices. The graph is thus admissible if and only if these equations have solutions in $\mathbb{Z}^n \backslash S$, this arithmetic analysis is too difficult to perform and we study wether they have solutions in $\mathbb{R}^n \backslash S$. The idea is that if these equations are independent then they can be at most n. In fact for a geometrically non degenerate graph the condition of independence is fulfilled when $d \leq n$, the case $d > n$ has been treated completely by methods of algebraic geometry in [12], in the same paper we proved only a partial result on degenerate graphs. Here, by restricting to the case $q = 1$, we are able to show that, for generic choices of S, a resonant graph gives a system which has no solutions in $\mathbb{R}^n \backslash S$. Note that a *resonance*, namely a relation between the vertices of the graph, implies a linear relation among the linear terms of the system of equations. Such a relation may correspond either to a relation on the equations or an incompatibility condition for the system. So first we reduce to minimal cases (only one resonance), and then we study those graphs for which the equations are generically compatible. This produces two cases, either the system has only solutions in S or only in $\mathbb{C}^n \backslash \mathbb{R}^n$, this concludes the proof.

The strategy follows these steps: first we reduce to the case of trees and describe the resonance in terms of edges (instead of vertices). Next we analyze in a combinatorial way all the possible minimal resonances (in this analysis the hypothesis $q = 1$ is essential). Then we prove that we can essentially reduce to those trees in which all the edges contribute to the resonance. Finally we show that such trees have at most two trivalent vertices (that is a vertex from which 3 edges originate), the other vertices have valency 1, 2. At this point one can deduce from the system a simple equation which has only solutions in S or only in $\mathbb{C}^n \backslash \mathbb{R}^n$ by inspection.

The proof of Proposition 2.2 is the content of Part 1, the proof we found is rather complex and takes a good 20 pages of detailed combinatorial analysis.

1.1.2. Dynamical consequences. From the fact that the characteristic polynomials of the matrix blocks are described through finitely many graphs we shall be able to show the existence of a *discriminant variety* also in the infinite dimensional setting and show:

COROLLARY 1.2. *There exists an algebraic hypersurface $\mathscr{A}$, in the space $\mathbb{R}^m$ of the parameters ξ, and a finite number of algebraic functions $\theta_i(\xi)$ homogeneous of degree 1 on the region $\mathbb{R}^m \setminus \mathscr{A}$, so that the eigenvalues of $Q := -\frac{1}{2}\mathrm{i}ad(N)$ on $F^{0,1}$ are of the form $n + \theta_j(\xi) + a(\xi)$, $a(\xi) = \sum_j n_j \xi_j$, $n_j \in \mathbb{Z}$, $\sum_j n_j = -1$, $n \in \mathbb{N}$. In particular the eigenvalues are all distinct and non-zero outside the countable union of hypersurfaces $\theta_i(\xi) - \theta_j(\xi) - a(\xi) \neq 0$ for all $i \neq j$ and $a(\xi)$.*

PROOF. We know that $F^{0,1}$ decomposes into the direct sum of infinitely many blocks corresponding to the connected components of the graph Λ_S defined in 2.12.

From Theorem 1 we have that the characteristic polynomials of the matrices $ad(N)$ in the various blocks are irreducible and distinct. In our case we have seen that, for two distinct blocks, this produces a non zero polynomial whose non vanishing is equivalent to the condition that the two blocks have distinct eigenvalues. In principle this gives countably many hypersurfaces. Since we know that our infinite list of matrices is obtained from a finite list by adding a scalar matrix of the form $(n + \sum_i n_i \xi_i)I$ we obtain a finite number of distinct algebraic function $\theta_i(\xi)$, outside an algebraic hypersurface $\mathscr{A}$, which are the eigenvalues of all the combinatorial blocks. The condition is $\theta_i(\xi) - \theta_j(\xi) - a(\xi) \neq 0$ for all $i \neq j$ and $a(\xi) = \sum_j n_j \xi_j$, $n_j \in \mathbb{Z}$, $\sum_j n_j = 0$. $\square$

In [11] we shall refine this Theorem by exhibiting a region of positive measure where the eigenvalues are explicitly bounded away from 0.

By construction of the matrix Q, real eigenvalues of Q correspond to imaginary eigenvalues of $ad(N)$. We have seen that outside a real hypersurface the eigenvalues of all the combinatorial blocks are distinct. Thus outside this hypersurface the cone of the ξ_i decomposes into open regions where the number of real roots is constant. We can furthermore show (see §2.14.1) that

PROPOSITION 1.3. *The open region where all the eigenvalues of Q are real is non empty in R_+^m.*

As a consequence of Proposition 1.2 one easily sees that one can perform a symplectic coordinate change so that the Hamiltonian is in *diagonal canonical form*, that is we have an infinite sum $\sum_k \theta_k |z_k|^2$ corresponding to the real eigenvalues, plus a (possibly empty, depending on the connected region of $O_\delta \setminus \mathscr{A}$ where O_δ is a small hypercube), finite sum of hyperbolic terms corresponding to the complex eigenvalues. Then Proposition 1.3 ensures that on an open region of parameters the Hamiltonian is diagonal and elliptic.

REMARK 1.4. No knowledge of the NLS is necessary in order to understand the Theorems of this paper which may be formulated as purely geometric questions.

REMARK 1.5. We should remark that only finitely many of the infinite blocks are not self adjoint matrices. If one restricts the analysis to the self adjoint blocks the proofs simplify drastically, in particular this is true for the first part which admits a far reaching generalization (cf. Theorem 3).

REMARK 1.6. The restriction to $q = 1$ plays a major role in both parts of the paper. However for any q and dimension $n = 1$ all the results of this paper have been proved in the Ph. D. Thesis of Nguyen Bich Van.

REMARK 1.7. In general ($q > 1, n > 1$) although we do not know that the eigenvalues are distinct we can use a *Fitting decomposition* with blocks corresponding to distinct eigenvalues. It turns out that these blocks are uniformly bounded for generic S.

REMARK 1.8. In Proposition 1.3 we have pointed out the existence of an elliptic region. It is easy to exhibit large regions where there are complex eigenvalues, which however can be at most a finite number bounded by a function of n, m.

2. Preliminaries

We start by presenting an elementary geometric problem which originates from the NLS but can be explained and treated in a completely independent way. Then we briefly describe the NLS normal form and show the origin and importance of the geometric problem in this context.

2.1. An elementary geometric problem. Given a point p in a sphere in Euclidean space $\mathbb{R}^n$ we can consider its *antipode* or *mirror* point p'. A similar construction holds in the case of two parallel hyperplanes H_1, H_2. Given a point p in one of them, say for instance H_1, we can construct a *mirror point* $p' \in H_2$ by drawing the line r perpendicular to H_1 through p and taking as p' the point of intersection between r and H_2. If we have several spheres $S_1, \dots, S_a$ and pairs of parallel hyperplanes $(H_1^1, H_2^1), \dots, (H_1^b, H_2^b)$ we have, for a point in the intersection of h such hypersurfaces, h mirror points. Each of them in turn could have several mirror points. The combinatorics resulting is encoded by a 2-colored graph, having as vertices the points of $\mathbb{R}^n$ and two types of edges; the edges colored black represent mirror pairs in parallel hyperplanes while edges colored red represent antipode points in one of the spheres. The edges are understood as purely combinatorial and not as segments of $\mathbb{R}^n$. The combinatorics of this graph can be extremely complicated and reflects partially the complex relative positions of all the given hypersurfaces.

In our case a configuration of previous type is associated to a set S (the tangential sites) as follows: given two distinct elements $v_i, v_j \in S$ construct the sphere $S_{i,j}$ having the two vectors as opposite points of a diameter and the two Hyperplanes, $H_{i,j}$, $H_{j,i}$, passing through v_i and v_j respectively, and perpendicular to the line though the two vectors v_i, v_j.

From this configuration of spheres and pairs of parallel hyperplanes we deduce, by the previous rules, a *combinatorial colored graph*, denoted by Γ_S, with vertices the points in $\mathbb{R}^n$ and two types of edges, which we call *black* and *red*.

- A black edge connects two points $p \in H_{i,j}$, $q \in H_{j,i}$, such that the line p, q is orthogonal to the two hyperplanes, or in other words $q = p + v_j - v_i$.
- A red edge connects two points $p, q \in S_{i,j}$ which are opposite points of a diameter $(p + q = v_i + v_j)$.

The Problem The problem consists in the study of the connected components of this graph. Of course the nature of the graph depends upon the choice of S but one expects a relatively simple behavior for S *generic*.

It is immediate by the definitions that the points in S are all pairwise connected by black and red edges and it is not hard to see that, for generic values of S, the set S is itself a connected component which we call the *special component*.

What we expect to have, as explained in §3.2 and proved in Part 1, is:

PROPOSITION 2.2. *For generic choices of S the connected components of this graph, different from the special component, are formed by affinely independent points.*

In particular each component has at most $n+1$ points.

In the next paragraph we explain how this problem arises in the NLS. The NLS considered in [12] depend upon an integer parameter q but here we concentrate in the simplest case when $q = 1$, which is connected to the previous geometric problem, and we have the *cubic NLS* the remaining cases are essentially open.

2.3. Some background. The cubic NLS on a torus is a Hamiltonian system, the symplectic variables are the Fourier coefficients of the functions $u(\varphi) := \sum_{k \in \mathbb{Z}^n} u_k e^{\mathrm{i}(k,\varphi)}$, the symplectic structure is $\mathrm{i} \sum_{k \in \mathbb{Z}^n} du_k \wedge d\bar{u}_k$ and the Hamiltonian is

$$H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k \pm \sum_{k_i \in \mathbb{Z}^n : \sum_{i=1}^4 (-1)^i k_i = 0} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}. \tag{3}$$

We shall choose the sign $+$ for simplicity of notations. We perform a step of "Resonant Birkhoff normal form". Denote by $K := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k$. A monomial $\prod_i u_{k_i}^{\alpha_i} \bar{u}_{k_i}^{\beta_i}$ in the u_k, $\bar{u}_k$ is an eigenvector for $\{K, -\}$ of eigenvalue $\sum_i (\alpha_i - \beta_i)|k_i|^2$ and such a step is a symplectic change of variables under which we cancel all or some of the quartic terms which do not Poisson commute with K, to the cost of introducing higher order terms which are then treated as *a perturbation*. The condition of commuting with K is $\sum_{i=1}^4 (-1)^i |k_i|^2 = 0$. Dropping the perturbation one has a *restricted model*.

$$H := \sum_{k \in \mathbb{Z}^n} |k|^2 u_k \bar{u}_k + \sum_{\substack{k_i \in \mathbb{Z}^n : \sum_{i=1}^4 (-1)^i k_i = 0, \\ \sum_{i=1}^4 (-1)^i |k_i|^2 = 0}} u_{k_1} \bar{u}_{k_2} u_{k_3} \bar{u}_{k_4}. \tag{4}$$

Note that the two conditions $\sum_{i=1}^4(-1)^i k_i = 0$, $\sum_{i=1}^4(-1)^i|k_i|^2 = 0$ have a geometric interpretation, that is the four points k_1, k_2, k_3, k_4 are the vertices of a *rectangle*.

As it is well known (cf. Colliander-Tao [6] and Grébert-Thomann [7]) this restricted model admits infinitely many invariant subspaces defined by requiring $u_k = 0$ for all $k \notin S$ where $S = \{v_1, \dots, v_m\}$, *tangential sites*, is some (arbitrarily large) subset of $\mathbb{Z}^n$ satisfying a *completeness condition* (cf. [12], 2.1.1). The dynamics on these subspaces depends in a subtle way on the geometric properties of S and, for generic choices of S the behavior is integrable (cf. [12], Proposition 1). In order to understand how to pass from solutions of the restricted model to true solutions of the NLS one has to have some structural stability result that is, as we explained before, control of the dynamics on the normal bundle to the family of invariant tori in the given invariant subspace. In coordinates we set

$$
\begin{aligned}
(5)\qquad & u_k := z_k \quad \text{for } k \in S^c,\\
& u_{v_i} := \sqrt{\xi_i + y_i}e^{\mathrm{i}x_i} = \sqrt{\xi_i}\Big(1 + \frac{y_i}{2\xi_i} + \cdots\Big)e^{\mathrm{i}x_i} \quad \text{for } i = 1, \dots m,
\end{aligned}
$$

considering the $\xi_i > 0$ as parameters, with $|y_i| < \xi_i$, while $y, x, w := (z, \bar z)$ are dynamical variables. In these variables the Hamiltonian can be decomposed as

$$
H \circ \Phi_\xi = (\omega(\xi), y) + \sum_{k \in S^c} |k|^2|z_k|^2 + \mathcal{Q}(\xi, x, w) + P(\xi, y, x, w) = N + P.
$$

Where $N := (\omega(\xi), y) + \sum_{k \in S^c}|k|^2|z_k|^2 + \mathcal{Q}(\xi, x, w)$, with $\mathcal{Q}(\xi, x, w)$ quadratic, is the *normal form* and P the *perturbation*.

We use systematically the fact that this Hamiltonian commutes with *momentum M* and *mass L*:

$$
(6)\quad M = \sum_i \xi_i v_i + \sum_i y_i v_i + \sum_{k \in S^c} k|z_k|^2, \quad L = \sum_i \xi_i + \sum_i y_i + \sum_{k \in S^c} |z_k|^2,
$$

We have, after some renormalizing, $\omega_i(\xi) := |v_i|^2 - 2\xi_i$. Finally the quadratic form is

$$
\begin{aligned}
(7)\quad \mathcal{Q}(\xi, x, w) = {}& 4 \sideset{}{^*}\sum_{\substack{1 \le i \ne j \le m\\ h,k \in S^c}} \sqrt{\xi_i\xi_j}e^{\mathrm{i}(x_i - x_j)} z_h \bar z_k + 2 \sideset{}{^{**}}\sum_{\substack{1 \le i < j \le m\\ h,k \in S^c}} \sqrt{\xi_i\xi_j}e^{-\mathrm{i}(x_i + x_j)} z_h z_k\\
& + 2 \sideset{}{^{**}}\sum_{\substack{1 \le i < j \le m\\ h,k \in S^c}} \sqrt{\xi_i\xi_j}e^{\mathrm{i}(x_i + x_j)} \bar z_h \bar z_k.
\end{aligned}
$$

Here $\sum^*$ denotes that (h, k, v_i, v_j) satisfy:

$$
\{(h, k, v_i, v_j) \mid h + v_i = k + v_j, |h|^2 + |v_i|^2 = |k|^2 + |v_j|^2\}.
$$

and $\sum^{**}$, that (h, v_i, k, v_j) satisfy:

$$\{(h, v_i, k, v_j) \mid h + k = v_i + v_j, |h|^2 + |k|^2 = |v_i|^2 + |v_j|^2\}.$$

Notice that in the sums $\sum^{**}$ each term appears twice. These constraints describe exactly the two types of rectangles in which two vertices lie in S and the others in S^c, thus these last two vertices are joined, by definition, by a black edge in the first case (in which they are vertices of a side of the rectangle) and a red in the second (in which they are opposite vertices of the rectangle). Note that the edges correspond to interacting sites.

We have described a very complicated infinite dimensional quadratic Hamiltonian which we wish to decompose into infinitely many decoupled finite dimensional blocks, corresponding to the components of the geometric graph Γ_S defined in the previous paragraph. In [12] we show that this is possible and we also proved the existence of a symplectic change of variables which makes the angles disappear.

2.4. The operator $ad(N)$.

Definition 2.5. Denote by $\mathbb{Z}^m := \{\sum_{i=1}^m a_i e_i, a_i \in \mathbb{Z}\}$ the lattice with basis the elements e_i.

Consider the *mass* η and the *momentum* π (the name comes from dynamical considerations):

$$\eta : \mathbb{Z}^m \to \mathbb{Z}, \quad \eta(e_i) := 1, \quad \pi : \mathbb{Z}^m \to \mathbb{Z}^n, \quad \pi_S = \pi : e_i \mapsto v_i.$$

At this point it is useful to formalize the idea of *energy transfer* in a combinatorial way. Let $S^2[\mathbb{Z}^m] := \{\sum_{i,j=1}^m a_{i,j} e_i e_j\}$, $a_{i,j} \in \mathbb{Z}$ be the polynomials of degree 2 in the e_i with integer coefficients. We extend the map π and introduce a linear map from $\mathbb{Z}^m$ to $S^2(\mathbb{Z}^m)$ denoted $a \mapsto a^{(2)}$ as:

$$\pi(e_i) = v_i, \quad \pi(e_i e_j) := (v_i, v_j), \quad *^{(2)} : \mathbb{Z}^m \to S^2(\mathbb{Z}^m), \quad e_i \mapsto e_i^2. \tag{8}$$

We have $\pi(AB) = (\pi(A), \pi(B))$, $\forall A, B \in \mathbb{Z}^m$.

Remark 2.6. Notice that we have $a^{(2)} = a^2$ if and only if a equals 0 or one of the variables e_i.

2.6.1. The space $F^{0,1}$. We start from the space $V^{0,1}$ of functions with basis the elements

$$e^{\mathrm{i}\sum_j v_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j v_j x_j} \bar{z}_k, \quad k \in S^c.$$

In this space the conditions of commuting with momentum, resp. with mass select the elements, called *frequency basis*

$$(9) \qquad F_B = e^{\mathrm{i}\sum_j \nu_j x_j} z_k, \quad e^{-\mathrm{i}\sum_j \nu_j x_j} \bar{z}_k; \quad k \in S^c$$

$$\sum_j \nu_j v_j + k = \pi(\nu) + k = 0, \quad \text{resp.} \quad \sum_j \nu_j + 1 = 0.$$

Denote by $F^{0,1}$ the subspace of $V^{0,1}$ commuting with momentum and mass.*

An element of F_B is completely determined by the value of ν and the fact that the z variable may or may not be conjugated. By construction $\nu \in \mathbb{Z}_c^m$ where

$$(10) \qquad \mathbb{Z}_c^m := \{\mu \in \mathbb{Z}^m \,|\, -\pi(\mu) \in S^c\}.$$

Denote by $\Theta \subset \mathbb{Z}^m$ the kernel of $\pi_S : e_i \mapsto v_i$ then, by Formula (10), we have $\mathbb{Z}_c^m = \mathbb{Z}^m \setminus \bigcup_i -e_i + \Theta$.

Now $ad(N)$ acts on $F^{0,1}$, its matrix representation, in the frequency basis, decomposes into infinitely many finite dimensional blocks described by matrices with coefficients quadratic polynomials in the variables $\sqrt{\xi_i}$. One easily sees that in the characteristic polynomial of each one of these matrices the square roots disappear (Lemma 2.14).

2.7. The Cayley graphs. We recall how we have found useful to cast some of the description of the operator $ad(N)$ into the language of group theory and in particular of the *Cayley graph* (cf. [9]). In fact to a matrix $C = (c_{i,j})$ we can always associate a graph, with vertices the indices of the matrix, and an edge between i, j if and only if $c_{i,j} \neq 0$. For the matrix of $ad(N)$ in the frequency basis the relevant graph comes from a special Cayley graph.

Let G be a group and $X = X^{-1} \subset G$ a subset.

DEFINITION 2.8. An X-marked graph is an oriented graph Γ such that each oriented edge is marked with an element $x \in X$.

$$a \xrightarrow{x} b \qquad a \xleftarrow{x^{-1}} b$$

We mark the same edge, with opposite orientation, with x^{-1}. Notice that if $x^2 = 1$ we may drop the orientation of the edge.

A typical way to construct an X-marked graph is the following. Consider an action $G \times A \to A$ of G on a set A, we then define.

DEFINITION 2.9 (Cayley graph). The graph A_X has as vertices the elements of A and, given $a, b \in A$ we join them by an oriented edge $a \xrightarrow{x} b$, marked x, if $b = xa$, $x \in X$.

In our setting the relevant group is the group of transformations of $\mathbb{Z}^m$ generated by translations $a : x \mapsto x + a$ and *sign change* $\tau : x \mapsto -x$. Thus $G := \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ is the *semidirect product*, and $\tau := (0, -1)$, $G = \mathbb{Z}^m \cup \mathbb{Z}^m\tau$ and the

* this convention is different from [12] where we only impose commutation with momentum

product rule is $a\tau = -\tau a$, $\forall a \in \mathbb{Z}^m$ (notice that this implies $(a\tau)^2 = (0,1)$). We think of an element $a = e^{\mathrm{i}\sum_j v_j x_j} z_k$ as being associated to the group element which, by abuse of notation, we still denote by $a = \sum_j v_j e_j \in \mathbb{Z}^m$. Then $\bar{a} = e^{-\mathrm{i}\sum_j v_j x_j} \bar{z}_k$ is associated to the group element $a\tau = \left(\sum_j v_j e_j\right)\tau \in \mathbb{Z}^m\tau$.

Thus the frequency basis is indexed by elements of $G^1 \setminus \bigcup_i \{-e_i + \Theta, (-e_i + \Theta)\tau\}$ where

$$G^1 := \{a, a\tau, a \in \mathbb{Z}^m \mid \eta(a) = -1\}.$$

We now consider the Cayley graph G_X of G with respect to the elements

$$X^0 := \{e_i - e_j, i \neq j \in [1, \dots, m]\}, \quad X^{-2} := \{(-e_i - e_j)\tau, i \neq j \in [1, \dots, m]\}.$$

If $p \in \mathbb{Z}$ it is easily seen that the set $G_p := \{a, \eta(a) = 0, a\tau \mid \eta(a) = p\}$ form a subgroup. In particular

Remark 2.10. G_{-2} is generated by the elements $X := X^0 \cup X^{-2}$, its right cosets are the connected components of the Cayley graph.

In the action of G_{-2} on $\mathbb{Z}^m$ the orbit of 0 is identified to G_{-2} and it is formed by the elements $a \in \mathbb{Z}^m \mid \eta(a) \in \{0, -2\}$. We can thus identify the Cayley graph on G_{-2} with the corresponding graph on this set of elements.

We distinguish the edges by *color*, as X^0 to be *black* and X^{-2} *red*, hence the Cayley graph is accordingly colored; by convention we represent red edges with an unoriented double line: $g = (-e_i - e_j)\tau$, $a \overset{g}{=} ga$ (recall that $g = g^{-1}$).

The set G^1 is also a right coset of G_{-2} and thus it is also a connected component of the Cayley graph G_X.

2.10.1. The matrix structure of $ad(N) := 2\mathrm{i}Q$. This is encoded in part by the Cayley graph G_X of G with respect to the elements $X := \{e_i - e_j, (-e_i - e_j)\tau\}$.

Given $a = \sum_i a_i e_i$, $\sigma = \pm 1$ set for $u = (a, \sigma)$

$$C((a,\sigma)) := \frac{\sigma}{2}(a^2 + a^{(2)}) = \frac{\sigma}{2}\left(\left(\sum_i a_i e_i\right)^2 + \sum_i a_i e_i^2\right), \tag{11}$$

$$K((a,\sigma)) := \pi(C(u)) = \frac{\sigma}{2}\left(\left|\sum_i a_i v_i\right|^2 + \sum_i a_i |v_i|^2\right).$$

Sometimes we call $K(u)$ the *quadratic energy* of u, notice that $C(u)$ has integer coefficients. In particular if $a \in \mathbb{Z}^m_c$ we have $K(a\tau) = -K(a)$ and we set for $a, b \in \mathbb{Z}^m_c$

$$Q_{a,a} = K(a) - \sum_j a_j \xi_j, \quad Q_{a\tau, a\tau} = K(a\tau) + \sum_j a_j \xi_j \tag{12}$$

$$Q_{a\tau, b\tau} = -2\sqrt{\xi_i \xi_j}, \quad Q_{a,b} = 2\sqrt{\xi_i \xi_j}, \tag{13}$$

if a, b are connected by an edge $e_i - e_j$

$$Q_{a,b\tau} = -2\sqrt{\xi_i\xi_j}, \quad Q_{a\tau,b} = 2\sqrt{\xi_i\xi_j}, \tag{14}$$
$$\text{if } a,\ b\tau \text{ are connected by an edge } (-e_i - e_j)\tau$$

We have shown in [12] that the blocks Q on $F^{0,1}$ come into pairs of conjugate Lagrangian blocks Γ, $\Gamma\tau$. With respect to the frequency basis the blocks are described as the connected components of a graph Λ_S which we now describe.

Definition 2.11. Given an edge $u \xrightarrow{x} v$, $u = (a, \sigma)$, $v = (b, \rho) = xu$, $x \in X_q$, we say that the edge is *compatible* with S or π if $K(u) = K(v)$.

Remark now that, if $g \in G$ we have $C(g) = 0$ if and only if $g = -e_i, -e_i\tau$. We call the elements $\{-e_i, -e_i\tau\}$ the *special component*.

Definition 2.12. The graph Λ_S is the subgraph of G_X inside $G^1 \setminus \bigcup_i \{-e_i + \Theta, (-e_i + \Theta)\tau\}$ in which we only keep the compatible edges.

Observe that the graph Λ_S is invariant under translations by Θ. We then have

Theorem 2. *The indecomposable blocks of the matrix Q in the frequency basis correspond to the connected components of the graph Λ_S.*

In a block the entries of Q are given by (12), (13), (14).

The fact that in the graph Λ_S we keep only compatible edges implies in particular that the *scalar part* $K((a, \sigma))$ (which is an integer) is constant on each block. On the other hand, in general, there are infinitely many blocks with the same scalar part. It will be convenient to ignore the scalar term $diag(K((a, \sigma)))$, given a compatible connected component A we hence define the matrix $C_A = Q_A - diag(K((a, \sigma)))$.

One of the main ingredients of our work is to understand the possible connected components Γ of the graphs Λ_S for S generic (but not necessarily fixed), we do this by choosing a vertex $u \in \Gamma$ which we call the *root* and analyzing such a component as a translation $\Gamma = Au$ where A is now a complete subgraph of the Cayley graph contained in G_{-2} and containing the element $(0, +) = 0$. If $u \in \mathbb{Z}^m$ the matrix C_{Au} is obtained from C_A by adding the scalar matrix $-u(\xi) = -(u, \xi)$ while $C_{A\tau} = -C_A$.

Example 2.13. Consider the following complete subgraph containing $(0, +)$.

$$A = (-e_1 - e_2, -) \overset{(-e_1-e_2)\tau}{=\!=\!=\!=} (0, +) \xrightarrow{e_1 - e_2} (e_1 - e_2, +).$$

A translation by an element $(u, +)$ is hence

$$A(u, +) = (-e_1 - e_2 - u, -) \overset{(-e_1-e_2)\tau}{=\!=\!=\!=} (u, +) \xrightarrow{e_1 - e_2} (e_1 - e_2 + u, +)$$

so we get that the matrices associated to these graphs are:

$$C_A = \begin{pmatrix} -\xi_1 - \xi_2 & 2\sqrt{\xi_1\xi_2} & 0 \\ -2\sqrt{\xi_1\xi_2} & 0 & 2\sqrt{\xi_1\xi_2} \\ 0 & 2\sqrt{\xi_1\xi_2} & \xi_2 - \xi_1 \end{pmatrix},$$

$$C_{Au} = \begin{pmatrix} -\xi_1 - \xi_2 - u(\xi) & 2\sqrt{\xi_1\xi_2} & 0 \\ -2\sqrt{\xi_1\xi_2} & -u(\xi) & 2\sqrt{\xi_1\xi_2} \\ 0 & 2\sqrt{\xi_1\xi_2} & \xi_2 - \xi_1 - u(\xi) \end{pmatrix}$$

In particular we have shown (cf. [12], §9) that A can be chosen among a finite number of graphs which we call *combinatorial*. Note that we do not impose the compatibility constraint on A but only on its translations. It is convenient, in drawing the graphs to drop the labels on the edges since they can be deduced from the vertices. In a combinatorial graph the color of a vertex is black if its mass is 0 and red if it is -2. Then in the vertices we drop the sign $\pm$, since this information can be deduced from the mass or from the parity (number of red edges) of the path connecting the vertex with the root. So the graph of the previous example will be denoted by:

$$A = -e_1 - e_2 = 0 \to e_1 - e_2.$$

Note that in all the combinatorial graphs the root is by convention set to 0.

Let us show that:

LEMMA 2.14. *The characteristic polynomial of a matrix C_A is in $\mathbb{Z}[\xi_1, \dots, \xi_m, t]$ (the square roots disappear).*

PROOF. By definition the determinant of an $n \times n$ matrix with entries $a_{i,j}$ is the sum with sign, over all permutations σ of the n indices, of the products $a_{1,\sigma(1)} \dots a_{n,\sigma(n)}$. It is convenient to rearrange this product using the cycle structure of σ, each cycle $(i_1, \dots, i_k)$ determines a factor $a_{i_1,i_2} \dots a_{i_k,i_1}$. Let us show that in each of these factors the square roots disappear. In fact, if the cycle is reduced to a single element it corresponds to a diagonal entry, which has no roots. Otherwise it corresponds to a sequence of edges forming a closed path. Then, by the definitions and compatibility, one sees that each index appearing in the edges appears an even number of times in such a closed path, hence the claim follows from the formula $\pm 2\sqrt{\xi_i\xi_j}$ of the entry corresponding to each edge. □

2.14.1. Proof of Proposition 1.3. We are ready to prove Proposition 1.3:

PROOF. We proceed by induction on the number m of the parameters, for $m = 1$ the statement is trivial, so assume the statement is true for $m - 1$ parameters. Let Γ be one of the combinatorial graphs, $A(\Gamma)$ the corresponding matrix and $(a_1, \dots, a_k)$ the vertices of Γ.

Let $\bar{A}$ be the matrix obtained from $A(\Gamma)$ by setting $\xi_m = 0$. We claim that this matrix is the one associated to the not necessarily connected colored graph $\overline{\Gamma}$ in $m-1$ coordinates obtained by dropping the last coordinate in all the vertices a_i, this is just a consequence of the definitions (see §2.4).

The first thing to be verified is that the vertices of $\overline{\Gamma}$ are all distinct (as colored vertices). In fact given a vertex $a \in \mathbb{Z}^m$ let $\bar{a} \in \mathbb{Z}^{m-1}$ be the vertex obtained by dropping the last coordinate a_m. We can reconstruct a from $\bar{a}$ and its color using the mass since $\eta(a) = \eta(\bar{a}) + a_m$.

Now we claim that the graphs appearing give characteristic polynomials which are distinct, for this we apply Proposition 9.2. If we had two connected components of $\overline{\Gamma}$ giving the same characteristic polynomial we should have two elements $\bar{a}$ black and $\bar{b}$ red so that $\bar{b} = \tau\bar{a} = -\bar{a}\tau$ red. We have $a = \bar{a} - \eta(\bar{a})e_m$ while $\tau\bar{a} = -\bar{a}\tau$ comes from $b = (-\bar{a} + (\eta(\bar{a}) - 2)e_m)\tau = (-a - 2e_m)\tau$. Thus in the graph Γ we cannot have these two vertices, since the presence of two vertices $b + a = -2e_m$ implies that the graph is not allowable by Definition 3.13.

Now we apply the fact that we know that all the blocks appearing in $A(\overline{\Gamma})$ are distinct and depend on $m-1$ variables, furthermore two different blocks have different characteristic polynomials by the previous remark and Lemma 9.2. From the hypotheses made there is an open region $\mathscr{B}_{m-1}$ in the complement of the discriminant variety for $m-1$ variables where for each of the finitely many combinatorial blocks all the eigenvalues are distinct and real.

Now this condition is stable so that for $A(\Gamma)$ there is a non empty open region complement of the discriminant variety for $A(\Gamma)$ where all the eigenvalues are distinct and real containing $\mathscr{B}_{m-1}$, since we have finitely many combinatorial graphs Γ we find an open component of the complement of the discriminant variety for all graphs Γ, containing $\mathscr{B}_{m-1}$, where all the eigenvalues are real. We further remove the resultants and have that they are also all distinct. $\square$

Part 1. Sphere and hyperplanes problem

In order to understand the possible components of the graph Λ_S we relate it to the geometric graph Γ_S.

3. The geometric problem

The condition for two points p, q to be the vertices of an edge is given by algebraic equations. Visibly $p \in H_{i,j}$ means that $(p - v_i, v_i - v_j) = 0$, the corresponding $q = p + v_j - v_i$, while $p \in S_{i,j}$ is given by $(p - v_i, p - v_j) = 0$ and the corresponding opposite point q is given by $p + q = v_i + v_j$.

We thus have two types of constraints describing when two points are joined by an edge, a linear $q - p = v_j - v_i$ or $p + q = v_i + v_j$ and a quadratic constraint $(p - v_i, v_i - v_j) = 0$ or $(p - v_i, p - v_j) = 0$. The fact that a point x belongs to a component described by the combinatorial graph is thus expressed by a list of linear and quadratic equations for x deduced by eliminating all the other vertices using the linear constraints.

We describe the linear constraints again through a Cayley graph. The group G also acts on $\mathbb{R}^n$ by setting

$$ak := -\pi(a) + k, \quad k \in \mathbb{R}^n, a \in \mathbb{Z}^m, \quad \tau k = -k \tag{15}$$

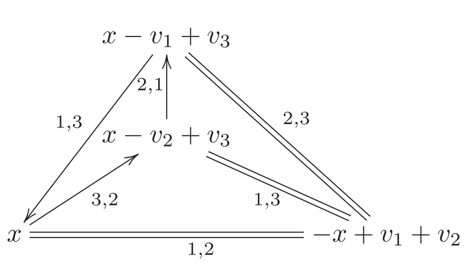
We then have that

REMARK 3.1. X defines also a Cayley graph on $\mathbb{R}^n$ and in fact the graph Γ_S is a subgraph of this graph.

3.2. Equations for the root. From the very construction of the graph it is convenient to *mark* the edges by $v_j - v_i$ in the first case and $v_j + v_i$ in the second (notice the sign change due to Formula (9)). In fact we use a more combinatorial way of marking which is illustrated in the next example. It is then clear that each connected component of this graph has a combinatorial description which encodes the information on the various types of edges which connect the vertices of the component.

The connection with the graph Λ_S comes from the fact that these equations are exactly the ones which define compatible edges.

EXAMPLE 3.3. The equations that x has to satisfy are:



$$(x, v_2 - v_3) = |v_2|^2 - (v_2, v_3)$$

$$|x|^2 - (x, v_1 + v_2) = -(v_1, v_2)$$

$$(x, v_1 - v_3) = |v_1|^2 - (v_2, v_3)$$

In fact it should be clear that a graph in Γ_S is obtained starting from a point x and then applying the elements of a complete sub graph $A \subset G_X$ of the Cayley graph containing 0. One the results of [12] (Theorem 3) is that in this fashion we have always isomorphisms between components of Λ_S and components of Γ_S.

The question is thus to understand when, given $x \in \mathbb{R}^n$, the elements hx, $h \in A$ describe the vertices of a corresponding geometric graph with *root* x in Γ_S.

One can easily verify that

PROPOSITION 3.4. *The elements hx, $h \in A$ describe the vertices in a component C of the geometric graph Γ_S if and only if, for each $h = (a, \sigma) \in A$ we have:*

$$\begin{cases} (x, \pi(a)) = K(h) & \text{if } \sigma = 1 \\ |x|^2 + (x, \pi(a)) = K(h) & \text{if } \sigma = -1 \end{cases}. \tag{16}$$

Therefore the question that we have to address is: for which graphs $A \subset \Gamma_X$ we can say that these equations have a solution in $\mathbb{R}^n \backslash S$ for generic values of the points v_i? Such a graph is called *compatible*.

A main result in [12] is that if the edges of the combinatorial graph span a lattice of dimension $> n$ then the only geometric realizations of this graph can be in the special component S.

It remains to analyze graphs with linearly dependent edges. In order to address this question we need to develop a more combinatorial approach.

3.5. Relations. Take a connected complete subgraph A, in the subgroup G_{-2} of G generated by X, of the Cayley graph G_X. By taking the first coordinates we identify its vertices with a subset, still denoted by A, of the set of elements in $\mathbb{Z}^m$ with $\eta(a) = 0, -2$ (the orbit of 0 under G_{-2}).

Definition 3.6.

- A graph A with $k+1$ vertices is said to be of *dimension* k.
- We call the dimension of the affine space spanned by A in $\mathbb{R}^m$ the *rank*, $\operatorname{rk} A$, of the graph A.
- If the rank of A is strictly less than the dimension of A we say that A *is degenerate*.

Once we choose a root r for A we can translate A so that $r = 0$ then instead of the affine space spanned by A we may consider the lattice spanned by the nonzero elements in A, it is natural to color all remaining vertices with the rule that a vertex a is *black* if $\eta(a) = 0$ or, equivalently, it is joined to the root by an even path and *red* otherwise. if $\eta(a) = -2$. Then we can extend the notion of *black* or *red* rank, and corresponding degeneracy. When we change the root we have a simple way of changing colors that we leave to the reader and the two ranks may just be exchanged.

If A is degenerate then there are non trivial relations, $\sum_a n_a a = 0$, $n_a \in \mathbb{Z}$ among the elements $a \in A$.

Remark 3.7. It is also useful to choose a maximal tree T in Γ. There is a triangular change of coordinates from the vertices a to the markings of T. Hence the relation can be also expressed as a relation between these markings.

We must have by linearity, for every relation $\sum_a n_a a = 0$, $n_a \in \mathbb{Z}$ that $0 = \sum_a n_a a^{(2)}$, $0 = \sum_a n_a \pi(a)$ and moreover we have:

$$0 = \sum_{a, \mid \eta(a) = -2} n_a. \tag{17}$$

Applying Formula (16) we deduce that we must have, with $a = g_a x$ for all vertices a

$$\begin{aligned} \sum_a n_a K(g_a) &= 2\Big(x, \sum_a n_a \pi(g_a)\Big) + \Bigg[\sum_{a \mid \eta(a) = -2} n_a\Bigg](x)^2 \\ &= 2\Big(x, \sum_a n_a \pi(g_a)\Big) = 0. \end{aligned} \tag{18}$$

The expression $\sum_a n_a K(g_a)$ is a linear combination with integer coefficients of the scalar products (v_i, v_j). We can prevent the occurrence of the component Γ by imposing it as avoidable resonance. We need to formalize the setting.

Let us use for the elements of G in the subgroup G_2 just their coordinate $a \in \mathbb{Z}^m$, $\eta(a) \in \{0, -2\}$. Then we have $\sum_a n_a K(a) = \pi\big(\sum_a n_a C(a)\big)$ hence we easily deduce:

Proposition 3.8. *The equation* (18) *is a non trivial constraint if and only if* $\sum_a n_a C(g_a) \neq 0$. *In this case we say that the graph has an* avoidable resonance.

Corollary 3.9. *If we have an avoidable resonance of previous type associated to* Γ *then, for a generic choice of the* $S := \{v_i\}$, Γ *as no geometric realizations.*

The main Theorem on this topic proved in [12] is:

Theorem 3. *Given a compatible connected X-marked graph, with a chosen root and of rank k for a given color, then either it has exactly k vertices of that color or it produces an avoidable resonance.*

Proof. Let us recall the proof for convenience of our treatment. Assume by contradiction that we can choose $k+1$ distinct vertices $(a_0, a_1, \ldots, a_k)$, different from 0 of the given color so that we have a non trivial relation $\sum_i n_i a_i = 0$ and the elements a_i, $i = 1, \ldots, k$ are linearly independent. Set $n_a = n_i$, if $a = a_i$ and $n_a = 0$ otherwise. If all these vertices have sign $+$, we have $\sum_a n_a a^2 = 0$. Similarly, if they are have sign $-$ we have $-\sum_a n_a a = \sum_a n_a \sigma(a) a = 0$ and also $\sum_a n_a a^{(2)} = 0$ so again $\sum_a n_a a^2 = 0$.

We can consider thus the elements $x_i := a_i, i = 1, \ldots, k$ as new variables and then we write the relations $\sum_a n_a a = \sum_a n_a a^2 = 0$ as

$$0 = a_{k+1} + \sum_{i=1}^{k} p_i x_i, \;\Rightarrow\; \Big(\sum_{i=1}^{k} p_i x_i\Big)^2 + \sum_{i=1}^{k} p_i x_i^2 = 0.$$

Now $\sum_{i=1}^{k} p_i x_i^2$ does not contain any mixed terms $x_h x_k$, $h \neq k$ therefore this equation can be verified if and only if the sum $\sum_{i=1}^{k} p_i x_i$ is reduced to a single term $p_i x_i$, and then we have $p_i = -1$ and $a_0 = a_i$, a contradiction. $\square$

Unfortunately there are examples of unavoidable resonances as we shall discuss in the next paragraph.

3.10. Degenerate resonant graphs.

Definition 3.11. We say that a graph A is *degenerate-resonant*, if it is degenerate and, for all the possible linear relations $\sum_i n_i a_i = 0$ among its vertices we have also $\sum_i n_i C(a_i) = 0$.

What we claim is that a degenerate-resonant graph A has no geometric realizations outside the special component.

REMARK 3.12. One may easily verify that the previous condition, although expressed using a chosen root, does not depend on the choice of the root.

One of the obstacles we have is that the proof of Theorem 3 breaks down in general since in fact there are non trivial degenerate-resonant graphs, the simplest of them is the *minigraph*

$$\begin{tikzcd}[arrows=-] (-e_2+e_1) \arrow[r, equal] \arrow[d] & (-2e_1) \arrow[d] \\ 0 \arrow[r, equal] & (-e_2-e_1) \end{tikzcd} \qquad \begin{tikzcd}[arrows=-] (-e_2+e_1)+a \arrow[r, equal] \arrow[d] & (-2e_1)-a \arrow[d] \\ a \arrow[r, equal] & (-e_2-e_1)-a \end{tikzcd} \tag{19}$$

Relation is $(-e_2+e_1)-(-e_2-e_1)+(-2e_1)=0$, we have

$$C(-e_2+e_1)=e_1^2-e_1e_2, \quad C(-e_2-e_1)=-e_1e_2, \quad C(-2e_1)=-e_1^2$$
$$e_1^2-e_1e_2-(-e_1e_2)-e_1^2=0.$$

A more complex example is

$$\begin{tikzcd}[arrows=-] & & & e_2-e_3 \arrow[d, "e_2-e_3"] & \\ -3e_1+e_2 \arrow[r, equal, "-e_1-e_4"] & 2e_1-e_2-e_4 \arrow[r, "e_1-e_4"] & e_1-e_2 \arrow[r, "e_1-e_2"] & 0 \arrow[r, equal, "-e_2-e_3"] & -e_2-e_3 \end{tikzcd}.$$

What is common of these two examples is that in each there is a pair of vertices a, b, of distinct colors, with $a+b=-2e_i$ for some index i.

DEFINITION 3.13. We shall say that a connected graph G is *allowable* if there is no pair of vertices $a, c \in G$ with $ac^{-1}=c^{-1}a=(-2e_i,\tau)$, or $(-3e_i+e_j,\tau)$, otherwise it is *not allowable*.

We may assume $a \in \mathbb{Z}^m$ black and $c=b\tau$, $b \in \mathbb{Z}^m$ red. We then easily see that

PROPOSITION 3.14. *If a graph is not allowable then it has no geometric realization outside the special component* (*i.e. it is not compatible*).

PROOF. We write the quadratic equation (16), for a vertex x, corresponding to the root a, given by the vertex $b=-2e_i$. Since $C(-2e_i)=-e_i^2$, $K(-2e_i)=-|v_i|^2$ we get

$$0=|x|^2+(x,\pi(-2e_i))-K(-2e_i)=|x|^2-2(x,v_i)+|v_i|^2=|x-v_i|^2.$$

Hence the only real solution of $|x-v_i|^2=0$ is $x=v_i$. Then we apply Remark 15 of [12] where we have shown that the special component is an isolated component of the graph.

In the other case x is in a sphere whose square radius is $\pi(A)$

$$A = \frac{(-3e_i + e_j)^2}{4} + C(-3e_i + e_j) = -\frac{1}{4}[(-3e_i + e_j)^2 + 2(-3e_i^2 + e_j^2)]$$

$$= -\frac{1}{4}[9e_i^2 - 6e_ie_j + e_j^2 - 6e_i^2 + 2e_j^2] = -\frac{3}{4}[e_i - e_j]^2$$

clearly $\pi(A) = -\frac{3}{4}|v_i - v_j|^2 < 0,\ \forall v_i \neq v_j$. □

What we conjectured and shall prove in this paper is (cf. §5):

THEOREM 4. *A degenerate-resonant graph A is* not allowable *hence it has no geometric realizations outside the special component.*

From this Theorem Proposition 2.2 follows.

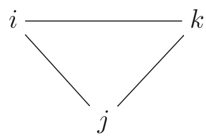
4. RESONANT GRAPHS

4.1. Encoding graphs. In order to understand relations, consider the complete graph T_m on the vertices $1, \ldots, m$. If we are given a marked graph Γ we associate to it the subgraph Λ of T_m, called its *encoding graph* in which we join the vertices i, j with a black edge if Γ contains an edge marked $e_j - e_i$ and by a red edge edge if Γ contains an edge marked $-e_j - e_i$. We mark $=$ the red edges.

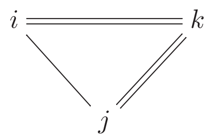
For each connected component of the encoding graph consider the subspace spanned by its edges. It is easily seen that these subspaces form a direct sum. Hence the encoding graph of a minimal relation is connected. Moreover a circuit in the encoding graph corresponds to a relation between the corresponding edges if and only if it contains an even number of red edges and we call it an *even circuit*.

This follows from the basic relations with which we can substitute two consecutive edges with a single one:

$$(e_i - e_j) + (e_j - e_k) + (e_k - e_i) = 0,$$



$$(e_i - e_j) - (-e_j - e_k) + (-e_k - e_i) = 0,$$



$$-2e_i = -(e_i - e_j) + (-e_i - e_j).$$

Thus for each index i of an odd circuit a sum, with coefficients ± 1, of its edges equals to $-2e_i$. The edges of an even circuit have a linear relation (unique up to sign) given by a sum with coefficients ± 1 equal 0. If we have a list of edges of Γ

which are linearly dependent and minimal (with respect to this property) then we claim that the corresponding elements in the encoding graph from a circuit, with some provisos due to the presence of red edges. More precisely we may have a *simple circuit* in which an even number of red edges appear or *two odd circuits* joined by a segment (possibly reduced to a point).

EXAMPLE 4.2. An even and a doubly odd encoding graph:

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, dash] & 10 \arrow[r, dash] & 9 \arrow[r, dash] & 8 \arrow[r, dash] & 7 \arrow[d, equal] \\
2 \arrow[r, equal] & 3 \arrow[r, equal] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6
\end{tikzcd}$$

$$\begin{tikzcd}
12 \arrow[r, dash] \arrow[d, dash] & 13 \arrow[r, equal] & 14 \arrow[r, dash] \arrow[d, dash] & 15 \arrow[r, equal] & 16 \arrow[d, dash] \\
11 \arrow[d, dash] & & 19 \arrow[r, dash] & 18 \arrow[r, dash] & 17 \\
1 \arrow[r, equal] \arrow[d, dash] & 10 \arrow[r, dash] & 9 \arrow[r, dash] & 8 \arrow[r, equal] & 7 \arrow[d, equal] \\
2 \arrow[r, equal] & 3 \arrow[r, equal] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6
\end{tikzcd}$$

This can be easily justified. Recall that the *valency* of a vertex is the number of edges which admit it as vertex. If the given edges give a minimal relation their encoding graph must be connected, furthermore it cannot have any vertex of valency 1 since the corresponding edge is clearly linearly independent from the others. Finally it cannot have more than 2 simple circuits otherwise we easily see that we have at least 2 relations.

For a connected graph the number c of independent circuits is the dimension of its first homology group and thus given, using the Euler characteristic, by $c = e - v + 1$ where e, v are the number of edges and vertices respectively. In our setting all vertices have valency ≥ 2 and we denote the valency of the vertex i by $V_i = v_i + 2$ (with $v_i \geq 0$). We have $2e = \sum_i V_i = \sum_i v_i + 2v$ so that we have $\sum_i v_i = 2c - 2$. If $c = 1$ the encoding graph is a simple circuit. If $c = 2$ we deduce that $\sum_i v_i = 2$ hence we have either only one vertex of valency 4 and the others of valency 2 or two vertices of valency 3 and the others of valency 2. The first case gives two loops joined in one vertex the second gives either two loops joined by a segment or two vertices joined by 3 segments. This last case is not possible since two of these segments will have the same parity and generate an even loop contradicting minimality.

4.3. Minimal relations. We want to study a minimal degenerate resonant graph Γ. Observe that for such a graph any proper subgraph is non-degenerate. In particular we have one and only one relation among the edges of a given maximal tree T in the graph and a corresponding relation for the vertices.

A minimal degenerate graph has a special type of relation which comes from the fact that in a maximal tree we have a minimum number of dependent edges. Such a situation arises when these edges, call their set $\mathscr{E}$, form in the encoding graph, a *even circuit* (where we allow the possibility that we have two odd circuits matching) as in the previous paragraph. Call $|\mathscr{E}|$ the subgraph of T formed by the edges $\mathscr{E}$, of course it need not be a priori connected but only a *forest* inside T.

In an even circuit the relation is a sum of edges $\sum_j \delta_j \ell_j = 0$, with signs $\delta_j = \pm 1$ in two odd matching circuits we may have some $\delta_j = \pm 2$ corresponding to the edges appearing in the segment connecting the two odd loops. In any case we list the edges appearing as ℓ_i. Each ℓ_i black is $\ell_i = a_i - b_i$ with a_i, b_i, its vertices of the same color while a red is $\ell_i = a_i + b_i$ with a_i red and b_i black its vertices.

The relation is thus

$$\sum_{i \text{ black}} \delta_i(a_i - b_i) + \sum_{j \text{ red}} \delta_j(a_j + b_j) = 0. \tag{20}$$

Notice that, by minimality, all the end points of T must be in $|\mathscr{E}|$. We may think of (20) as a formal relation on the vertices (instead of on the edges), note that a vertex in $\mathscr{E}$ need not appear in (20) however all end-points in $\mathscr{E}$ must appear and, if a vertex v has coefficient k in the relation, it must be the vertex of at least k of the given edges (in the case $\delta_i = \pm 1$).

4.3.1. Basic formulas. We work with G_{-2} identified with elements in $\mathbb{Z}^m$ either with $\eta(a) = 0$, *black* or $\eta(a) = -2$ *red*. We have set $C(a) = \frac{1}{2}(a^2 + a^{(2)})$ for a black and $C(a) = -\frac{1}{2}(a^2 + a^{(2)})$ for a red.

In our computations we use always the rules:

- for u, v black, we have $u + v$ black and

$$C(u+v) = \frac{1}{2}((u+v)^2 + (u+v)^{(2)}) = C(u) + C(v) + uv$$

- for u black v red, we have $u + v$ red and

$$C(u+v) = -\frac{1}{2}((u+v)^2 + (u+v)^{(2)}) = -C(u) + C(v) - uv$$

- for u, v red, we have $u - v$ black and

$$\begin{aligned} C(u-v) &= \frac{1}{2}((u-v)^2 + (u-v)^{(2)}) = \frac{1}{2}((u^2 + v^2 - 2uv + (u-v)^{(2)}) \\ &= \frac{1}{2}((u^2 + v^2 - 2uv + (u-v)^{(2)}) = -C(u) + C(v) + v^2 - uv \end{aligned}$$

- for u black, we have $-u$ black and

$$C(-u) = C(u) - u^{(2)}.$$

5. The resonance

5.1. The resonance relation. This chapter is devoted to the proof of Theorem 4. In order to prove it we take a minimal degenerate resonant graph Γ and inside it a maximal tree T and then we start studying it. In fact it would be possible to classify these trees, we arrive a little short of this since we need only to show 4.

5.1.1. Relations. Associated to T we have its encoding graph and the encoding graph of the edges $\mathscr{E}$ involved in the relation. We index the edges in the relation and set $\ell_i = \vartheta_i e_i - e_{i+1}$ where $\vartheta_i = \pm 1$ (depending on the color of the edge). As we explain in course of the proofs we will need to identify some vertices e_i.

We distinguish two cases, if the encoding graph of the relation is 1) an even or 2) a doubly odd loop. The simplest case to treat is case 1) which then suggests how to deal with the other cases.

Case 1. Up to changing notations we may assume that the loop is formed by the edges $\ell_i = \vartheta_i e_i - e_{i+1}$, $i = 1, \dots, k-1$, $\ell_k = \vartheta_k e_k - e_1$, (here we identify $e_1 = e_{k+1}$). Set

$$\delta_i := \prod_{j \le i} \vartheta_i = \vartheta_i \delta_{i-1},$$

we assume we have an even number of $\vartheta_i = -1$, by assumption $\delta_k = 1$.

We call δ_i the *parity* of i.

Lemma 5.2. *We have the relation:*

$$A = \sum_{i=1}^{k} \delta_i \ell_i = 0.$$

Proof. Consider an index $i > 1$, the coefficient of e_i in A is $-\delta_{i-1} + \delta_i \vartheta_i$. Since $\delta_i = \delta_{i-1}\vartheta_i$ for this e_i the coefficient is 0. For e_1 the coefficient comes from $\delta_1 \ell_1 + \delta_k \ell_k$, we have $\delta_1 = \vartheta_1$, $\delta_k = 1$ so we also get coefficient 0. □

Set $\zeta : \mathbb{Z}^m \to \mathbb{Z}$, $\zeta(e_i) = \delta_{i-1}$ (by convention $\delta_0 = 1$) so that, by linearity, $\zeta(\ell_i) = \vartheta_i \delta_{i-1} - \delta_i = 0$.

Lemma 5.3. *The ℓ_i span the codimension 1 subspace of the space $e_1, \dots, e_k$ formed by the vectors a such that*

$$a = \sum_i \alpha_i e_i \,|\, \zeta(a) = \sum_i \delta_{i-1} \alpha_i = 0. \tag{21}$$

Proof. $\zeta(\ell_i) = 0$, so the ℓ_i are in this subspace, but they span a subspace of codimension 1 hence the claim. □

Case 2. For a double loop with k edges, we have either one or two vertices in the encoding graph of valency > 2 separating the two odd loops, we call these vertices *critical*. We start from a odd loop and a critical vertex which we may assume to be 1. We call $A = \{1, \dots, h\}$ the indices in the first loop. We then list the edges $\ell_1, \dots, \ell_h$ in a circular way and

LEMMA 5.4. *We may choose the signs $\delta_i = \pm 1$ so that for any index $j \le h$ we have:*

$$\sum_{i=1}^{j} \delta_i \ell_i = -\delta_j e_{j+1} - e_1, \quad \sum_{i=1}^{h} \delta_i \ell_i = -2e_1, \quad \sum_{i=j+1}^{h} \delta_i \ell_i = -e_1 + \delta_j e_{j+1}. \tag{22}$$

PROOF. From the first Formula the others follow. We define $\delta_i = \vartheta_i \delta_{i-1}$ if $i > 1$ and set $\delta_1 = -\vartheta_1$. Then if $j = 1$, $\delta_1 \ell_1 = \delta_1(\vartheta_1 e_1 - e_2) = -e_1 - \delta_1 e_2$ and this follows from the definitions. By induction

$$\sum_{i=1}^{j+1} \delta_i \ell_i = -\delta_j e_{j+1} - e_1 + \delta_{j+1} \ell_{j+1}$$

$$-\delta_j e_{j+1} - e_1 + \delta_{j+1} \vartheta_{j+1} e_{j+1} - \delta_{j+1} e_{j+2} = -e_1 - \delta_{j+1} e_{j+2}. \qquad \square$$

For notational convenience we identify $e_{h+1} = e_1$. If we have two critical vertices, call $b \ge h+2$ the other, we have then a segment joining them formed by a string of elements $\ell_{h+1} = \vartheta_{h+1} e_{h+1} - e_{h+2}, \dots, \ell_{b-1} = \vartheta_{b-1} e_{b-1} - e_b$. We call B this set of indices and assign to these edges *signs* $\delta = \pm 2$ so that $\sum_{i=h+1}^{b-1} \delta_i \ell_i = \sum_{i \in B} \delta_i \ell_i = 2[e_1 + \vartheta e_b]$ where $\vartheta = 1$ if and only if this segment is odd.

We finish with the other odd loop, call C the corresponding set of indices and assign, as before, signs ± 1 so that $\sum_{i=b}^{k} \delta_i \ell_i = \sum_{i \in C} \delta_i \ell_i = -2\vartheta e_b$. With these choices the relation is

$$R := \sum_{i=1}^{k} \delta_i \ell_i = -2e_1 + 2[e_1 + \vartheta e_{b+1}] - 2\vartheta e_{b+1} = 0. \tag{23}$$

We have chosen the indices so that we order the edges as they occur in one way of *walking* the cycle, starting from the critical vertex 1. We say that an index is critical if the corresponding vertex is critical. Here $1, h+1, b$ are critical.

REMARK 5.5. The non critical indices are divided in 2 or 3 sets (depending if we have only one critical vertex or two). If u is not critical we have $\delta_u = \vartheta_u \delta_{u-1}$.

LEMMA 5.6. *The ℓ_i span the sublattice of the lattice spanned by $e_1, \dots, e_k$ formed by those vectors*

$$a = \sum_i \alpha_i e_i \,|\, \eta(a) = \sum_i \alpha_i \cong 0, \quad \textit{modulo } 2. \tag{24}$$

PROOF. $\eta(\ell_i) \cong 0$ modulo 2, so the ℓ_i are in this sub-lattice, the fact that they span is easily seen by induction. $\square$

5.6.1. Signs. We choose a root r in T and then each vertex x acquires a color $\sigma_x = \pm 1$. The color of x is red and $\sigma_x = -1$ if the path from the root to x has an odd number of red edges, the color is black and $\sigma_x = 1$ if the path is even.

An edge ℓ_i is connected to the root r by a unique path p_i ending with ℓ_i we denote its final vertex x_i and we set $\sigma_i := \sigma_{x_i}$. If ℓ_i is black we set $\lambda_i = 1$ if the edge is equioriented with the path, that is it points outwards, $\lambda_i = -1$ if it points inwards. Finally we set $\lambda_i = 1$ if the edge is red.

$$r \cdots\cdots \ldots \quad \xrightarrow{\ell_i} x \quad \lambda_i = 1, \qquad r \cdots\cdots \ldots \quad \xleftarrow{\ell_i} x \quad \lambda_i = -1. \tag{25}$$

DEFINITION 5.7. Once we choose a root in T, each red edge ℓ_i (that is $\vartheta_i = -1$) appears as edge with one end denoted by a_i red and the other denoted by b_i black, we have $\ell_i = a_i + b_i$. For a black edge $\vartheta_i = 1$ we define a_i, b_i so that instead $a_i = b_i + \ell_i$, and a_i, b_i have the same color. We thus write $\ell_i = a_i - \vartheta_i b_i$.

In particular for the resonant trees:

PROPOSITION 5.8.

$$\mathcal{R} := \sum_{i \mid \vartheta_i = -1} \delta_i(-a_i^{(2)} - \ell_i a_i + e_i e_{i+1}) + \sum_{i \mid \vartheta_i = 1} \delta_i \sigma_i(-e_{i+1}^2 + e_i e_{i+1} + \ell_i a_i) = 0. \tag{26}$$

$$\sum_{i \mid \vartheta_i = -1} \delta_i(b_i^{(2)} + \ell_i b_i - e_i e_{i+1}) + \sum_{i \mid \vartheta_i = 1} \delta_i \sigma_i(e_i^2 - e_i e_{i+1} + \ell_i b_i) = 0$$

PROOF. We start from the relation $\sum_i \delta_i \ell_i = 0$ and substitute the previous formulas, we deduce

$$R := \sum_{i \mid \vartheta_i = -1} \delta_i(a_i + b_i) + \sum_{i \mid \vartheta_i = 1} \delta_i(a_i - b_i) = 0. \tag{27}$$

We next have by the resonance hypothesis

$$\sum_{i \mid \vartheta_i = -1} \delta_i(C(a_i) + C(b_i)) + \sum_{i \mid \vartheta_i = 1} \delta_i(C(a_i) - C(b_i)) = 0.$$

We next apply the formulas 4.3.1.

For $a_i, \ell_i = -e_i - e_{i+1}$ red, we have $b_i + a_i = \ell_i$ and b_i is black:

$$\begin{aligned} C(a_i) + C(b_i) &= -1/2(a_i^2 + a_i^{(2)}) + 1/2(b_i^2 + b_i^{(2)}) \\ &= -1/2(a_i^2 + a_i^{(2)}) + 1/2((\ell_i - a_i)^2 + \ell_i^{(2)} - a_i^{(2)}) \\ &= -1/2(a_i^2 + a_i^{(2)}) + 1/2(\ell_i^2 - 2\ell_i a_i + a_i^2 + \ell_i^{(2)} - a_i^{(2)}) \\ &= -a_i^{(2)} - \ell_i a_i + e_i e_{i+1} = -a_i^{(2)} - \ell_i a_i + e_i e_{i+1}. \end{aligned}$$

For $a_i = b_i + \ell_i$ and $\ell_i = e_i - e_{i+1}$ black we have:

$$\begin{aligned} C(a_i) - C(b_i) &= \sigma_i[1/2(a_i^2 + a_i^{(2)}) - 1/2(b_i^2 + b_i^{(2)})] \\ &= \sigma_i[1/2(a_i^2 + a_i^{(2)}) - 1/2((a_i - \ell_i)^2 - \ell_i^{(2)} + a_i^{(2)}) \\ &= \sigma_i[-1/2(\ell_i^2 - 2\ell_i a_i - \ell_i^{(2)})] = \sigma_i[-e_{i+1}^2 + e_i e_{i+1} + \ell_i a_i]. \end{aligned}$$

The second identity follows from the first by substituting. □

5.8.1. Some reductions. Denote by $b_i = \sum_{h=1}^m b_{i,h} e_h$ and expand the second Formula (26). Observe that the coefficients of the mixed terms $e_i e_j$, $i \neq j$ come all from the sum

$$B := \sum_{i \mid \vartheta_i = -1} \delta_i(\ell_i b_i - e_i e_{i+1}) + \sum_{i \mid \vartheta_i = 1} \delta_i \sigma_i(-e_i e_{i+1} + \ell_i b_i).$$

If $h \notin [1, \dots, k]$, the coefficient of e_h in B (which must be equal to 0) is

$$\sum_{i \mid \vartheta_i = -1} \delta_i \ell_i b_{i,h} + \sum_{i \mid \vartheta_i = 1} \delta_i \sigma_i \ell_i b_{i,h} = 0.$$

By the uniqueness of the relation it follows that this relation is a multiple of (23) hence the numbers $b_{i,h}$, $\vartheta_i = -1$ and $\sigma_i b_{i,h}$, $\vartheta_i = 1$ are all equal. Since now we can choose as root one of the elements b_i we deduce that all these coefficients $b_{i,h}$ equal to 0. Thus, with this choice of root, b_i, a_i have support in the vertices of the encoding graph.

As a consequence we claim that:

LEMMA 5.9. *In case* 2) *the edges of the tree coincide with the edges* ℓ_i *of the relation.*

In case 1) *either the edges of the tree coincide with the edges* ℓ_i *of the relation or we can reduce to the case in which the tree* T *consists only of the edges involved in the relation, plus a single special extra edge* E *with* $\zeta(E) = 2$ (*see Lemma* 5.3 *for the definition of* ζ).

E *is either a red edge of the form* $-e_i - e_j$ *with* i, j *of the same value of* ζ *or a black edge of the form* $-e_i + e_j$ *with* i, j *of the opposite value of* ζ.

PROOF. Let T' be the forest support of the edges ℓ_i, if this is a tree it must coincide with T by minimality and we are done, if T' is not a tree there is at least one segment S in T joining two end points in T'. All the edges in S by definition are not in the relation. Their sum with suitable signs is supported in $[1, \dots, k]$ and in fact it is either the sum or the difference of two of the elements a_i, b_j, in particular it has the form $E = \sum_{i=1}^k \alpha_i e_i$.

If we are in case 2) then, by Lemma 5.6, $2E$ is a linear combination of the ℓ_i with integer coefficients. This is a new relation containing edges not supported in $[1, \dots, k]$ contradicting the hypotheses.

If we are in case 1) we must have $\zeta(E) \neq 0$ otherwise E is in the span of the edges ℓ_i and we have another relation among the edges of T contradicting minimality. By the same reason we cannot have two such segments, since the ℓ_i span a subspace of codimension 1 and we still would have a new relation.

Finally we claim that E is an edge.

We look at the encoding graph U of the edges in S, we want to show that they form a path joining two points in $[1, 2, \dots, k]$ so that the loop they generate in this way is odd.

First remark that every end vertex of U appears with non zero coefficient ± 1 in the vector E hence all end points of U lie in $[1, 2, \dots, k]$.

Next if U contains two different paths joining points in $[1, 2, \dots, k]$ each such path gives rise by summing with suitable signs to a non-zero linear combination of elements in $[1, 2, \dots, k]$. Since the span of the edges ℓ_i has codimension 1 in the span of the elements e_i, if we have two more paths we deduce a new relation. We deduce that U is either a single path joining two vertices $u, v \in [1, 2, \dots, k]$ and not meeting any other point of $[1, 2, \dots, k]$ or it may also be a single loop originating from a vertex u in $[1, 2, \dots, k]$. In this case the loop must be odd otherwise we have another relation, then we see that if we choose as root one of the two vertices of T joined by S the other vertex is $-2e_u$ and we are finished, since we have proved that the graph is not allowable i.e. we found the desired pair of Proposition 3.14.

Otherwise E is an element of mass either 0 or -2 has support in two elements of $[1, \dots, k]$ with coefficients ± 1 hence it is an edge, since we are assuming that it does not appear in the relation the only possibility is that it must be of the form $e_u - e_v, -e_u - e_v$. $u, v \in [1, 2, \dots, k]$ and linearly independent from the edges ℓ_h, this means, by Formula (24), that u, v must have opposite parity in the first case and the same parity in the second. If S is not equal to the edge E we claim that the complete graph Γ we started from was not minimal. Indeed we construct a tree T' by replacing the path S by the single edge E. This is a proper subgraph of Γ by completeness. The complete graph associated to T' is resonant-degenerate (it contains all the vertices appearing in the relation). This is a contradiction.

(28) I)

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, equal] & 10 \arrow[r, equal] & 9 \arrow[r, dash] & 8 \arrow[r, equal] & 7 \arrow[d, equal] \\
2 \arrow[r, equal] & 3 \arrow[u] \arrow[r, equal] & 4 \arrow[r, dash] & 5 \arrow[r, equal] & 6
\end{tikzcd}$$

II)

$$\begin{tikzcd}
1 \arrow[r, equal] \arrow[d, equal] & 10 \arrow[r, equal] & 9 \arrow[r, equal] & 8 \arrow[r, dash] & 7 \arrow[d, equal] \\
2 \arrow[r, equal] & 3 \arrow[ur, equal] \arrow[r, dash] & 4 \arrow[r, equal] & 5 \arrow[r, equal] & 6
\end{tikzcd}$$

□

REMARK 5.10. In the case 1) with an extra edge joining the indices i, j we shall say that i, j are critical and divide accordingly the remaining indices in two sets and all edges in two sets A, B accordingly. Note that in this case for all indices one has $\delta_u = \vartheta_u \delta_{u-1}$.

Remark 5.11. In case 2) we divide the edges in three sets A, B, C where A are the edges of the first loop, B (possibly empty) the edges of the segment and C the ones of the second loop. In case 1) with an extra edge we divide the edges in two sets A, B separated by the extra edge E.

As for a non critical index u we shall say that $u \in A$ resp. $u \in B, C$ if the two edges ℓ_{u-1}, ℓ_u are in A (resp. B, C).

5.11.1. Some geometry of trees. Let us collect some generalities which will be used in the course of the proof. In all this section T will be a tree, for the moment with no further structure and later related to the Cayley graph.

Given a set A of edges in T let us denote by $\langle A \rangle$ the minimal tree contained in T and containing A, we call it the *tree generated by* A. The simplest trees are the *segments* in which no vertex has valency > 2. In fact in a segment we have exactly two end points of valency 1 and the *interior points* of valency 2.

Lemma 5.12. 1) *If A consists of* 2 *edges then $\langle A \rangle$ is a segment, more generally if A consists of* 2 *segments S_1, S_2 with the interior vertices of valency* 2 *then again $\langle A \rangle$ is a segment, if moreover $S_1 \cap S_2$ contains an edge, then $S_1 \cup S_2 = \langle S_1, S_2 \rangle$ and all its interior vertices have valency* 2*.*

If we only assume that S_2 has interior vertices of valency 2 *but we also assume that $S_1 \cap S_2$ contains at least one edge then*

2) *$\langle S_1, S_2 \rangle = S_1 \cup S_2$ and it is a segment.*

Proof. 1) Consider $S_1 \cap S_2$, if this is empty, there is a unique segment joining two points in S_1, S_2 and disjoint from them, then this must join two end points by the hypothesis on the valency and the statement is clear.

2) Let A be a segment connected component of $S_1 \cap S_2$. Unless $S_2 \subset S_1$ one of the end points a of A is an internal vertex of S_1, since this has valency 2 this is possible only if a is an end point of S_1, if also the other end point of A is an internal vertex of S_1 the same argument shows that $S_1 \subset S_2$. The final case is that the other end of A is also an end point of S_2 and then the statement is clear. $\square$

6. The contribution of an index u

6.0.1. The strategy. We want to exploit Formula (26) in order to understand the graph. We proceed as follows.

Definition 6.1. Given a quadratic expression Q in the elements e_i and any index u we set $e_u C_u(Q)$ to be the sum of all terms in Q which contain e_u but not e_u^2.

Notice that C_u is a linear map from quadratic expressions to linear expressions in the e_i, $i \neq u$. By Formula (26) we have $C_u(\mathscr{R}) = 0$. We observe that only the terms $\ell_i a_i$ or $-e_i e_{i+1}$ may contribute to $C_u(\mathscr{R})$ hence:

$$C_u(\mathscr{R}) = \sum_{i \mid \vartheta_i = -1} \delta_i(-C_u(\ell_i a_i) + C_u(e_i e_{i+1})) + \sum_{i \mid \vartheta_i = 1} \delta_i \sigma_i (C_u(e_i e_{i+1}) + C_u(\ell_i a_i)) = 0.$$

We choose an index u which appears only in $\ell_{u-1} = \vartheta_{u-1}e_{u-1} - e_u$ and in $\ell_u = \vartheta_u e_u - e_{u+1}$. This is any index in case 1) with no extra edge while it excludes the *critical indices* in the other cases (see Remarks 5.5 and 5.10).

We separately compute the contributions of

$$-\mathscr{R}' := \sum_{i\,|\,\vartheta_i=-1} \delta_i e_i e_{i+1} + \sum_{i\,|\,\vartheta_i=1} \delta_i\sigma_i e_i e_{i+1}, \quad \mathscr{R}'' := -\sum_{i\,|\,\vartheta_i=-1} \delta_i\ell_i a_i + \sum_{i\,|\,\vartheta_i=1} \delta_i\sigma_i\ell_i a_i,$$

since $C_u(\mathscr{R}) = C_u(\mathscr{R}'') - C_u(\mathscr{R}')$.

We need the following formula for the elements a_j, easily proved by induction, where the black edges ℓ are oriented outwards from the root and σ_ℓ denotes the color of the endpoint of the segment ending with ℓ:

$$a_j = \begin{cases} -\sum_{\ell\preceq\ell_j}\sigma_\ell\ell, & \sigma_j = -1, \quad \ell_j \text{ red} \\ -\sum_{\ell\prec\ell_j}\sigma_\ell\ell, & \sigma_j = 1, \quad \ell_j \text{ red} \\ \sigma_j\sum_{\ell\preceq\ell_j}\sigma_\ell\ell, & \lambda_j = 1, \quad \ell_j \text{ black} \\ \sigma_j\sum_{\ell\prec\ell_j}\sigma_\ell\ell, & \lambda_j = -1, \quad \ell_j \text{ black} \end{cases} \tag{29}$$

If $i \neq u-1, u$ set $\mu_u(i)$ to be the coefficient of e_u in a_i then

LEMMA 6.2. *If $i \neq u-1, u$ we have $C_u(\ell_i a_i) = \mu_u(i)\ell_i$.*

The contribution $C_u(\mathscr{R}')$ depends on the two colors of ℓ_{u-1}, ℓ_u according to the following table (see Remarks 5.5, 5.10):

(30)

colors		*contribution of $\mathscr{R}'$*
rr	$\delta_{u-1} = -\delta_u$	$-\delta_{u-1}e_{u-1} - \delta_u e_{u+1} = \delta_u[e_{u-1} - e_{u+1}]$
rb	$\delta_{u-1} = \delta_u$	$-\delta_{u-1}e_{u-1} - \sigma_u\delta_u e_{u+1} = -\delta_u[e_{u-1} + \sigma_u e_{u+1}]$
br	$\delta_{u-1} = -\delta_u$	$-\delta_{u-1}\sigma_{u-1}e_{u-1} - \delta_u e_{u+1} = \delta_u[\sigma_{u-1}e_{u-1} - e_{u+1}]$
bb	$\delta_{u-1} = \delta_u$	$-\delta_{u-1}\sigma_{u-1}e_{u-1} - \sigma_u\delta_u e_{u+1} = -\delta_u[\sigma_{u-1}e_{u-1} + \sigma_u e_{u+1}]$

PROOF. The first statement is clear since the edge ℓ_i does not contain the term e_u. For the second we see that the contribution to $C_u(\mathscr{R}')$ comes from the two terms $e_{u-1}e_u$, $e_u e_{u+1}$. The term $e_{u-1}e_u$ if $\theta_{u-1} = -1$, i.e. ℓ_{u-1} is red, appears from $C_u(-\delta_{u-1}e_{u-1}e_u) = -\delta_{u-1}e_{u-1}$. If $\theta_{u-1} = 1$, i.e. ℓ_{u-1} is black, appears from $C_u(-\sigma_{u-1}\delta_{u-1}e_{u-1}e_u) = -\sigma_{u-1}\delta_{u-1}e_{u-1}$.

The term $e_u e_{u+1}$, if $\theta_u = -1$, i.e. ℓ_u is red, gives rise to $C_u(-\delta_u e_u e_{u+1}) = -\delta_u e_{u+1}$ if $\theta_u = 1$, i.e. ℓ_u is black, gives rise to $C_u(-\sigma_u\delta_u e_u e_{u+1}) = -\sigma_u\delta_u e_{u+1}$.

We then use the fact that $\delta_u = \delta_{u-1}$ if δ_u is black, while $\delta_u = -\delta_{u-1}$ if δ_u is red. □

We thus write

$$0 = -C_u(\mathscr{R}) = \sum_{i\,|\,\vartheta_i=-1,\, i\neq u-1,\, u} \delta_i\mu_u(i)\ell_i - \sum_{i\,|\,\vartheta_i=1,\, i\neq u-1,\, u} \delta_i\sigma_i\mu_u(i)\ell_i + L_u$$

where L_u is the contribution from $C_u(\mathscr{R}')$ and from the terms associated to $a_{u-1}\ell_{u-1}$, $a_u\ell_u$.

We now choose the root so that the segment S_u, generated by the two edges ℓ_{u-1}, ℓ_u, appears as follows:

$$S_u := r \xrightarrow{\ell_u} \dots \qquad \xrightarrow{\ell_{u-1}} x_{u-1}. \tag{31}$$

The value of L_u depends upon 3 facts, 1) the two colors of ℓ_{u-1}, ℓ_u. 2) The orientation λ of the edges ℓ_{u-1}, ℓ_u which are black. 3) The color σ_{u-1} of x_{u-1}. We thus obtain 18 different cases described in §6.2.3.

6.2.1. The contribution of $a_u\ell_u$. If $\ell_u = -e_u - e_{u+1}$ is red we have $a_u = \ell_u$ and $C_u(\delta_u\ell_u a_u) = 2\delta_u e_{u+1}$. If $\ell_u = e_u - e_{u+1}$ is black we have $\sigma_u = 1$, if $\lambda_u = 1$ we have $a_u = \ell_u$ and $C_u(-\delta_u\sigma_u\ell_u a_u) = 2\delta_u e_{u+1}$. If $\lambda_u = -1$ we have $a_u = 0$ and $C_u(-\delta_u\sigma_u\ell_u a_u) = 0$. Summarizing:

$$\begin{aligned} C_u(\delta_u\ell_u a_u) &= 2\delta_u e_{u+1}, && \ell_u \text{ is red} \\ C_u(-\delta_u\sigma_u\ell_u a_u) &= 2\delta_u e_{u+1}, && \ell_u \text{ is black} \quad \lambda_u = 1 \\ C_u(-\delta_u\sigma_u\ell_u a_u) &= 0, && \ell_u \text{ is black} \quad \lambda_u = -1 \end{aligned} \tag{32}$$

6.2.2. The contribution of $a_{u-1}\ell_{u-1}$. In a_{u-1} consider the part $\bar{a}_{u-1}$ of the sum formed by the edges ℓ_i, $\ell_u \prec \ell_i \prec \ell_{u-1}$.

We have $a_{u-1} = \bar{a}_{u-1} + \tilde{a}_{u-1}$ where

$$\tilde{a}_{u-1} = \begin{cases} -\sigma_u\lambda_u\ell_u + \ell_{u-1}, & \text{if } \sigma_{u-1} = -1, \quad \ell_{u-1} \text{ red} \\ -\sigma_u\lambda_u\ell_u, & \text{if } \sigma_{u-1} = 1, \quad \ell_{u-1} \text{ red} \\ \sigma_{u-1}\sigma_u\lambda_u\ell_u + \ell_{u-1}, & \text{if } \lambda_{u-1} = 1, \quad \ell_{u-1} \text{ black} \\ \sigma_{u-1}\sigma_u\lambda_u\ell_u, & \text{if } \lambda_{u-1} = -1, \quad \ell_{u-1} \text{ black} \end{cases} \tag{33}$$

We then have

$$C_u(\ell_{u-1}a_{u-1}) = -\bar{a}_{u-1} + C_u(\ell_{u-1}\tilde{a}_{u-1})$$

Finally

$$C_u(\ell_{u-1}\ell_u) = \vartheta_{u-1}\vartheta_u e_{u-1} + e_{u+1}, \quad C_u(\ell_{u-1}^2) = -\vartheta_{u-1}2e_{u-1}.$$

$$C_u(\ell_{u-1}\tilde{a}_{u-1}) = \begin{cases} -\sigma_u\lambda_u C_u(\ell_{u-1}\ell_u) + C_u(\ell_{u-1}^2), & \sigma_{u-1} = -1, \quad \ell_{u-1} \text{ red} \\ -\sigma_u\lambda_u C_u(\ell_{u-1}\ell_u), & \sigma_{u-1} = 1, \quad \ell_{u-1} \text{ red} \\ \sigma_{u-1}\sigma_u\lambda_u C_u(\ell_{u-1}\ell_u) + C_u(\ell_{u-1}^2), & \lambda_{u-1} = 1, \quad \ell_{u-1} \text{ black} \\ \sigma_{u-1}\sigma_u\lambda_u C_u(\ell_{u-1}\ell_u), & \lambda_{u-1} = -1, \quad \ell_{u-1} \text{ black} \end{cases}$$

$$C_u(\ell_{u-1}\tilde{a}_{u-1}) = \begin{cases} -\sigma_u\lambda_u(-\vartheta_u e_{u-1} + e_{u+1}) + 2e_{u-1}, & \sigma_{u-1} = -1, \quad \ell_{u-1} \text{ red} \\ -\sigma_u\lambda_u(-\vartheta_u e_{u-1} + e_{u+1}), & \sigma_{u-1} = 1, \quad \ell_{u-1} \text{ red} \\ \sigma_{u-1}\sigma_u\lambda_u(\vartheta_u e_{u-1} + e_{u+1}) - 2e_{u-1}, & \lambda_{u-1} = 1, \quad \ell_{u-1} \text{ black} \\ \sigma_{u-1}\sigma_u\lambda_u(\vartheta_u e_{u-1} + e_{u+1}), & \lambda_{u-1} = -1, \quad \ell_{u-1} \text{ black} \end{cases}$$

If ℓ_{u-1} is red we then compute the contribution of $\delta_{u-1}\ell_{u-1}a_{u-1}$ getting (recall that σ_u is -1 if ℓ_u is red, one otherwise)

$$-\delta_{u-1}\bar{a}_{u-1}+\delta_{u-1}\begin{cases} e_{u+1}+3e_{u-1}, & \sigma_{u-1}=-1, \quad \ell_u \text{ red}\\ e_{u+1}+e_{u-1}, & \sigma_{u-1}=1, \quad \ell_u \text{ red}\\ -\lambda_u[e_{u+1}-e_{u-1}]+2e_{u-1}, & \sigma_{u-1}=-1, \quad \ell_u \text{ black}\\ -\lambda_u[e_{u+1}-e_{u-1}], & \sigma_{u-1}=1, \quad \ell_u \text{ black}\end{cases} \tag{34}$$

If ℓ_{u-1} is black we then compute the contribution of $-\sigma_{u-1}\delta_{u-1}\ell_{u-1}a_{u-1}$ getting

$$\sigma_{u-1}\delta_{u-1}\bar{a}_{u-1}-\sigma_{u-1}\delta_{u-1}\begin{cases} -\sigma_{u-1}[e_{u+1}-e_{u-1}]-2e_{u-1}, & \lambda_{u-1}=1, \quad \ell_u \text{ red}\\ -\sigma_{u-1}[e_{u+1}-e_{u-1}], & \lambda_{u-1}=-1, \quad \ell_u \text{ red}\\ \sigma_{u-1}\lambda_u[e_{u-1}+e_{u+1}]-2e_{u-1}, & \lambda_{u-1}=1, \quad \ell_u \text{ black}\\ \sigma_{u-1}\lambda_u[e_{u-1}+e_{u+1}], & \lambda_{u-1}=-1, \quad \ell_u \text{ black}\end{cases} \tag{35}$$

We thus write if ℓ_{u-1} is red

$$\begin{aligned} 0=-C_u(\mathscr{R}) = & \sum_{i\,|\,\vartheta_i=-1,\, i\neq u-1,u} \delta_i\mu_u(i)\ell_i \\ & - \sum_{i\,|\,\vartheta_i=1,\, i\neq u-1,u} \delta_i\sigma_i\mu_u(i)\ell_i - \delta_{u-1}\bar{a}_{u-1}+L \end{aligned} \tag{36}$$

If ℓ_{u-1} is black

$$\begin{aligned} 0=-C_u(\mathscr{R}) = & \sum_{i\,|\,\vartheta_i=-1,\, i\neq u-1,u} \delta_i\mu_u(i)\ell_i \\ & - \sum_{i\,|\,\vartheta_i=1,\, i\neq u-1,u} \delta_i\sigma_i\mu_u(i)\ell_i + \sigma_{u-1}\delta_{u-1}\bar{a}_{u-1}+L. \end{aligned} \tag{37}$$

In both cases by L we denote the contribution from the Formulas (30), (32), and (34) or (35).

6.2.3. The 18 *cases.* So now we expand L

1) ℓ_{u-1},ℓ_u both red $\sigma_{u-1}=1$.

$$\delta_u[e_{u-1}-e_{u+1}]+2\delta_u e_{u+1}-\delta_u(e_{u+1}+e_{u-1})=0.$$

2) ℓ_{u-1}, ℓ_u both red $\sigma_{u-1}=-1$.

$$\delta_u[e_{u-1}-e_{u+1}]+2\delta_u e_{u+1}-\delta_u[e_{u+1}+3e_{u-1}]=-2\delta_u e_{u-1}.$$

3) ℓ_{u-1} red, ℓ_u black $\sigma_{u-1}=1$, $\lambda_u=1$.

$$-\delta_u[e_{u-1}+e_{u+1}]+2\delta_u e_{u+1}-\delta_u[e_{u+1}-e_{u-1}]=0$$

4) ℓ_{u-1} red, ℓ_u black $\sigma_{u-1} = -1$, $\lambda_u = 1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u+1} - e_{u-1}] - \delta_u 2e_{u-1} = -2\delta_u e_{u-1}$$

5) ℓ_{u-1} red, ℓ_u black $\sigma_{u-1} = 1$, $\lambda_u = -1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + \delta_u[e_{u+1} - e_{u-1}] = -2\delta_u e_{u-1}$$

6) ℓ_{u-1} red, ℓ_u black $\sigma_{u-1} = -1$, $\lambda_u = -1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + \delta_u[e_{u+1} - e_{u-1}] + \delta_u 2e_{u-1} = 0$$

7) ℓ_{u-1} black, ℓ_u red $\sigma_{u-1} = 1$, $\lambda_{u-1} = 1$.

$$\delta_u[e_{u-1} - e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u+1} - e_{u-1}] - 2\delta_u e_{u-1} = 0$$

8) ℓ_{u-1} black, ℓ_u red $\sigma_{u-1} = -1$, $\lambda_{u-1} = 1$.

$$\delta_u[-e_{u-1} - e_{u+1}] + 2\delta_u e_{u+1} + \delta_u[e_{u+1} - e_{u-1}] + 2\delta_u e_{u-1} = 2\delta_u e_{u+1}.$$

9) ℓ_{u-1} black, ℓ_u red $\sigma_{u-1} = 1$, $\lambda_{u-1} = -1$.

$$\delta_u[e_{u-1} - e_{u+1}] + 2\delta_u e_{u+1} + \delta_u[e_{u+1} - e_{u-1}] = 2\delta_u e_{u+1}$$

10) ℓ_{u-1} black, ℓ_u red $\sigma_{u-1} = -1$, $\lambda_{u-1} = -1$.

$$\delta_u[-e_{u-1} - e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u+1} - e_{u-1}] = 0.$$

11) ℓ_{u-1}, ℓ_u both black, $\sigma_{u-1} = 1$, $\lambda_{u-1} = 1$, $\lambda_u = 1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u-1} + e_{u+1}] + 2\delta_u e_{u-1} = 0$$

12) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = -1$, $\lambda_{u-1} = 1$, $\lambda_u = 1$.

$$-\delta_u[-e_{u-1} + e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u-1} + e_{u+1}] - 2\delta_u e_{u-1} = -2\delta_u e_{u-1}$$

13) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = 1$, $\lambda_{u-1} = -1$, $\lambda_u = 1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u-1} + e_{u+1}] = -2\delta_u e_{u-1}$$

14) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = -1$, $\lambda_{u-1} = -1$, $\lambda_u = 1$.

$$-\delta_u[-e_{u-1} + e_{u+1}] + 2\delta_u e_{u+1} - \delta_u[e_{u-1} + e_{u+1}] = 0$$

15) ℓ_{u-1}, ℓ_u both black, $\sigma_{u-1} = 1$, $\lambda_{u-1} = 1$, $\lambda_u = -1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + \delta_u[e_{u-1} + e_{u+1}] + 2\delta_u e_{u-1} = 2\delta_u e_{u-1}$$

16) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = -1$, $\lambda_{u-1} = 1$, $\lambda_u = -1$.

$$-\delta_u[-e_{u-1} + e_{u+1}] + \delta_u[e_{u-1} + e_{u+1}] - 2\delta_u e_{u-1} = 0$$

17) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = 1$, $\lambda_{u-1} = -1$, $\lambda_u = -1$.

$$-\delta_u[e_{u-1} + e_{u+1}] + \delta_u[e_{u-1} + e_{u+1}] = 0$$

18) ℓ_{u-1}, ℓ_u both black $\sigma_{u-1} = -1$, $\lambda_{u-1} = -1$, $\lambda_u = -1$.

$$-\delta_u[-e_{u-1} + e_{u+1}] + \delta_u[e_{u-1} + e_{u+1}] = 2\delta_u e_{u-1}$$

By inspection we see that we have proved the following remarkable:

COROLLARY 6.3. *The contribution of L equals to* 0 *if and only if* $\sigma_{u-1} = \lambda_{u-1}\lambda_u$. *In this case the coefficient of* e_u *in the end point* x_{u-1} *of the segment* S_u *is* 0.

If $\sigma_{u-1} = -\lambda_{u-1}\lambda_u$ *the contribution of L equals to* $\pm 2e_{u\pm 1}$. *In this case the coefficient of* e_u *in the end point* x_{u-1} *of the segment* S_u *is* ± 2.

PROOF. The first is by inspection, as for the second we check a few cases. This coefficient comes from the two contributions of ℓ_{u-1}, ℓ_u. They appear by $\sigma_{u-1}[\sigma_u\lambda_u\ell_u + \sigma_{u-1}\lambda_{u-1}\ell_{u-1}]$. Now $\sigma_u\lambda_u\ell_u = -\ell_u = e_u + e_{u+1}$ if ℓ_u is red and similarly $\sigma_{u-1}\lambda_{u-1}\ell_{u-1} = e_u + e_{u-1}$ if ℓ_{u-1} is red and $\sigma_{u-1} = -1$. This is case 2). If ℓ_{u-1} is black then the coefficient of e_u in $\sigma_{u-1}\lambda_{u-1}\ell_{u-1}$ is 1 if and only if $\sigma_{u-1}\lambda_{u-1} = -1$ and in this case this is equivalent to $\sigma_{u-1} = -\lambda_{u-1}\lambda_u$. These are cases 8, 9.

Similar argument when ℓ_u is black. $\square$

COROLLARY 6.4. *If* $\ell_{u-1} \prec \ell_j$ *we have* $\mu_u(j) = 0$ *if the contribution of L is* 0, *otherwise* $\mu_u(j) = \pm 2$.

6.4.1. Contribution of L equals to 0. We say that u is of type I. We deduce that the other edges ℓ_i satisfy a relation, i.e. either (36) or (37). This is impossible unless this is the trivial relation with all coefficients 0. Let us draw the implications of this. Recall that S_u is the minimal segment containing the edges ℓ_u, ℓ_{u-1} (cf. Formula (31)).

Notice that any edge ℓ_j comparable with ℓ_u and not with ℓ_{u-1} appears in the relation, only from the term $\mu_u(j)\ell_j$ (indeed in this case $\bar{a}_{u-1}$ does not depend on ℓ_j). Since then $\mu_u(j) = \pm 1$ this is a contradiction. Thus no edge is comparable with ℓ_u and not with ℓ_{u-1}. This means that all internal vertices of S_u have valency 2, moreover all edges ℓ_j with $\ell_u \prec \ell_j \prec \ell_{u-1}$ appear with coefficient $\pm\delta_{u-1} \pm \delta_j$, coming from $\bar{a}_{u-1}\delta_{u-1}$ and from $\pm\delta_j\ell_j\mu_u(j)$ (see formulas (36)–(37)) we thus must have that this sum equals zero.

Now, in case 2) if we start from $u \in A \cup C$ (see Remark 5.11) this implies that it is not possible that $j \in B$ since the sum of these two coefficients is odd and so it is not zero, so the segment S_u is all formed by elements in $A \cup C$. If we start from $u \in B$ it is not possible that $j \in A \cup C$ since again $\pm\delta_{u-1} \pm \delta_j$ is odd, so the segment S_u is all formed by elements in B.

Finally in case 1) with an extra edge E it is not possible that E is in between ℓ_{u-1}, ℓ_u otherwise E would appear and only in $\bar{a}_{u-1}$. Hence the value of ζ of the relation would be ± 2.

6.4.2. Contribution of L equals to $\pm 2\delta_u e_{u\pm 1}$. We say that u is of type II. We thus have, from (36) or (37), a relation expressing $\pm 2\delta_u e_{u\pm 1}$ as linear combination of the edges $\ell_j \neq \ell_{u-1}, \ell_u$. Now these edges are linearly independent so such an expression if it exists it is unique. Let us assume for instance that the relation expresses $2e_{u-1}$, the other case is identical.

In order to understand which elements appear in C_u, first remark that the only edges that may contribute to the expression of C_u are those for which $\ell_u \prec \ell_j$. If ℓ_j is not comparable with ℓ_{u-1} they contribute by $\pm\delta_j$. If $\ell_u \prec \ell_j \prec \ell_{u-1}$ they contribute by $\pm\delta_j \pm \delta_{u-1}$. Finally if $\ell_{u-1} \prec \ell_j$ they contribute by 0, $\pm 2\delta_j$ by Corollary 6.4.

Case 1A (single loop) no extra edge: such a relation does not exist. For instance if $2e_{u-1}$ is a linear combination $\sum_j c_j \ell_j$ of the edges $\ell_j \neq \ell_{u-1}, \ell_u$ since e_{u-1} only appears in ℓ_{u-2} with sign -1 we must have that $c_{u-2} = -2$ and then $2e_{u-2}$ is a linear combination $\sum_j c_j \ell_j$ of the edges $\ell_j \neq \ell_{u-2}, \ell_{u-1}, \ell_u$, continuing by induction we reach a contradiction.

Case 1B (single loop) an extra edge: we may assume that the extra edge $E = \vartheta e_1 - e_h$, this edge divides the loop into two parts A, B. The edges in $A := \{\ell_1, \dots, \ell_{h-1}\}$ and E form an odd loop as well as the edges in B and E. We may assume for instance that $h < u$ is an index in B. We know that, for an odd loop, we can write $2e_1$ uniquely as the sum of the edges of the odd loop A, E and then we write $2e_{u-1} = \pm\sum_{k=1}^{u-2} 2\delta_k \ell_k \pm 2e_1$, let us call $\mathscr{R}'$ this relation. The edges appearing in the relation are all the edges of A, E with coefficient ± 1 and all the edges ℓ_k, $h \leq k \leq u-2$ with coefficients ± 2. This relation must be proportional to either (36) or (37). Notice that E appears in this relation with coefficient ± 1.

This is possible if and only if $E \prec \ell_{u-1}$. Moreover we know that all the edges in A appear with coefficient ± 1 hence by Corollary 6.4 it follows that they must be comparable with ℓ_u but not with ℓ_{u-1}. Finally for the edges in B we have that the ℓ_k with $h \leq k \leq u-2$ are comparable with ℓ_u and, since they appear with coefficient ± 2 in $\mathscr{R}'$, we must have either $\ell_k \prec \ell_{u-1}$ or $\ell_{u-1} \prec \ell_k$. All the others are not comparable with ℓ_u.

Denote by T_A and T_B the two minimal trees generated by A, B respectively. We have:

Corollary 6.5. 1) *If the indices of A and B are all of type I then either T_A and T_B form two disjoint segments separated by E, or the edges in $A \cup B$ form a segment, the extra edge is outside this segment so the graph is not minimal degenerate.*

2) *If there is an index in B (resp. in A) of type II, the two minimal trees T_A and T_B generated by A, B respectively are segments and can intersect only in a vertex or in the edge E. If they intersect in a vertex then all $v \in A$ (resp. all $v \in B$) have type I and the vertex is an end point of E.*

Proof. 1) In this case we know that all the segments S_u for u non critical are segments which do not contain E and with the interior vertices of valency 2. By a simple induction we have that $\bigcup_{u \in A} S_u$ and $\bigcup_{v \in B} S_u$ are segments which do not contain E and with the interior vertices of valency 2 (cf. Lemma 5.12). If these two segments have an edge in common then, by the same Lemma, their union is a segment not containing E and thus this segment gives a minimal degenerate graph and the one we started from is not minimal. The same happens if they meet in an end point of both. The only remaining case is that T_A and T_B form two disjoint segments separated by E.

2) We have just seen that all the edges in A lie in branches originating from vertices of the segment S_u different from the last vertex of ℓ_{u-1}. On the other hand the edges in B are in S_u and possibly in the other branches originating from the end points of S_u. This implies that the two trees T_A and T_B can only have an intersection inside S_u.

Take any non critical index $v \in A$, if v is of type I the segment S_v is either disjoint from S_u or it may intersect S_u in a vertex, since S_v has the interior vertices of valency 2 and it cannot overlap with S_u otherwise one or both of its ending edges, both in A would also be in S_u which on the contrary is all formed by edges in B. If v is of type II we can apply the same analysis to v and deduce that the segment S_v intersects S_u in the edge E.

If all indices in A are of type I by the previous analysis the tree they generate can meet S_u (and also T_B) only in one vertex so they lie in a single branch. Applying Lemma 5.12 it follows that the tree T_A is a segment and it intersects S_u in a vertex.

Now suppose that this vertex v is not an end point of E. Call S the segment from v to E. If $\ell_j \in S$ we must have that if j is not a critical index 1, h it must be of type I (otherwise we could not have that the edges in A follow ℓ_j) and thus $\ell_{j-1} \in S$. Also $\ell_{j+1} \in S$ otherwise it should be of type II but then we have again that the vertex v is outside the segment S_{j+1}, by induction we arrive at a contradiction $\ell_u \in S$.

As for T_B we have now proved that it is formed only by the edges in B and by E. By induction we see that $T_B = \bigcup_{a \in B} S_a$ and in fact it is a segment. In fact let $T_B^i := \bigcup_{h<j\le i\le k} S_i$, assume T_B^i is a segment and consider $T_B^{i+1} = T_B^i \cup S_{i+1}$. By induction and construction these two segments intersect at least in the edge ℓ_i. If at least one of the two is only formed from indices of type I we see again by induction that its interior vertices have valency 2 and by Lemma 5.12 we have that their union is a segment. If $i+1$ is of type II as well as one of the indices j with $h < j \le i$ we have that T_B^i contains E. By the previous analysis it follows that inside the segment T_B^i and S_{i+1} all interior vertices have valency 2 hence again Lemma 5.12 applies. □

Case 2. A doubly odd loop is divided in 3 (or 2) parts: the two odd loops A, C and the segment B (possibly empty) joining them. We divide this into two subcases:

Assume first $u \in A$ (the case $u \in C$ is similar). We have $\pm 2\delta_u e_{u\pm 1}$ a linear combination of the edges in B, C with coefficient δ_i (or all $-\delta_i$) equal to $2e_1$ plus, (cf. Formula (22)), $2\sum_{i=1}^{u-2} \delta_i \ell_i = -2\delta_{u-2}e_{u-1} - 2e_1$ from which we have the required expression for $-2e_{u-1}$, similarly for $-2e_{u+1}$. This is the unique expression $\mathscr{R}'$ as linear combination of the linearly independent edges $\ell_j \neq \ell_{u-1}, \ell_u$.

As before this relation must be proportional to either (36) or (37). Inspecting these relations we first observe that, if $j \in B, C$ the edge ℓ_j must have coefficient $\pm\delta_j$. By Corollary 6.4 if $\ell_{u-1} \prec \ell_j$ we have that $\mu_u(j) = \pm 2$ hence by inspection we deduce that $\ell_{u-1} \not\prec \ell_j$.

If $\ell_u \prec \ell_j \prec \ell_{u-1}$ the coefficient of ℓ_j in the two possible relations comes from two terms, a term $\pm\delta_j$ coming from the first two summands (since in this case $\mu_u(j) = \pm 1$), and a term $\pm\delta_{u-1}$ from $\bar{a}_{u-1}$, hence no index in B or C can appear in $\bar{a}_{u-1}$ by parity. Since these edges appear in the relation $\mathscr{R}'$ we deduce that all ℓ_j, $j \in B \cup C$ are in branches which originate from internal vertices of S_u. Inside the segment S_u there are only edges of A. If we are in the case $L = \pm 2\delta_u e_{u-1}$ all edges ℓ_j with $\ell_{u-1} \prec \ell_j$ appear with coefficient $\pm 2\delta_j$ hence they are in the set $i \in A$, $i \le u-2$. The remaining edges ℓ_i in A with $i > u$ do not appear hence they either satisfy $\ell_u \prec \ell_i \prec \ell_{u-1}$ or are not comparable with ℓ_u. Similar discussion for $L = \pm 2\delta_u e_{u+1}$. A similar consideration holds if $u \in C$.

Assume $u \in B$. If $u \in B$ the contribution of L is $\pm 4e_{u\pm 1}$. The two cases are similar.

i) If the contribution is $\pm 4e_{u-1}$, this comes from a sum $2\sum_{i \in A} \delta_i \ell_i = \pm 4e_1$ plus $2\sum_{j \in B, j \le u-2} \delta_j \ell_j = \pm 4[e_{u-1} \pm e_1]$.

ii) The contribution $\pm 4e_{u+1}$, comes from the sum $2\sum_{i \in C} \delta_i \ell_i = \pm 4e_b$ plus a sum of $2\sum_{j \in B, j \ge u+1} \delta_j \ell_j = \pm 4[e_{u+1} \pm e_b]$.

This formula for L must coincide with that given by (36) or (37).

We claim that there is no edge ℓ_j with $\ell_u \prec \ell_j$ and ℓ_j is not comparable with ℓ_{u-1}. Indeed this edge would have $\mu_u(j) = \pm 1$ and would not appear in $\bar{a}_{u-1}$. This is incompatible with the fact that the coefficient must be $\pm 2\delta_j$. Thus we deduce that all internal vertices of the segment S_u have valency 2.

Finally if $\ell_u \prec \ell_j \prec \ell_{u-1}$ we have that the coefficient of ℓ_j in the relation associated to Formulas (36) or (37) is $\pm\delta_j \pm \delta_{u-1}$. Note that $u \in B$ is not critical and hence $\ell_u, \ell_{u-1} \in B$ so $\delta_{u-1} = \pm 2$. If $j \in A \cup C$ we have that this number is odd so it cannot be one of the coefficients appearing in the relation i) or ii). In case i) finally we deduce that if $j \in A$ we have $\ell_{u-1} \prec \ell_j$ while all the $j \in C$ lie in the branches of the tree from the root different from the one containing ℓ_u.

Corollary 6.6. 1) *The edges in B always form a segment, its internal vertices have valency* 2.

2) *If there is an index of type II in B all edges in A and all edges in C are separated and lie in the two segments originating from the two end points of S_u.*

3) *If there is an index of type II in A* (*or C*) *all edges in A and all edges in C are separated and lie in two segments which can be disjoint or meet in one vertex.*

4) *If all indices are of type I then either all edges in A and all edges in C are separated and lie in the two segments originating from the two end points of S_u. or the edges of $A \cup C$ form a segment.*

Proof. 1) The proof is similar to that of Corollary 6.5. We already know that, if $j \in B$ is of type I inside the segment S_u there are only edges ℓ_j with $j \in B$ and its internal vertices have valency 2, we have proved this now also for type II. The claim follows from Lemma 5.12.

2) Assume there is an index $u \in B$ of type II with contribution $\pm 4e_{u-1}$. Analyzing the corresponding relation we have then that all edges ℓ_i with $i \le u-2$ and all edges in C precede ℓ_u, all edges in A follow ℓ_{u-1}.

Finally if $\ell_{u-1} \prec \ell_j$ then ℓ_j appears in the relation so since in the relation appear either all the edges in C and none of the edges in A or conversely we must have that these two blocks lie in the two branches originating from the two end points of S_u.

3) Assume there is an index of type II in A, we then have seen that T_C is formed by branches originating from interior points of S_u. Now if $u \in C$ is of type I the segment S_u cannot contain edges in A otherwise it would contain interior vertices of valency > 2. If $u \in C$ is of type II the segment S_u does not contain edges in A by the previous argument.

4) If all indices are of type I we have seen that all segments S_u for $u \in A \cup C$ are formed by edges in $A \cup C$ and their interior vertices have valency 2. Finally the statement that we have segments follows from Lemma 5.12 as in Corollary 6.5. □

6.6.1. All indices are of type I, $L = 0$. We have already seen (Case 1) that the case of the single loop and all indices are of type I is not possible. Let us thus treat the special case when we are in the doubly odd loop and still all indices of $A \cup C$ are of type I or when just the indices of A are of type I but we know that they form a segment.

If neither S_A, S_B, S_C contains a critical vertex we have seen that the graph spanned by $A \cup C$ is a segment as well as S_B and we have.

$$a')\ 0 \xrightarrow{S_{A\cup C}} v \xrightarrow{S_B} w. \tag{38}$$

In this segment we take as root one on its end points and denote by $\bar\sigma_i$, $\bar\lambda_i$ the corresponding values of color and orientation (with respect to this root). Recall that the notation σ_i, λ_i is relative to the segment S_u as in the previous discussion (see Formula (31)). In the next Lemma we analyze the 9 cases in which $L = 0$.

LEMMA 6.7. *We claim that every edge ℓ_j, $j \in A$ (resp. $j \in C$) has the property that $\delta_j = \delta\bar\sigma_j$ if red and $\delta_j = \delta\bar\lambda_j\bar\sigma_j$ if black for $\delta = \delta_1\bar\sigma_1$ (resp. $\delta = \delta_h\bar\sigma_h$ where h is the minimal element in C).*

PROOF. By induction $\delta_{u-1} = \delta\bar\sigma_{u-1}$ if red and $\delta_{u-1} = \delta\bar\lambda_{u-1}\bar\sigma_{u-1}$ if black.

Look at S_u. If ℓ_{u-1}, ℓ_u are both red $\sigma_{u-1} = 1$, (Case 1))

$$\delta_u = -\delta_{u-1} = -\delta\bar\sigma_{u-1} = \delta\bar\sigma_u\sigma_{u-1} = \delta\bar\sigma_u$$

If ℓ_{u-1} is red and ℓ_u is black we are in Cases 3), 6) and we have $\sigma_{u-1} = \lambda_u$, $\delta_u = \delta_{u-1} = \delta\bar\sigma_{u-1}$. We also have $\sigma_{u-1} = -\bar\sigma_{u-1}\bar\sigma_u$ if $\ell_{u-1} \prec \ell_u$ and $\sigma_{u-1} = \bar\sigma_{u-1}\bar\sigma_u$ if $\ell_u \prec \ell_{u-1}$.

$$\delta_u = \begin{cases} -\delta\bar\sigma_u\lambda_u = \delta\bar\sigma_u\bar\lambda_u & \ell_{u-1} \prec \ell_u \\ \delta\bar\sigma_u\lambda_u = \delta\bar\sigma_u\bar\lambda_u & \ell_u \prec \ell_{u-1} \end{cases}.$$

If ℓ_{u-1} is black and ℓ_u is red we are in Cases 7), 10) and we have $\sigma_{u-1} = \lambda_{u-1}$. If $\ell_{u-1} \prec \ell_u$ we have $\lambda_{u-1}\bar{\lambda}_{u-1} = -1$, $\bar{\sigma}_{u-1} = \bar{\sigma}_u\sigma_{u-1}$

$$\delta_u = -\delta_{u-1} = -\delta\bar{\sigma}_{u-1}\bar{\lambda}_{u-1} = -\delta\bar{\sigma}_u\sigma_{u-1}\bar{\lambda}_{u-1} = -\delta\bar{\sigma}_u\lambda_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u.$$

If $\ell_u \prec \ell_{u-1}$ we have $\lambda_{u-1}\bar{\lambda}_{u-1} = 1$, $\bar{\sigma}_{u-1} = -\bar{\sigma}_u\sigma_{u-1}$

$$\delta_u = -\delta_{u-1} = -\delta\bar{\sigma}_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\sigma_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\lambda_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u.$$

If ℓ_{u-1}, ℓ_u are both black we are in Cases 11), 14), 16), 17) and we have $\sigma_{u-1} = \lambda_u\lambda_{u-1}$ by Corollary 6.3. If $\ell_{u-1} \prec \ell_u$ (in the order of the total segment) we have $\lambda_{u-1}\bar{\lambda}_{u-1} = -1$, $\bar{\sigma}_{u-1} = \bar{\sigma}_u\sigma_{u-1}$

$$\delta_u = \delta_{u-1} = \delta\bar{\sigma}_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\sigma_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\lambda_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u.$$

$$\delta_u = -\delta_{u-1} = -\delta\bar{\sigma}_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\sigma_{u-1}\bar{\lambda}_{u-1} = \delta\bar{\sigma}_u\lambda_{u-1}\lambda_u\bar{\lambda}_{u-1}$$

Now clearly $\lambda_{u-1}\lambda_u\bar{\lambda}_{u-1} = \bar{\lambda}_u$. □

Now we take the left vertex of $S_{A\cup C}$ as in (38) as root, that is we consider it as the 0 vertex and want to compute first the value of the other end vertex v of $S_{A\cup C}$ and then the end vertex w of the total segment appearing in (38). Recall that we have an even number of red edges so that the end vertex is black, let us say that this vertex belongs to the last edge ℓ_j. We can compute it by using the various options of formula (29). If ℓ_j is red or if it is black and $\lambda_j = -1$ we have that the last vertex is $v = b_j$ and not a_j, in the remaining case $v = a_j$. In all cases a simple analysis shows that $v = \pm\sum_j \bar{\lambda}_j\bar{\sigma}_j\ell_j$. By Lemma 6.7 we have $\bar{\lambda}_j\bar{\sigma}_j = \delta\delta_j$ hence $\sum_{j\in A} \bar{\lambda}_j\bar{\sigma}_j\ell_j = \pm 2e_1$ and similarly $\pm\sum_{j\in C} \bar{\lambda}_j\bar{\sigma}_j\ell_j = \pm 2e_b$. We thus have that $v = \pm 2(e_1 - e_b)$ or $v = \pm 2(e_1 + e_b)$ but this is impossible for a black vertex which has mass 0.

Now a similar argument on the segment S_B gives as value of S_B either $\pm(e_1 - e_b)$ or $-e_1 - e_b$.

In the first case we take as root the point v. Now the left and right hand vertices are $a = \pm(e_1 - e_b)$, $b = \pm 2(e_1 - e_b)$. The relation is $b = \pm 2a$ so the resonance must be $C(b) = \pm 2C(a)$ which we see immediately is not valid.

It remains the possibility $a = -e_1 - e_b$, $b = \pm 2(e_1 - e_b)$, in this case fixing one end vertex to be 0 the other is $a + b = -e_1 - e_b \pm 2(e_1 - e_b)$ which also gives a non allowable graph from Definition 3.13 and Proposition 3.14.

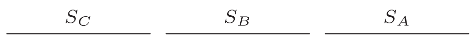
If the edges in A form a segment and are of type I the same argument shows that fixing the root at one end the other end vertex is $-2e_i$ for some i. We deduce

COROLLARY 6.8. *The case of all indices of type I does not occur or it produces a not-allowable graph* 3.13.

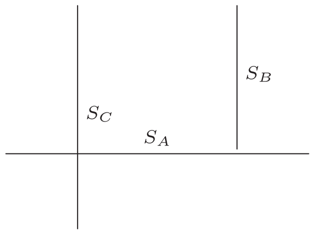
6.8.1. Indices of type II. If there is at least one index of type II the case analysis that we have performed shows that between two edges in A there are only edges in A and the edges in A form a segment, the same happens for

B, C. Denoting S_A, S_B, S_C these segments their union is a tree, the internal vertices of S_B have valency 2, so their relative position a priori can be only one of the following.

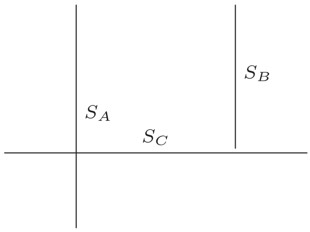
a)



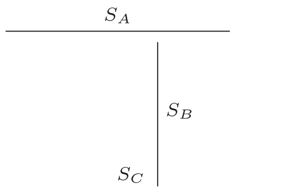
b)



c)

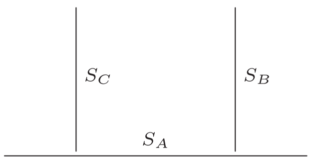


d)

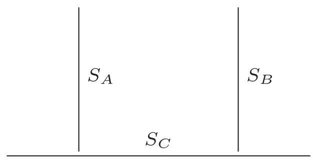


where if only one of S_A, S_C contains a critical vertex we have the special cases

b′)

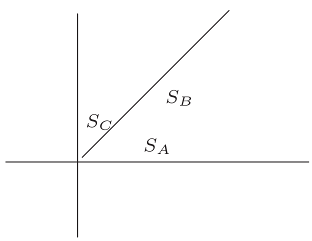


c′)



In all these cases it is possible that the two critical vertices coincide as in

b″)



In all these cases we may also have that B is empty so S_B does not appear.

2) If A contains no index of type II) we apply to it Lemma 6.7 and deduce that the segment equals $\delta \sum_{i \in A} \delta_i \ell_i = -2\delta e_1$. Since the mass of a segment can only be 0, -2 we deduce that if one extreme is set to be 0 the other is $-2e_1$.

3) is similar to 2).

Notice that at this point we have proved for the doubly odd loop Theorem 4 in all cases except b), c), d), b′). Of course b) and c) are equivalent and in fact b′) is a special case of b).

4) Let us treat the case in which $u \in A$ gives a contribution to L equal $\pm 2e_{u-1}$ (the other is similar), from our analysis in our setting all edges ℓ_j, $j \leq u-2$ must be comparable with ℓ_u.

In all cases we have that S_A and S_C have a unique critical vertex which divides the segment.

So S_A is divided into two segments, one X ending with a red vertex x the other Y with a black vertex y since in S_A there is an odd number of red edges which are distributed into the two segments.

We choose as root the critical vertex. With this choice we denote by $\bar{\sigma}$, $\bar{\lambda}$ the corresponding values on the edges (in order to distinguish from the ones σ, λ we have used where the root is at the beginning of S_u).

Lemma 6.9. i) *The edges in Y, X have the property that, $\delta_j \bar{\sigma}_j \bar{\lambda}_j = \delta$ is constant.*

ii)

$$y = \sum_{j \in Y} \bar{\sigma}_j \bar{\lambda}_j \ell_j = \delta \sum_{j \in Y} \delta_j \ell_j; \quad x = -\sum_{j \in X} \bar{\sigma}_j \bar{\lambda}_j \ell_j = -\delta \sum_{j \in X} \delta_j \ell_j$$
$$\delta = -1, \quad x - y = -2e_1$$

Proof. i) We want to prove that on X and Y the value $\delta_j \bar{\sigma}_j \bar{\lambda}_j$ is constant. For this by induction it is enough to see that the value does not change for ℓ_u, ℓ_{u-1}. When they are not separated we can use Lemma 6.7. When separated we first compare the values that we call $\bar{\sigma}_j$ when we place the root at the critical vertex with the values σ_j when we place the root at the beginning of ℓ_u and we easily see that $\bar{\sigma}_u \bar{\sigma}_{u-1} = \sigma_{u-1}$. In order to prove that $\delta_j \bar{\sigma}_j \bar{\lambda}_j$ is constant we need to show that when ℓ_u, ℓ_{u-1} are separated

$$1 = \delta_{u-1} \bar{\sigma}_{u-1} \bar{\lambda}_{u-1} \delta_u \bar{\sigma}_u \bar{\lambda}_u = \delta_{u-1} \sigma_{u-1} \bar{\lambda}_{u-1} \delta_u \bar{\lambda}_u.$$

We have $\bar{\lambda}_{u-1} = \lambda_{u-1}$ while $\bar{\lambda}_u = -\vartheta_u \lambda_u$. In other words we need

$$-\delta_{u-1} \vartheta_u \sigma_{u-1} \lambda_{u-1} \delta_u \lambda_u = 1.$$

Since by definition $\delta_{u-1} \vartheta_u = \delta_u$ we have to verify that

$$-\delta_{u-1} \vartheta_u \sigma_{u-1} \lambda_{u-1} \delta_u \lambda_u = -\sigma_{u-1} \lambda_{u-1} \lambda_u = 1.$$

This is in our case the content of the second part of Corollary 6.3.

ii) By definition

$$y = \sum_{j \in Y} \bar{\sigma}_j \bar{\lambda}_j \ell_j = \delta \sum_{j \in Y} \delta_j \ell_j; \quad x = -\sum_{j \in X} \bar{\sigma}_j \bar{\lambda}_j \ell_j = -\delta \sum_{j \in X} \delta_j \ell_j$$

hence $x - y = -\delta \sum_{j \in A} \delta_j \ell_j = \delta 2e_1$. But $\eta(x) = -2$, $\eta(y) = 0$ implies $\delta = -1$. □

If we take as root the vertex x the other vertex of S_A is $x+y$.

Proposition 6.10. *If the graph is resonant $x+y=-2e_j$ for some j.*

Proof. We choose as root the critical vertex of S_A. We have $x-y=-2e_1=\sum_{j\notin A}\delta_j\ell_j$. This is a linear combination of the edges outside the segment S_A therefore the resonance relation has the form:

$$C(x)-C(y)=\sum \alpha_i C(v_i)$$

where the vertices v_i are linear combination of the edges not in A. Therefore these vertices have support which intersects the support of the vertices in S_A only in e_1, hence we must have $C(x)-C(y)=\alpha e_1^2$ for some α. Applying the mass η we see that $\eta(C(y))=0$, $\eta(C(x))=-1$ hence $\alpha=-1$.

We now apply the rules of the operator C to x red, y black

$$-e_1^2=C(-2e_1)=-C(-y)+C(x)+xy=-C(y)+y^{(2)}+C(x)+xy$$

and get that $C(x)-C(y)=-e_1^2-y^{(2)}-xy$. Thus if the graph is resonant we must have $y^{(2)}+xy=0$. One easily verifies that $y^{(2)}$ is an irreducible polynomial unless y is of the form $y=\beta(e_i-e_j)$. In this case from the factorization $y^{(2)}=-xy$ and the fact that $\eta(x)=-2$ we deduce that $x=-e_i-e_j$. Since $x-y=-2e_1$ we must have that $\beta=\pm 1$ and if $\beta=1$ we have $e_i=e_1$, $x+y=-2e_j$. If $\beta=-1$ we have $e_j=e_1$, $x+y=-2e_1$. $\square$

We have thus verified that the graph is not-allowable by Definition 3.13 for the two extremes of the segment S_A, a similar analysis would apply to S_C.

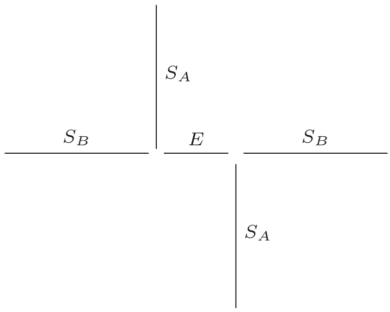
6.11. The extra edge. We treat now case 1) with an extra edge $E=\vartheta e_1-e_h$, $\vartheta=\pm 1$. We have the function ζ such that $\zeta(e_1)=1$, $\zeta(\ell_i)=0$, $\forall i$ and $\zeta(E)=2\vartheta$. In this case the even loop is divided into two odd paths. We divide the indices different from the two critical indices $1, h$ in two blocks $A=(2,\dots,h-1)$, $B=(h+1,\dots,k-1)$ and argue as in the previous section.

From Corollary 6.5 it follows that, either the extra edge is outside the segment spanned by the ℓ_i, this may happen if we are in a situation as (up to symmetry between A, B)

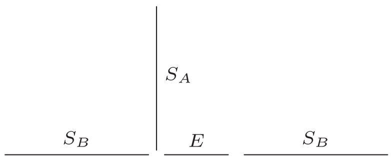
a) b)

In these cases the edge E can be removed and the graph is not minimal. Otherwise it could separate the two segments spanned by the two blocks A, B or it could appear in one or both of these segments according to the following pictures:

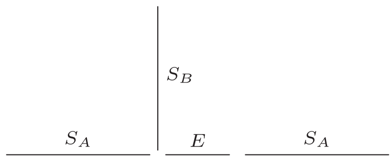
c)



d)



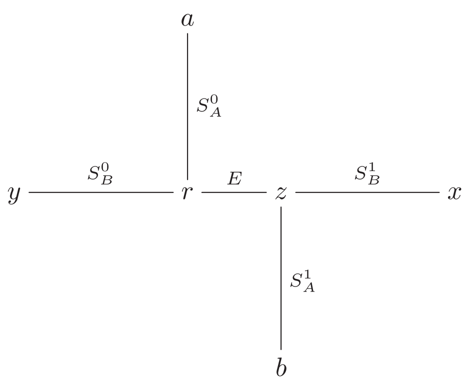
e)



Cases d), e) are special cases of c), and in fact follow from previous results, so we treat case c).

6.11.1. $E = e_1 - e_h$ *is black.* We look at the picture c).

c)



We can fix the signs δ_i so that

$$\sum_{i=1}^{h-1} \delta_i \ell_i = -e_1 - e_h, \quad \sum_{i=h}^{k} \delta_i \ell_i = e_1 + e_h.$$

Of the two vertices y, x one is black the other is red. The same for a, b.

Case 1: a, y **black** b, x **red** gives for the various paths:

$$S_B^1 = z + x, \quad S_B^0 = y, \quad S_A^0 = a, \quad S_A^1 = z + b$$

$$y = \sum_{j \in S_B^0} \sigma_j \lambda_j \ell_j = \delta \sum_{j \in S_B^0} \delta_j \ell_j, \quad x = -E - \sum_{j \in S_B^1} \sigma_j \lambda_j \ell_j = -E - \delta \sum_{j \in S_B^1} \delta_j \ell_j$$

$$a = \sum_{j \in S_A^0} \sigma_j \lambda_j \ell_j = \delta' \sum_{j \in S_A^0} \delta_j \ell_j, \quad b = -E - \sum_{j \in S_A^1} \sigma_j \lambda_j \ell_j = -E - \delta' \sum_{j \in S_A^1} \delta_j \ell_j$$

$$x - y = -\delta \sum_{i \in B} \delta_i \ell_i - E = -\delta(e_1 + e_h) - e_1 + e_h,$$

$$b - a = -\delta' \sum_{i \in B} \delta_i \ell_i - E = \delta'(e_1 + e_h) - e_1 + e_h$$

for two signs δ, δ'. Applying the mass η we see that $\delta = 1, \delta' = -1$ hence $x - y = b - a = -2e_1$ is the relation among the vertices of the graph. By resonance

$$x - y = b - a, \ \Rightarrow \ C(x) - C(y) = C(b) - C(a).$$

We now apply the rules of the operator C to x red, y black

$$-e_1^2 = C(-2e_1) = -C(-y) + C(x) + xy = -C(y) + y^{(2)} + C(x) + xy$$

and get that $C(x) - C(y) = -e_1^2 + y^{(2)} + xy = -e_1^2 + y^{(2)} + (y - 2e_1)y$. On the other hand this element is a quadratic polynomial in the elements e_i appearing in the edges of B which must be equal by the resonance relation to a quadratic polynomial in the elements e_i appearing in the edges of A. Now the edges of A have in common with the edges of B only the elements e_1, e_h, so $-e_1^2 + y^{(2)} + (y - 2e_1)y$ must contain only these indices, it easily follows that if an element e_i, $i \neq 1, h$ appears in y with coefficient α we must have $\alpha = -1$, moreover if e_i appears in y no e_j, $j \neq 1$ can appear in y otherwise we have a mixed term in y^2 of type $2e_ie_j$ which does not cancel. Next we can only have $y = e_1 - e_i$ in order to cancel the mixed term from $-2e_1y$.

In this case the segment from y to x has value $x - (-y) = x - y + 2y = -2e_1 + 2(e_1 - e_i) = -2e_i$ and the result is proved.

The other possibility is that $y = \alpha(e_1 - e_h)$ for some α, since y is in any case a sum of edges in B this is actually not possible by computing the value of ζ.

a, y **red** b, x **black** is symmetric to the previous case.

Case 2: a, x **black** b, y **red** gives, as in the previous case, the value $b - a = -2e_1$. Then:

$$S_B^1 + z = x, \quad S_B^0 = y, \quad S_A^0 = a, \quad S_A^1 - z = b$$

$$y = -\sum_{j \in S_B^0} \sigma_j \lambda_j \ell_j = -\delta \sum j \in S_B^0 \delta_j \ell_j,$$

$$x = E + \sum_{j \in S_B^1} \sigma_j \lambda_j \ell_j = E + \delta \sum j \in S_B^1 \delta_j \ell_j$$

$$x - y = \delta \sum_{i \in B} \delta_i \ell_i + E = \delta(e_1 + e_h) + e_1 - e_h,$$

by mass $\delta = 1$ and $y - x = -2e_1$, we argue as before.

6.11.2. $E = -e_1 - e_h$ is red. In this case the even loop is divided into two even paths. We can fix the signs δ_i so that

$$\sum_{i=1}^{h-1} \delta_i \ell_i = e_1 - e_h, \quad \sum_{i=h}^{k} \delta_i \ell_i = -e_1 + e_h.$$

We still have a situation as in the previous analysis with some changes.

Case 1: a, y **black** b, x **red** gives for the various paths:

$$y = \sum_{j \in S_B^0} \sigma_j \lambda_j \ell_j = \delta \sum_{j \in S_B^0} \delta_j \ell_j, \quad x = E - \sum_{j \in S_B^1} \sigma_j \lambda_j \ell_j = E - \delta \sum_{j \in S_B^1} \delta_j \ell_j$$

$$a = \sum_{j \in S_A^0} \sigma_j \lambda_j \ell_j = \delta' \sum_{j \in S_A^0} \delta_j \ell_j, \quad b = E - \sum_{j \in S_A^1} \sigma_j \lambda_j \ell_j = E - \delta' \sum_{j \in S_A^1} \delta_j \ell_j$$

$$x - y = -\delta \sum_{i \in B} \delta_i \ell_i + E = -\delta(-e_1 + e_h) - e_1 - e_h,$$

$$b - a = -\delta' \sum_{i \in B} \delta_i \ell_i + E = \delta'(e_1 - e_h) - e_1 - e_h$$

for two signs δ, δ'. Thus $x - y$, $b - a$ can take the values $-2e_1$, $-2e_h$. If they take the same value we have $x - y = b - a$ and we argue as in the previous section. Otherwise up to symmetry we may assume that $x - y = -2e_1$, $b - a = -2e_h$ and $x - y = b - a + 2z$ is the relation among the vertices of the graph. By resonance

$$\begin{aligned} x - y = b - a + 2z, \Rightarrow\ C(x) - C(y) &= C(b) - C(a) + 2C(E) \\ &= C(b) - C(a) - 2e_1 e_h. \end{aligned}$$

We now apply the rules of the operator C to x red, y black

$$-e_1^2 = C(-2e_1) = -C(-y) + C(x) + xy = -C(y) + y^{(2)} + C(x) + xy$$

and get that $C(x) - C(y) = -e_1^2 + y^{(2)} + xy = -e_1^2 + y^{(2)} + (y - 2e_1)y$. On the other hand this element is a quadratic polynomial in the elements e_i appearing in the edges of B which must be equal by the resonance relation to a quadratic polynomial in the elements e_i appearing in the edges of A. Now the edges of A have in common with the edges of B only the elements e_1, e_h, so $-e_1^2 + y^{(2)} + (y - 2e_1)y$ must contain only these indices, it easily follows that if an element e_i, $i \neq 1, h$ appears in y with coefficient α we must have $\alpha = -1$, moreover two distinct elements of this type cannot appear otherwise we have a mixed term in y^2 of type $2e_i e_j$ which does not cancel. Next we can only have $y = e_1 - e_i$ in order to cancel the mixed term from $-2e_1 y$.

In this case the segment from y to x has value $x - (-y) = x - y + 2y = -2e_1 + 2(e_1 - e_i) = -2e_i$ and the result is proved.

The other possibility is that $y = \alpha(e_1 - e_h)$ for some α, this is possible only if $\alpha = \pm 1$ and $y = \sum_{j \in S_B} \sigma_j \lambda_j \ell_j$ all edges are involved, and $x = z$. Then the segment from y to $x = z = E$ has values $E - y = -e_1 - e_h \pm (e_1 - e_h) = -2e_1, -2e_h$.

a, y **red** b, x **black** is symmetric to the previous case.

Case 2: a, x **black** b, y **red** gives as in the previous case the value $b - a = -2e_1, -2e_h$. Then:

$$y = -\sum_{j \in S_B^0} \sigma_j \lambda_j \ell_j = -\delta \sum j \in S_B^0 \delta_j \ell_j,$$

$$x = -E + \sum_{j \in S_B^1} \sigma_j \lambda_j \ell_j = -E + \delta \sum j \in S_B^1 \delta_j \ell_j$$

$$y - x = -\delta \sum_{i \in B} \delta_i \ell_i + E = -\delta(-e_1 + e_h) - e_1 - e_h \in \{-2e_1, -2e_h\}.$$

We argue again as before.

Part 2. The irreducibility theorem

7. The matrices

The operator $ad(N) = 2iQ$ under study acts on the space spanned by the frequency basis and here it decomposes into blocks corresponding to the connected components of the Cayley graph G_X restricted by Defnition 2.12 (Theorem 2).

For each such component A we have seen that Q acts as a scalar $K(a)$ plus a matrix C_A homogeneous of degree 1 in the variables ξ_i. According to Formulas (12), (13), (14) the entries of $C_A = (c_{a,b})$ are the following. If $a \in A$, $a = \sum_i a_i e_i \in \mathbb{Z}^m$ the diagonal entry $c_{a,a} = -a(\xi) = -\sum_i a_i \xi_i$. If $a \in A$, $a = \left(\sum_i a_i e_i\right)\tau \in \mathbb{Z}^m \tau$ the diagonal entry $c_{a,a} = a(\xi) = \sum_i a_i \xi_i$.

If $a, b \in A$ are not connected by an edge $c_{a,b} = 0$. If $a, b \in \mathbb{Z}^m$ are connected by a black edge $e_i - e_j$ then $c_{a,b} = 2\sqrt{\xi_i \xi_j}$, if $a, b \in \mathbb{Z}^m \tau$ are connected by a black edge $e_i - e_j$ then $c_{a,b} = -2\sqrt{\xi_i \xi_j}$, finally if a, b are connected by a red edge $-e_i - e_j$ then one of them is in $\mathbb{Z}^m$ the other in $\mathbb{Z}^m \tau$ and we have $c_{a,b} = -2\sqrt{\xi_i \xi_j}$ if $a \in \mathbb{Z}^m$, $b \in \mathbb{Z}^m \tau$ and $c_{a,b} = 2\sqrt{\xi_i \xi_j}$ in the other case. If red edges are not present the matrix is symmetric.

Notice then some rules, if $b \in \mathbb{Z}^m$ we have $C_{Ab} = C_A - b(\xi) Id$, finally $C_{A\tau} = -C_A$.

By Lemma 2.14, when we expand the characteristic polynomial of such a matrix the square roots disappear and we get a polynomial, denoted $\chi_A(t)$ (or sometimes just χ_A) monic in t and with coefficients polynomials in the variables ξ_i with integer coefficients. Our goal is to prove that

Theorem 5 (irreducibility theorem). *If A is a non-degenerate allowable graph in G_X the polynomial $\chi_A(t)$ is irreducible as polynomial in $\mathbb{Z}[t, \xi]$.*

We prove furthermore that the graph A is determined by $\chi_A(t)$, this we call the *separation Lemma* 9.2.

In fact in this form the statement is not true, we need to use the fact that mass is conserved. This is enough for the dynamical consequences. In algebraic terms the conservation of mass consists in restricting to the coset of G_2 (one of the connected components of the Cayley graph) of elements a, $a\tau \in G$, $a \in \mathbb{Z}^m$, $\eta(a) = -1$. We also need to use systematically Theorem 4 which tells us that we can restrict to those graphs in which the vertices are affinely independent.

Remark 7.1. The hypothesis that the graph is non-degenerate is necessary. In the simple example of

$$0 \xrightarrow{1,2} e_2 - e_1 \xrightarrow{1,2} 2e_2 - 2e_1$$

one easily verifies that the characteristic polynomial is not irreducible.

On the other hand it is likely that the condition to be allowable is not necessary in order to prove irreducibility and separation. To avoid it complicates the proofs and, since we do not need the stronger result, we have not tried to discuss it.

8. Irreducibility and separation

8.1. Preliminaries. Observe first that, given $g \in G$, $A \subset G$ we have that $\chi_A(t)$ is irreducible if and only if $\chi_{Ag}(t)$ is irreducible.

Consider a projection $\pi_i : \mathbb{Z}^m \rtimes \mathbb{Z}/(2) \to \mathbb{Z}^{m-1} \rtimes \mathbb{Z}/(2)$ where we remove the i^{th} coordinate $\pi_i[(a_1, \dots, a_m), \delta] \mapsto [(a_1, \dots, \check{a}_i, \dots a_m), \delta]$. Take now a set $A \subset \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$ of vertices and consider the graph obtained from Γ_A by removing all the edges which contain i in its marking, call this new graph Γ_A^i. Even if A is connected this new graph Γ_A^i may well not be connected. We now claim

Proposition 8.2. *If A is connected the map π_i, restricted to Γ_A^i, is injective and a graph isomorphism with $\Gamma_{\pi_i(A)}$, a graph in $\mathbb{Z}^{m-1} \rtimes \mathbb{Z}/(2)$.*

If A is non degenerate each connected component of $\Gamma_{\pi_i(A)}$ is non degenerate.

Proof. We know that the mass $\ell = \eta(a)$ depends only on the color of a so that we have $a_i = \eta(a) - \eta(\pi_i(a))$ and thus if a, b are black vertices (or red vertices), $\pi_i(a) = \pi_i(b) : \eta(a) = \eta(b)$ hence $a_i = b_i \Rightarrow a = b$. Otherwise, if a is black, b is red then it is clearly $\pi_i(a) \neq \pi_i(b)$ because $\pi_i(a)$ is black, $\pi_i(b)$ is red. If we decompose $X = X_m$ into the elements containing the index i and the complement X_m^i we see that π_i establishes a 1–1 correspondence between X_m^i and X_{m-1} from which the second claim since π_i is a group homomorphism. The third claim follows easily from the definitions. □

A simple corollary of this proposition is that.

Corollary 8.3. *If we set* $\xi_i = 0$ *in the matrix* C_A *we have the matrix* $C_{\pi_i(A)}$, *hence*

$$\chi_A(t)|_{\xi_i=0} = \chi_{\pi_i(A)}(t)$$

Let $B_1, \dots, B_k$ *be the connected components of* $\pi_i(A)$. *We have*

$$\prod_{j=1}^{k} \chi_{B_j}(t) = \chi_{\pi_i(A)}(t) = \chi_A(t)|_{\xi_i=0}.$$

As a consequence, we have the following inductive step.

Corollary 8.4. *Assume that* A *is non degenerate and that we have already proved the irreducibility theorem for* $m-1$ *or for* $n < |A|$. *We deduce that the factors* $\chi_{B_j}(t)$ *of* $\chi_{\pi_i(A)}(t)$ *are the irreducible monic factors of* $\chi_A(t)|_{\xi_i=0}$.

We want to prove Theorem 1 by induction as follows. We assume irreducibility and separation in dimension $n-1$ and prove first the separation in dimension n and finally irreducibility in dimension n.

Take a connected A and let ℓ be the mass of a black vertex of A, then the mass of a red vertex is $-2-\ell$.

Lemma 8.5 (Parity test). i) *If we compute* t *at a number* $g \not\cong \ell$ mod (2), *we have* $\chi_A(g) \neq 0$.

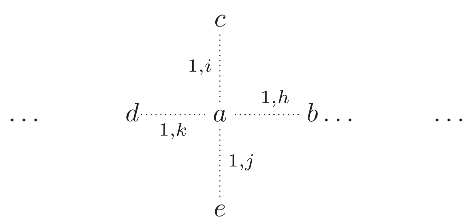
ii) *If a linear form* $t + \sum_i a_i \xi_i$, $a_i \in \mathbb{Z}$ *divides* $\chi_A(t)$ *we must have* $\sum_i a_i \cong \ell$ mod (2).

Proof. i) The matrix C_A modulo 2 is diagonal and $\chi_A(t) \cong \prod_i (t + a_i(\xi))$ mod (2). If we compute modulo 2 and set all $\xi_i = 1$, we get $\chi_A(t) \cong (t+\ell)^m$ mod (2), hence $\chi_A(g) \cong (g+\ell)^m \cong g + \ell$ mod (2).

ii) A linear form $t + \sum_i a_i \xi_i$, $a_i \in \mathbb{Z}$ divides $\chi_A(t)$ if and only if we have $\chi_A\left(-\sum_i a_i \xi_i\right) = 0$, then set $\xi_i = 1$ and use the first part. □

We shall use the parity test as follows.

Lemma 8.6. *Suppose we have a connected set* A *in* $\mathbb{Z}^m$, *in which we find a vertex* a *and an index, say* 1, *so that the graph* Γ_A *has the following properties:*



we have:

- 1 *appears in all and only the edges having a as vertex.*
- *When we remove a (and the edges meeting a) we have a connected graph* $\mathcal{A}$ *with at least* 2 *vertices.*
- *When we remove the edges associated to any index, the factors described in Corollary* 8.3 *are irreducible.*

Then the polynomial $\chi_A(t)$ *is irreducible.*

PROOF. We take a as root, and translate the set A so that $a = 0$. Setting $\xi_1 = 0$ we have by Corollary 8.3 and the hypotheses, that $\chi_A(t) = tP(t)$ with $P = \chi_{\mathcal{A}}(t)$ irreducible of degree > 1. Thus, if the polynomial $\chi_A(t)$ factors, then it must factor into a linear $t - L(\xi)$ times an irreducible polynomial of degree > 1.

Moreover modulo $\xi_1 = 0$ we have that 0 and ℓ coincide, thus $L(\xi)$ is a multiple of ξ_1.

Take another index $i \neq 1, h$ if a is an end and the only edge from a is marked $(1, h)$ otherwise just different from 1 and set $\xi_i = 0$. Now the polynomial $\chi_A(t)$ specializes to the product $\prod_j \chi_{A_j}(t)$ where the A_j are the connected components of the graph obtained from A by removing all edges in which i appears as marking. By hypothesis $\{a\}$ is not one of the A_j.

If no factor is linear we are done. Otherwise there is an isolated vertex $d \neq a$ so that $\{d\}$ is one of the connected components A_j. The linear factor associated is $t + d(\xi)|_{\xi_i=0}$. Clearly we have that the coefficient of ξ_1 in $d(\xi)$ is ± 1 (since the marking 1 appears only once). This implies that $L(\xi) = \pm\xi_1$ and this is not possible by the parity test. □

9. THE SEPARATION LEMMA

Given a connected graph $G \subset G_X$ consider $\tau G = \{(-a, -\delta) \mid (a, \delta) \in G\}$.

REMARK 9.1. τG is a connected graph, if and only if G contains only black edges.

PROOF. The connected components of the Cayley graph are the cosets $G_2 u$, $u \in G$. If there exists a red edge $(-e_i - e_j, \tau)$ connecting two elements $a, b \in G$ then $ba^{-1} = (-e_i - e_j, \tau) \Rightarrow \tau b(\tau a)^{-1} = (e_i + e_j, \tau) \notin G_2$. τb, τa are not in the same connected component of the Cayley graph. Instead $ba^{-1} = e_i - e_j \Rightarrow \tau b(\tau a)^{-1} = e_j - e_i$, τa, τb are connected by a black edge marked j, i in τG. □

LEMMA 9.2 (Separation lemma). *Given two connected non-degenerate allowable graphs* $G_1, G_2 \subset G_X$ *if* $\chi_{G_1} = \chi_{G_2}$, *then* $G_1 = G_2$ *or* $G_1 = \tau G_2$.

If we take $G \subset G^1$, then G is of mass -1 we have that τG is of mass 1, we deduce that a connected color marked graph G of mass -1 can be recovered from its characteristic polynomial.

PROOF. We will prove this lemma by induction. When $n = 0 : \chi_G(t) = t + a$, it is easy to see that $G = \{(a, +)\}$ or $G = \{(-a, -)\}$.

Induction process: $n > 1$. Suppose that we have the separation and the irreducibility for graphs of dimensions $k \leq n - 1$. Take a connected colored marked graph $G = \{(v_1, \delta_1), \dots, (v_{n+1}, \delta_m)\}$, $(v_i, \delta_i) \in \mathbb{Z}^m \rtimes \mathbb{Z}/(2)$, the associated matrix C_G and its characteristic polynomial χ_G.

Associate to G the list L of vectors $w_i := \delta_i v_i$, we see that these vectors are affinely independent. If the w_i have all the same mass then the graph G has only black edges and then it is either the graph with vertices w_i or with vertices τw_i as seen before, if they have different masses then the masses are of type k for black vertices and $k + 2$ for red and the graph G is thus reconstructed from L.

Therefore we need to show that, from the characteristic polynomial, we can recover the list $L := \{w_1, \dots, w_n\}$. Before starting the proof let us make a useful remark, the characteristic polynomial gives as information the trace of the matrix C_G and thus in particular the sum $\sum_{i=1}^n w_i(\xi)$ and the mass $s := \sum_{i=1}^n \eta(w_i)$. If we have a elements in the list of mass k and $(n - a)$ of mass $k + 2$ we have that $s = nk + 2b = n(k + 2) - 2(n - b)$. Thus if we know that a certain number h is the mass of a vertex we can deduce

LEMMA 9.3. *If $s = nh$ then all vertices in G have the same color. If $nh < s$ then h is the mass of the black vertices and there are b red vertices where $s = nh + 2b$. Similarly if $nh > s$ then h is the mass of the red vertices and there are b red vertices where $s = nh - 2(n - b)$.*

We set one of the variables $\xi_i = 0$ for instance $\xi_1 = 0$. We know that the matrix C_G specializes to the direct sum of the matrices C_{G_i} where the G_i correspond to the various connected components of the graph G which are obtained by removing all edges in which 1 appears as marking and dropping in each component the first coordinate of the various vertices. We have that specializing $\xi_1 = 0$ we specialize the polynomial χ_G to $\prod_i \chi_{G_i}$. Since we are assuming irreducibility in dimensions less than $n - 1$ the factors χ_{G_i} are all irreducible and thus can be determined by the unique factorization of polynomials. Therefore all the vectors of $\pi_1(L)$, that is the w_i with the first coordinate removed can be recovered uniquely (up to the sign) by induction and we obtain a list of n vectors $L^1 : \{(*, b_i, c_{3,i}, \dots, c_{m,i})\}$.

Now we set another variable, say $\xi_2 = 0$. By similar arguments as above all the w_i with the second coordinate removed can be recovered by induction giving a list $L^2 : \{(a_i, *, c_{3,i}, \dots, c_{m,i})\}$.

Now our problem is this: if we know the vectors obtained from L after removing the first or the second coordinate can we recover the given vectors? We shall need to perform a case analysis.

1) Recovering the list L:

We thus consider the vectors $L^{1,2}$ obtained from L by dropping the first two coordinates $(*, *, c_3, \dots, c_m)$ and collect the ones where $c_3, \dots, c_m$ are fixed. The first remark is that, if in this list a given vector $(*, *, c_3, \dots, c_m)$ appears only once then we know exactly from which vector it comes from the two lists L^1, L^2 and so we can reconstruct the vector v in L from which it arises. Then by Lemma 9.3 we can determine if in the graph all vertices have the same color or, if this is not the case, which is the mass of the black end red vertices and how many there are.

Next since the vectors in the graph, by assumption, are affinely independent, we have at most 3 vectors in L, giving the same vector $(*, *, c_3, \dots, c_m)$ in $L^{1,2}$ since 4 of such vectors lie in a 2-dimensional plane so they are not affinely independent.

a) Assume we have 3 vectors $v_1, v_2, v_3 \in L$ giving the same vector $\underline{c} = (*, *, c_3, \dots, c_m)$ in $L^{1,2}$ and let $c = \eta(\underline{c})$. We claim that v_1, v_2, v_3 cannot have the same color, in fact this would imply that they have the same mass and then they lie in a line and cannot be affinely independent. Let then a_1, a_2, a_3 resp. b_1, b_2, b_3 be the first, resp. second coordinates of these vectors (deduced from the two lists L^1, L^2) we need to be able to reconstruct the 3 vectors $v_1, v_2, v_3 \in L$ by matching the a_i with the b_j. First observe that we know the total mass m of v_1, v_2, v_3. This is $m = 3k + 2$ or $m = 3k + 4$ depending if we have two or 1 black vertices among v_1, v_2, v_3. Since $3k + 2$ is congruent to 2 modulo 3 while $3k + 4$ is congruent to 1 modulo 3, we can deduce both k and the number of black vertices from m.

Call $l := k - c$, now consider one of the vectors in L^1, start from $(a_1, *, \underline{c})$, if there is no b_i with $a_1 + b_i = l$ then there must necessarily be one, say b_1 with $a_1 + b_1 = l + 2$ and then $(a_1, *, \underline{c})$ comes from the red vector $(a_1, b_1, \underline{c})$. Similarly if there is no b_i with $a_1 + b_i = l + 2$ then there must necessarily be one, say b_1 with $a_1 + b_1 = l$ and then $(a_1, *, \underline{c})$ comes from the black vector $(a_1, b_1, \underline{c})$. In this case we can easily see how to match the other two vectors, in case the other two vectors have the same color we must match them so that $a_2 + b_i = l'$, $a_3 + b_j = l'$ where $l' = l$ if the color is black and $l + 2$ if red. We claim that only one match is possible, in fact if we had $a_2 + b_3 = a_3 + b_2 = a_2 + b_2 = a_3 + b_3$ we would have that the two vectors v_2, v_3 coincide.

Suppose now we know that the two colors are distinct, then as before, if there is no b_j, $j = 2, 3$ such that $a_2 + b_j = l$ we know that there is one, say b_2 for which $a_2 + b_2 = l + 2$ and we have reconstructed the two vectors $(a_2, b_2, \underline{c})$, $(a_3, b_3, \underline{c})$. Finally it is possible that $b_3 = b_2 + 2$ and $a_2 + b_2 = l$ then we have $a_3 + b_3 = l + 2$ which implies $a_3 = a_2 = a$ and again we reconstruct the two vectors (actually by Definition 3.13 this is not allowed).

It remains to analyze the case in which none of the a_i satisfies the condition that it cannot be paired uniquely.

So let us assume that, up to reordering b_1 is maximum. There is one a_i which must be paired with b_1 and we are assuming that it can also be paired with another b_i giving a different color. We must necessarily have that the value of this a_i, which we may assume reordering to be a_1 is $a_1 = l + 2 - b_1$, we have recovered a red vector $(a_1, b_1, \underline{c})$. The rest of the analysis follows as before.

b) There are in $L^{1,2}$ only 2 vectors of the form $(*, *, c_3, \dots, c_m)$ with $c_3, \dots, c_m$ fixed. For simplicity we denote $\underline{c} := (c_3, \dots, c_m)$ and their sum by c. We know then two vectors in $L^{1,2}$ of the form $(a_1, *, \underline{c})$, $(a_2, *, \underline{c})$ and two vectors in L^2 of the form $(*, b_1, \underline{c})$, $(*, b_2, \underline{c})$ which specialize in $L^{1,2}$ to the given vectors.

A priori in L we can either have $(a_1, b_1, \underline{c})$, $(a_2, b_2, \underline{c})$ or $(a_1, b_2, \underline{c})$, $(a_2, b_1, \underline{c})$. The first pair gives two vertices of the same color if and only if $a_1 + b_1 = a_2 + b_2$, similarly for the second. If we have $a_1 + b_1 = a_2 + b_2$, $a_1 + b_2 = a_2 + b_1$ we deduce that $a_1 = a_2$, $b_1 = b_2$ and this is impossible since it implies that in L we have two equal vectors, therefore in at least one of the two pairs we have different colors. We may thus assume (changing the indices if necessary) that $a_1 + b_2 = a_2 + b_1 + 2$, this implies $a_1 - a_2 = b_1 - b_2 + 2$. Write $a_1 + b_1 = a_2 + b_2 + x$, $x \in (-2, 0, 2)$ and thus $2(b_1 - b_2) = x - 2$. If $x = -2$ we have $b_1 - b_2 = -2$, $a_1 = a_2$ and we argue as before, this case is impossible.

If $x = 2$ we have $b_1 = b_2 = b$, $a_1 = a_2 + 2 = a + 2$ we have in the possible list of vectors $(a + 2, b, \underline{c})$, $(a, b, \underline{c})$. We know that this list is not allowed by Definition 3.13. Assume that $x = 0$ thus $b = b_1$, $b_2 = b + 1$, $a = a_2$, $a_1 = a + 1$ we have the two possibilities 1) $(a + 1, b, \underline{c})$, $(a, b + 1, \underline{c})$ or 2) $(a + 1, b + 1, \underline{c})$, $(a, b, \underline{c})$. In this case both cases are a priori possible, in fact if the graph were just a single edge marked $e_1 - e_2$ or $-e_1 - e_2$ the two cases cannot be recovered by the two specializations but only from the full characteristic polynomial.

$$
\begin{aligned}
&G_1 = (e_1, +) \xrightarrow{e_2 - e_1} (e_2, +) \qquad G_2 = (0, +) \xlongequal{-e_2 - e_1} (-e_1 - e_2, -), \\
(39) \qquad &C_{G_1} = \begin{vmatrix} -\xi_1 & 2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & -\xi_2 \end{vmatrix}, \quad C_{G_2} = \begin{vmatrix} 0 & -2\sqrt{\xi_1\xi_2} \\ 2\sqrt{\xi_1\xi_2} & -\xi_1 - \xi_2 \end{vmatrix}
\end{aligned}
$$

The characteristic polynomials are distinct:

$$t^2 + (\xi_1 + \xi_2)t - 3\xi_1\xi_2, \quad t^2 + (\xi_1 + \xi_2)t + 4\xi_1\xi_2$$

but the two specializations coincide.

So we need a deeper analysis. First let us assume that we know if all the vectors have the same mass or we know the mass of black and red vertices.

If we know that all vertices have the same mass then case 2) is excluded. Suppose then that we know the mass k of a black vertex.

If case 1) holds we must have that $a + b + c$ is either $k - 1$ or $k + 1$, if case 2) holds we must have that $a + b + c = k$. Thus we can determine in which case we are.

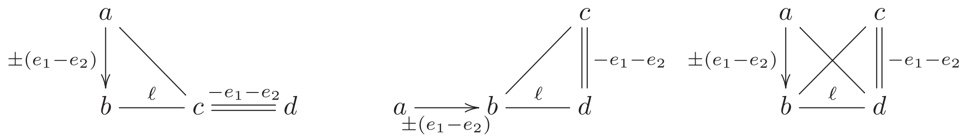
The other possibility is that we do not have the previous information but by the previous analysis this means that in the list $L^{1,2}$ each vector appears twice. If the list consists of just two vectors we can conclude by the explicit formulas of the characteristic polynomial.

Assume we have at least two pairs one u_1, u_2 giving $(*, *, \underline{c})$ the other v_1, v_2 giving $(*, *, \underline{d})$. In each case we know that the two vertices are connected either by the edge $e_1 - e_2$ or by $-e_1 - e_2$. We deduce that the only possibility at this point is that there are only two such lists so L has 4 elements and we must have both edges $e_1 - e_2$ and $-e_1 - e_2$.

The two edges involve two disjoint pairs of vertices so that the graph must be of the form

$$a \xrightarrow{\pm(e_1 - e_2)} b \overset{\ell}{\text{———}} c \xlongequal{-e_1 - e_2} d$$

if ℓ does not contain any of the indices 1, 2 or possibly of the form



if ℓ contains one of the indices 1, 2. The edge l can have either color (which determines the color of the further edge).

In particular the graph has either 3 black and one red vertex or 3 red and one black vertex so either $s = 4k + 6 = 4(k + 1) + 2$ or $s = 4k + 2$.

This gives two possible values for the mass of black vertices, k or $k + 1$. Finally specializing to $\xi_i = 0$ where $i \neq 1, 2$ appears in ℓ and to $\xi_1 = 0$ (or $\xi_2 = 0$) if 1 resp. 2 does not appear in ℓ we see that of the 4 vectors in $L^{1,2}$ at least one appears only once and we are back in the previous case which we have treated. □

10. Irreducibility theorem

We prove Theorem 5 by induction. Assume the separation and irreducibility in all dimensions less than n, we will prove the irreducibility in dimension n. Since this property is invariant under translation we often choose a vertex as the root and assume that it corresponds to 0. We thus always deal with combinatorial graphs and we may identify the black vertices as elements a in $\mathbb{Z}^m$ with $\eta(a) = 0$ and the red vertices as elements a in $\mathbb{Z}^m$ with $\eta(a) = -2$ (Remark 2.10).

Therefore from now on we assume that G is a combinatorial graph with $n + 1$ vertices and T a maximal tree in G with n linearly independent edges.

Lemma 10.1. *We have one of the following possibilities:*

i) *We have n indices all with multiplicity* 2.
ii) *We have at least two indices with multiplicity* 1 *in distinct edges.*
iii) *We have two indices with multiplicity* 1 *in the same edge the remaining with multiplicity* 2.
iv) *We have one index with multiplicity* 1 *one with multiplicity* 3 *and the remaining with multiplicity* 2.

Proof. We must have at least n distinct indices appearing in the edges, otherwise these edges span a subspace of dimension less than n. In total on the n edges of T appear $2n$ indices counted with multiplicity. If every index appears with multiplicity ≥ 2 we must have n indices all with multiplicity 2.

If we have at least 3 indices of multiplicity 1 we are in case ii), if we have only two indices of multiplicity 1 in the same edge, the remaining indices satisfy property i) for the remaining $n - 1$ edges. Assume finally that only one index appears with multiplicity 1. Of the remaining $k \geq n - 1$ indices appearing assume a have multiplicity ≥ 3 and b multiplicity 2 hence

$$a+b \geq n-1, \quad 3a+2b \leq 2n-1 \Rightarrow b \geq n-2$$

we deduce that $a=1$ and the multiplicity is 3, we are in the last case. □

We thus have to treat 4 cases.

REMARK 10.2.

- Dash lines mean that they may be black or red.
- Black edges are denoted by single lines, red edges-by double lines.
- $\bar{A}$ denotes the completed graph obtained from the graph A.

Sometimes given a combinatorial graph G by a *block* A of G we mean a connected complete subgraph A of G. If A is a block in a maximal tree T of G the completion $\bar{A}$ is a block in G. By abuse of notation we denote by $\chi_A(t) := \chi_{\bar{A}}(t)$ to be the characteristic polynomial of the matrix associated to $\bar{A}$. We now fix a maximal tree in G.

LEMMA 10.3. *If in T there are two blocks A, B and two indices i, j such that:*
i) *i, j do not appear in the edges of the blocks A, B.*
ii)

$$\chi_{\bar{A}} \cong \chi_{\bar{B}} \quad \textit{modulo } \xi_i = \xi_j = 0, \tag{40}$$

then $|B| = |A| = 1$, $A = \{(a, \sigma_1)\}$, $B = \{(b, \sigma_2)\}$, $a, b \in \mathbb{Z}^m$ *and* $b = \ell + \sigma_2\sigma_1 a$. *Where* $\ell = n_i e_i + n_j e_j$, $n_i + n_j = -1 + \sigma_2\sigma_1$.

Assume that i, j appear at most twice in the tree then if $\sigma_2\sigma_1 = 1$ *we may have* $\ell = \pm(e_i - e_j)$, $\pm 2(e_i - e_j)$. *If* $\sigma_2\sigma_1 = -1$ *we may have* $\ell = -e_i - e_j, -2e_i, -2e_j$.

PROOF. Since the degree of the characteristic polynomial is the number of vertices by assumption $|B| = |A|$. Choose the root in A. This gives to each vertex v a sign σ_v. Let $A = \{(a_1, \sigma_1), \dots, (a_r, \sigma_r)\}$; $B = \{(b_1, \delta_1), \dots, (b_r, \delta_r)\}$, then to these graphs we associate as in §9 the list L of vectors $v_h = \sigma_h a_h$ and $w_h = \delta_h b_h$. Since i, j do not appear in A (resp. B), the vectors v_h have the same i-th and j-th coordinates and we can write $v_h = \bar{v}_h + a$, similarly for B the vectors $w_h = \bar{w}_h + b$ where a, b are linear combinations of e_i, e_j and $\bar{v}_h$, $\bar{w}_h$ are linear combinations of the e_s, $s \neq i, j$.

The list of vectors $\bar{v}_h$ is the one associated to the graph $\bar{A}$ once we set equal to 0 the elements e_i, e_j hence it is the list of vectors associated to the polynomial $\chi_{\bar{A}}|_{\xi_i=\xi_j=0}$ similarly $\bar{w}_h$ is the one associated to $\chi_{\bar{B}}|_{\xi_i=\xi_j=0}$. Hence by the separation lemma up to reordering we may assume that $\bar{v}_h = \bar{w}_h$ hence $v_h = w_h + c$, $c = a - b = n_i e_i + n_j e_j$.

Clearly if $r > 1$ we have that $w_r = w_1 - v_1 + v_r$ so that the vectors (v_h, w_k) are not affinely independent contrary to the hypotheses.

We have thus proved that $|B| = |A| = 1$ hence $A = \{(a, \sigma_1)\}$, $B = \{(b, \sigma_2)\}$ and finally $b = n_i e_i + n_j e_j + \sigma_2\sigma_1 a$. Of course $n_i e_i + n_j e_j$ is the value up to sign of the path joining a, b. If $\sigma_2\sigma_1 = 1$ we have $\eta(a) = \eta(b)$ hence $\ell = n(e_i - e_j)$

if both indices i, j cannot appear more than twice in the path we have $|n| \leq 2$. If $\sigma_2\sigma_1 = -1$ we have $\eta(a+b) = -2$ hence $\ell = ne_i - (n+2)e_j$. A similar case analysis gives the possibilities $\ell = -e_i - e_j, -2\varepsilon_i, -2e_j$ if both indices i, j cannot appear more than twice in the path. $\square$

COROLLARY 10.4. *Under the assumptions of Lemma* 10.3 *the number of edges in the path from a to b in which appears any marking $h \neq i, j$ must be even. The parity of the number of edges in which appears i equals the parity of the number of edges in which appears j.*

In a maximal tree T in a graph Γ consider an edge ℓ containing the indices i, j. Denote by A, B the two connected components obtained by removing ℓ from T.

LEMMA 10.5. *Assume that the two connected components A, B do not have the index i in any edge. Then any other edge in Γ connecting A, B must contain the index i.*

PROOF. In a path which is a circuit you cannot have that an index appears only once (or even an odd number of times). $\square$

We now consider two edges ℓ_1, ℓ_2 containing the indices i, h and i, k respectively. When we remove these edges in T we have 3 connected components in T

$$A \overset{i,h}{\cdots} B \overset{i,k}{\cdots} C$$

in the complete graph $\bar{T}$ once we remove all the edges containing i the graph $\bar{B}$ is a connected component. Then we may either have other 2 components $\bar{A}$, $\bar{C}$ or a connected component $\overline{A \cup C}$. We shall use this fact systematically as follows. By induction in the first case we have $\chi_G(t)|_{\xi_i=0} = \chi_{\bar{A}}(t)\chi_{\bar{B}}(t)\chi_{\bar{C}}(t)$ modulo $\xi_i = 0$ is a factorization into irreducible factors, in the second case a factorization into irreducible factors is $\chi_G(t) \cong \chi_{\overline{A\cup C}}(t)\chi_{\bar{B}}(t)$ modulo $\xi_i = 0$.

Hence if G is not irreducible in the second case it can only factor into two irreducible factors $\chi_G(t) = UV$ with $U \cong \chi_{\bar{B}}(t)$, $V \cong \chi_{\overline{A\cup C}}(t)$ modulo $\xi_i = 0$, in the first case we may have either a factorization into 3 irreducible factors $\chi_G(t) = UVW$ with $U \cong \chi_{\bar{A}}(t)$, $V \cong \chi_{\bar{B}}(t)$, $W \cong \chi_{\bar{C}}(t)$ modulo $\xi_i = 0$ or 3 possible factorizations into 2 irreducible factors.

10.6. Indices appearing once.

LEMMA 10.7. *If there exists a pair of indices, say* $(1, i)$, *such that* 1 *appears only once in the maximal tree T and T has the form:*

$$A - \overset{1,h}{-} - B$$

Figure 1

where $i \neq h$, and i appears only in the block B. Then χ_G is irreducible.

Proof. Let the root be in A. Since 1 appears only once in T, every edge in G that connects A and B must have 1 in the indexing. We have:

$$\chi_G \cong \chi_{\bar{A}}\chi_{\bar{B}} \quad \text{modulo } \xi_1 = 0. \tag{41}$$

By the previous discussion if χ_G is not irreducible, it must factor into two irreducible polynomials: $\chi_G = UV$ such that $U \cong \chi_{\bar{A}}$ modulo $\xi_1 = 0$.

Let $B_1, \dots, B_s$ be the connected components obtained from B by deleting all the edges which have i in the indexing, B_1 be the component that is connected with A. We have:

$$\chi_G \cong \chi_{\overline{A\cup B_1}}\chi_{\overline{B_2}}\cdots\chi_{\overline{B_s}} \quad \text{modulo } \xi_i = 0. \tag{42}$$

Remark that $deg(U) = |A| < deg(\chi_{\overline{A\cup B_1}}) = |A| + |B_1|$. $U \cong \chi_{\bar{A}}$ is irreducible modulo $\xi_1 = \xi_i = 0$, then U must be irreducible modulo $\xi_i = 0$. Hence

$$U \cong \chi_{\overline{B_j}} \quad \text{modulo } \xi_i = 0 \text{ for some } j \in \{2, \dots, s\} \tag{43}$$

From $U \cong \chi_{\bar{A}}$ modulo $\xi_1 = 0$ and (43) we deduce $\chi_{\bar{A}} \cong \chi_{\overline{B_j}}$ modulo $\xi_1 = \xi_i = 0$. So, by Lemma 10.3, $|A| = |B_j| = 1$. Let $A = \{a\}$. Then by Lemma 8.6, for the vertex a and the index 1, χ_G is irreducible. □

Corollary 10.8. *If there are two indices which appear only once and not in the same edge in the maximal tree then χ_G is irreducible.*

We have thus treated one of the 4 cases of Lemma 10.1.

Lemma 10.9. *If there exists a pair of indices, say $(1, i)$, such that 1 appears only once in the maximal tree T while i appears twice and T has the form:*

$$A \overset{i,l}{- - -} B \overset{1,h}{- - -} C \overset{i,k}{- - -} D$$

Figure 2

then either χ_G is irreducible or $|A| = |C| = 1$ or $|B| = |D| = 1$.

Proof. We have $\chi_G \cong \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D}}$ modulo $\xi_1 = 0$ so if χ_G is not irreducible it has a factor $U \cong \chi_{\overline{A\cup B}}$ modulo $\xi_1 = 0$. This implies $U \cong \chi_{\bar{A}}\chi_{\bar{B}}$ modulo $\xi_1 = \xi_i = 0$. Now $\chi_G \cong \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C}}$ or $\chi_G \cong \chi_{\bar{A}}\chi_{\bar{D}}\chi_{\overline{B\cup C}}$ modulo $\xi_3 = 0$ and inspecting the two factorizations the claim follows from Lemma 10.3. □

10.10. Two indices appear only once and in the same edge. Let these two indices be 1, 2. If there exists another index, say 3, which appears only once, then we can replace 2 by 3 and we are back in the case of Corollary 10.8. Otherwise by Lemma 10.1 we have exactly $n - 1$ distinct indices different from 1, 2 and they appear twice. Take one of these indices, say 3. If we cannot apply Lemma 10.7 we must be in the case, in which the maximal tree T has the form

$$A \overset{3,k}{- - -} B \overset{1,2}{- - -} C \overset{3,h}{- - -} D$$

Figure 3

where the indices 1 and 3 do not appear elsewhere in the tree. By inspection of figure (3) all edges in G which connect A and C contain 1, 3 in the indexing, all edges in G which connect B and D contain 1, 3 in the indexing. Then we have:

$$\chi_G \cong \chi_{\overline{A \cup B}} \chi_{\overline{C \cup D}} \quad \text{modulo } \xi_1 = 0. \tag{44}$$

$$\chi_G \cong \chi_{\bar{A}} \chi_{\overline{B \cup C}} \chi_{\bar{D}} \quad \text{or} \quad \chi_G \cong \chi_{\overline{A \cup D}} \chi_{\overline{B \cup C}} \quad \text{modulo } \xi_3 = 0. \tag{45}$$

The second case holds when A, D are joined by some edge which does not contain 3. From (44) we see that if χ_G is not irreducible, then it has an irreducible factor $U \cong \chi_{\overline{A \cup B}}$ mod. $\xi_1 = 0$ which implies $U \cong \chi_{\bar{A}} \chi_{\bar{B}}$ modulo $\xi_1 = \xi_3 = 0$. Comparing (44) and (45) taking into account the degree and using the irreducibility of $\chi_{\bar{A}}, \chi_{\bar{B}}, \chi_{\bar{D}}$ modulo $\xi_1 = \xi_3 = 0$ we get the following possibilities

$$U \cong \chi_{\bar{A}} \chi_{\bar{D}}, \chi_{\overline{A \cup D}}, \chi_{\overline{B \cup C}} \quad \text{modulo } \xi_3 = 0. \tag{46}$$

In the first two cases of (46) we have

$$U \cong \chi_{\bar{A}} \chi_{\bar{B}} \cong \chi_{\bar{A}} \chi_{\bar{D}} \quad \text{modulo } \xi_1 = \xi_3 = 0$$

which implies

$$\chi_{\bar{B}} \cong \chi_{\bar{D}} \quad \text{modulo } \xi_1 = \xi_3 = 0 \tag{47}$$

Hence by Lemma 10.3 we must have: $B = \{b\}$, $D = \{d\}$. But the index 2 appears only once in the path from b to d contradicting Corollary 10.4.

In the last case of (46) we have

$$U \cong \chi_{\bar{A}} \chi_{\bar{B}} \cong \chi_{\bar{B}} \chi_{\bar{C}} \quad \text{modulo } \xi_1 = \xi_3 = 0$$

which implies

$$\chi_{\bar{A}} \cong \chi_{\bar{C}} \quad \text{modulo } \xi_1 = \xi_3 = 0 \tag{48}$$

We arrive at the same conclusions.

10.11. Only the index 1 *appears once in the tree.* From Lemma 10.1 there is only one index, say 3, which appears three times. All other indices, different from 1, 3, appear twice. We need to distinguish two subcases:

10.11.1. When 1, 3 *appear together in one edge.* If T has the form as in figure (4) then, by Lemma 10.7, χ_G is irreducible.

$$A \overset{1,3}{- - -} B \overset{2,k_1}{- - -} C \overset{2,k_2}{- - -} D$$

Figure 4

Therefore, assume that T has the form as in figure (5)

$$A \overset{2,k_1}{- - -} B \overset{1,3}{- - -} C \overset{2,k_2}{- - -} D$$

Figure 5

We start the discussion as in the previous paragraph

$$\chi_G \cong \chi_{\overline{A\cup B}} \chi_{\overline{C\cup D}} \quad \text{modulo } \xi_1 = 0. \tag{49}$$

$$\chi_G \cong \chi_{\bar{A}} \chi_{\overline{B\cup C}} \chi_{\bar{D}} \quad \text{or} \quad \chi_G \cong \chi_{\overline{A\cup D}} \chi_{\overline{B\cup C}} \quad \text{modulo } \xi_2 = 0. \tag{50}$$

The second case holds when A, D are joined by some edge which does not contain 2. From (49) we see that if χ_G is not irreducible, then it must factor into two irreducible polynomials: $\chi_G = UV$, $U \cong \chi_{\overline{A\cup B}}$ modulo $\xi_1 = 0$ implies $U \cong \chi_{\bar{A}} \chi_{\bar{B}}$ modulo $\xi_1 = \xi_2 = 0$. Comparing (49) and (50) taking into account the degree and using the irreducibility of $\chi_{\bar{A}}, \chi_{\bar{B}}, \chi_{\bar{D}}$ modulo $\xi_1 = \xi_2 = 0$ we get the following possibilities

$$U \cong \chi_{\bar{A}} \chi_{\bar{D}}, \chi_{\overline{A\cup D}}, \chi_{\overline{B\cup C}} \quad \text{modulo } \xi_2 = 0. \tag{51}$$

In the first two cases of (51) we have

$$U \cong \chi_{\bar{A}} \chi_{\bar{B}} \cong \chi_{\bar{A}} \chi_{\bar{D}} \quad \text{modulo } \xi_1 = \xi_2 = 0$$

which implies

$$\chi_{\bar{B}} \cong \chi_{\bar{D}} \quad \text{modulo } \xi_1 = \xi_2 = 0 \tag{52}$$

In the last case of (51) we have

$$U \cong \chi_{\bar{A}} \chi_{\bar{B}} \cong \chi_{\bar{B}} \chi_{\bar{C}} \quad \text{modulo } \xi_1 = \xi_2 = 0$$

which implies

$$\chi_{\bar{A}} \cong \chi_{\bar{C}} \quad \text{modulo } \xi_1 = \xi_2 = 0 \tag{53}$$

By symmetry we need to consider only case (53). By Lemma 10.3 we get $|A| = |C| = 1$, $A = \{0\}$, $C = \{c\}$, $c = \tau_{n_1 e_1 + n_2 e_2}(0)$. By inspection of Figure (5) $n_1, n_2 \in \{\pm 1\}$.

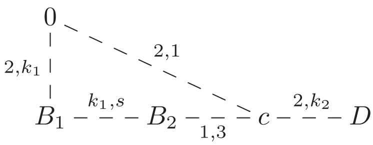
$$\eta(c) \in \{0, -2\} \;\Rightarrow\; c = \pm(e_1 - e_2), -e_1 - e_2 \tag{54}$$

We have thus proved:

Lemma 10.12. *Either $|A| = |C| = 1$ and there is an edge marked $(1, 2)$ that connects $A = 0$ and $c = C$. Or the same statement for B, D. Moreover, all indices, different from* 1, 2 *must appear an even number of times in every path from* 0 *to c* (*resp. b, d*).

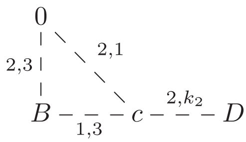
Assume $A = 0$, $C = c$, consider the index k_1.

i) If $k_1 \neq 3$, then k_1 must appear once more in the block B like:



Now we can apply 10.7 to the pair $(1, k_1)$ and get the irreducibility of χ_G.

ii) So we can assume that $k_1 = 3$, consider the index k_2.



A) If $k_2 \neq 3$, then either k_2 appears in the block D as in figure (7), and then by Lemma 10.7 for the pair $(1, k_2)$, χ_G is irreducible; or it appears in the block B as in figure (6).

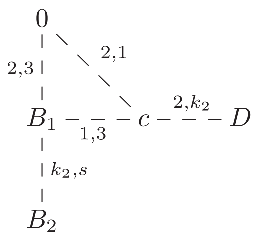


Figure 6

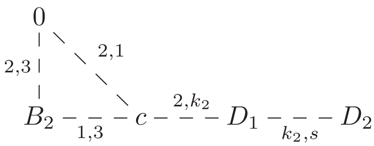
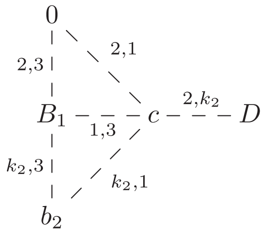


Figure 7

In the case of figure (6) we can apply Lemma 10.12 for 1, k_2. Since $|0 \cup B_1| > 1$ the only possibility is that $B_2 = b_2$ and there exists an edge with the marking $(1, k_2)$ that connects c and b_2.

Now we claim that we must have $s = 3$ in fact s must appear an even number of times in both paths from 0, c and from b_2, c, this is possible only for $s = 3$.



We now remove the two edges marked 1, 3 and k_2, 3. In the resulting maximal tree 3 appears once and we can apply Lemma 10.7 to the pair $(3, k_2)$, χ_G is irreducible.

B) If $k_2 = 3$ and $|B| > 1$. Let i be an index that appears in B. If i appears twice in B, then, by Lemma 10.7 we get the irreducibility of χ_G. Otherwise, i appears in this form:

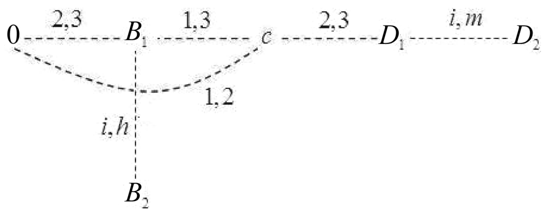


Figure 8

This case is excluded by Lemma 10.12 for the pair 1, i. The case $|D| > 1$ is treated similarly. So now we have to consider only the case, when $|B| = |D| = 1$.

C) $k_2 = 3$, $|B| = |D| = 1$. Up to symmetry, we have 4 subcases, displayed in figures (9)–(12).

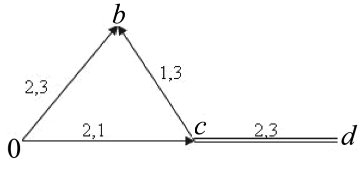


Figure 9

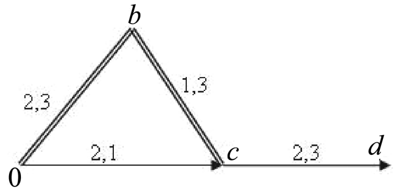


Figure 10

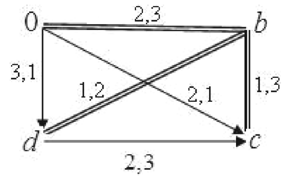


Figure 11

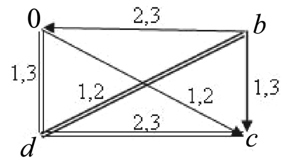


Figure 12

By using the program Mathematica we have verified that the characteristic polynomials of these graphs are irreducible.

10.12.1. When 1, 3 *do not appear together in any edge.* We have three possible cases (given in figures (13), (14), (15)).

1) When T up to symmetry has the form as in figure (13):

$$A \overset{1,2}{---} B$$

Figure 13

where 3 appears only in the block B then, by Lemma 10.7, for the pair $(1,3)$, χ_G is irreducible.

2) When T up to symmetry has the form as in figure (14):

$$A \overset{3,k_1}{---} B \overset{1,2}{---} C \overset{3,k_2}{---} D \overset{3,k_3}{---} E$$

Figure 14

We have

$$\text{modulo} \quad \xi_1 = 0, \quad \chi_G \cong \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D\cup E}} \tag{55}$$

$$\text{modulo} \quad \xi_3 = 0, \quad \chi_G = \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C}}\chi_{\bar{D}}\chi_{\bar{E}} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C}}\chi_{\bar{E}} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C\cup E}} \\ \chi_{\bar{A}}\chi_{\bar{D}}\chi_{\overline{B\cup C\cup E}} \end{cases} \tag{56}$$

Arguing as in previous cases, if χ_G factors then we can factor it as UV with $U_{\xi_1=0} = \chi_{\overline{A\cup B}}$. Analyzing the possible values of $U_{\xi_3=0}$ we have, comparing (55) and (56) and setting $\xi_1 = \xi_3 = 0$, the following possibilities:

$$\begin{matrix} U \\ \text{mod } \xi_3 = 0 \end{matrix} \cong \begin{cases} \chi_{\overline{B\cup C}} & \Rightarrow \ \chi_{\bar{A}} \cong \chi_{\bar{C}} \text{ mod. } \xi_1 = \xi_3 = 0 \\ \chi_{\overline{A\cup D}} \text{ or } \chi_{\bar{A}}\chi_{\bar{D}} & \Rightarrow \ \chi_{\bar{B}} \cong \chi_{\bar{D}} \text{ mod. } \xi_1 = \xi_3 = 0 \\ \chi_{\bar{A}}\chi_{\bar{E}} & \Rightarrow \ \chi_{\bar{B}} \cong \chi_{\bar{E}} \text{ mod. } \xi_1 = \xi_3 = 0 \\ \chi_{\bar{D}}\chi_{\bar{E}} \ \Rightarrow \ \begin{cases} \chi_{\bar{A}} \cong \chi_{\bar{D}}, \chi_{\bar{B}} \cong \chi_{\bar{E}} \text{ mod. } \xi_1 = \xi_3 = 0 \\ \chi_{\bar{A}} \cong \chi_{\bar{E}}, \chi_{\bar{B}} \cong \chi_{\bar{D}} \text{ mod. } \xi_1 = \xi_3 = 0 \end{cases} \end{cases} \tag{57}$$

It is enough to exclude the first 3 cases of (57).

Case 1. If $\chi_{\bar{A}} = \chi_{\bar{C}}$ modulo $\xi_1 = \xi_3 = 0$, by Lemma 10.3 and by inspection we deduce that $A = \{0\}$, $C = \{c\}$ and $c = \pm(e_1 - e_3)$, $-e_1 - e_3$. Hence there is an edge marked 1, 3 that connects 0 and c. We can then replace the maximal tree T with the one in which we keep this edge and remove the one marked 1, 2 and we find ourselves in the case treated in the previous paragraph.

Case 2. If $\chi_{\bar{B}} \cong \chi_{\bar{D}}$ modulo $\xi_1 = \xi_3 = 0$, then, by Lemma 10.3 $B = \{b\}$, $D = \{d\}$ and 2 should appear an even number of times between them, again a contradiction (we are in the case $k_2 = 2$).

Case 3. If $\chi_{\bar{B}} \cong \chi_{\bar{E}}$ modulo $\xi_1 = \xi_3 = 0$, then, by Lemma 10.3 and choosing the root at B we have $B = \{0\}$, $E = \{e\}$ we have the same contradiction as in the previous case.

3) When T has the form:

$$\begin{tikzcd} A \arrow[r, dash, dashed, "{3,k_1}"] & B \arrow[r, dash, dashed, "{1,2}"] & C \arrow[r, dash, dashed, "{3,k_3}"] \arrow[d, dash, dashed, "{3,k_2}"] & E \\ & & D & \end{tikzcd}$$

Figure 15

$$\chi_G \cong \chi_{\overline{A\cup B}}\chi_{\overline{C\cup D\cup E}} \quad \text{modulo } \xi_1 = 0 \tag{58}$$

From (58) we see that if χ_G is not irreducible, then $\chi_G = UV$, where U, V are irreducible, $U \cong \chi_{\overline{A\cup B}}$ modulo $\xi_1 = 0$, $\Rightarrow$ $U \cong \chi_{\bar{A}}\chi_{\bar{B}}$ modulo $\xi_1 = \xi_3 = 0$.

$$\text{(59)} \qquad \text{modulo} \quad \xi_3 = 0, \quad \chi_G \cong \begin{cases} \chi_{\bar{A}}\chi_{\overline{B\cup C}}\chi_{\bar{D}}\chi_{\bar{E}} \\ \chi_{\overline{A\cup D}}\chi_{\overline{B\cup C}}\chi_{\bar{E}} \\ \chi_{\overline{A\cup E}}\chi_{\overline{B\cup C}}\chi_{\bar{D}} \\ \chi_{\bar{A}}\chi_{\overline{B\cup C}}\chi_{\overline{D\cup E}} \\ \chi_{\overline{A\cup D\cup E}}\chi_{\overline{B\cup C}} \end{cases}$$

As for U it may be congruent modulo $\xi_3 = 0$ to

$$\chi_{\overline{B\cup C}}, \quad \chi_{\overline{A\cup D}}, \quad \chi_{\overline{A\cup E}},$$
$$\chi_{\bar{A}}\chi_{\bar{D}}, \quad \chi_{\bar{A}}\chi_{\bar{E}}, \quad \chi_{\bar{D}}\chi_{\bar{E}}, \quad \chi_{\overline{A\cup D\cup E}}$$

giving the following subcases: 1) $\chi_{\bar{C}} \cong \chi_{\bar{A}}$, 2) $\chi_{\bar{B}} \cong \chi_{\bar{D}}$, 3) $\chi_{\bar{B}} \cong \chi_{\bar{E}}$, 4) $\chi_{\bar{B}} \cong \chi_{\overline{D\cup E}}$ modulo $\xi_1 = \xi_3 = 0$. The fourth case can be excluded by cardinality. We treat the other 3 cases.

1) $\chi_{\bar{C}}|_{\xi_1=\xi_3=0} = \chi_{\bar{A}}$, by Lemma 10.3, $A = \{0\}$, $C = \{c\}$, and $c = \pm(e_1 - e_3)$, $-e_1 - e_3$. Hence there is an edge marked 1, 3 that connects 0 and c. We can then replace the maximal tree T with the one in which we keep this edge and remove the one marked 1, 2 and we find ourselves in the case treated in the previous paragraph.

2) $\chi_{\bar{B}} \cong \chi_{\bar{D}}$ modulo $\xi_1 = \xi_3 = 0$ by Lemma 10.3 $\Rightarrow |B| = |D| = 1$, $B = \{b\}$, $D = \{d\}$ and $\sigma_d d + \sigma_b b = \pm(e_1 - e_3)$, $-e_1 - e_3$. Hence there is an edge marked 1, 3 that connects b and d. We can then replace the maximal tree T with the one in which we keep this edge and remove the one marked 1, 2 and we find ourselves in the case treated in the previous paragraph.

3) $\chi_{\bar{B}} \cong \chi_{\bar{E}}$ modulo $\xi_1 = \xi_3 = 0$ is similar to case 2), changing the role of k_2 and k_3.

10.13. Every index appears twice in the tree.

LEMMA 10.14. *If χ_G is not irreducible the graph is a tree.*

PROOF. Assume there is a an edge marked i, j in the graph and not in the tree, then a segment in the tree together with this edge form a dependent circuit, thus we can remove an edge marked a, b in this segment and add the edge i, j in order to obtain another maximal tree. Clearly in a circuit there is at least an edge such that the indices i, j are distinct fro the indices a, b. This means that in the new maximal tree one of the indices i, j appears with multiplicity 1 and we are back to a previous case. □

From now on we thus assume that the graph is a tree T. We start with some special cases:

10.14.1. $n = 2$.

$$T: \quad -e_1 - e_2 = 0 \to e_1 - e_2$$

is not allowable (but its characteristic polynomial is irreducible).

10.14.2. $n = 3$. Up to symmetry of the indices T has the form as in figure (16) or as in figure (17):

$$0 \overset{1,2}{- - -} b \overset{2,3}{- - -} c \overset{1,3}{- - -} d$$

Figure 16

$$\begin{array}{ccccc} 0 & \overset{1,2}{- - -} & b & \overset{2,3}{- - -} & c \\ & & \vert\; {\scriptstyle 1,3} & & \\ & & d & & \end{array}$$

Figure 17

Remark 10.15. If all edges in T are black, or there are exactly two red edges then the edges are linearly dependent.

1) When the graph T has the form as in figure (16) a) If all edges are red, then $G = \overline{T}$ is not a tree:

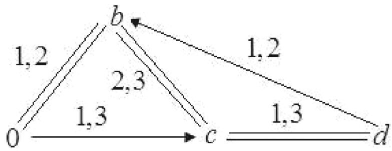


Figure 18

We need to consider the cases, when in T there is one red and two black edges. Up to symmetry we may assume the red edge is the first or the second.

b) When the red edge connects 0 and b:

b1) When T has the form:

$$0 \overset{1,2}{=\!=} b \overset{2,3}{\longrightarrow} c \overset{1,3}{\longrightarrow} d$$

We have

$$b = -e_1 - e_2, \quad c - b = e_2 - e_3 \;\Rightarrow\; c = -e_1 - e_2.$$

Hence $G = \overline{T}$ is not a tree.

b2) If T has the form:

$$0 \overset{1,2}{=\!=} b \overset{2,3}{\longleftarrow} c \overset{1,3}{\longrightarrow} d$$

We have $b - c = e_1 - e_3$, $d - c = e_1 - e_3 \Rightarrow d - b = e_1 - e_2$, i.e. in G there is a black edge marked $(1, 2)$ that connects b and d. Hence $G = \overline{T}$ is not a tree.

b3) If T has the form:

$$0 \overset{1,2}{=\!=\!=} b \xleftarrow{2,3} c \xleftarrow{1,3} d$$

$$\chi_T = \det\begin{pmatrix} t & 2\sqrt{\xi_1\xi_2} & 0 & 0 \\ -2\sqrt{\xi_1\xi_2} & t+\xi_1+\xi_2 & 2\sqrt{\xi_2\xi_3} & 0 \\ 0 & 2\sqrt{\xi_2\xi_3} & t+\xi_1+2\xi_2-\xi_3 & 2\sqrt{\xi_1\xi_3} \\ 0 & 0 & 2\sqrt{\xi_1\xi_3} & t+2\xi_1+2\xi_2-2\xi_3 \end{pmatrix}$$

By using the program Mathematica we computed χ_T and verified that it is irreducible.

c) When the red edge connects b and c:

c1) If T has the form:

$$0 \overset{1,2}{-\!\!-\!\!-} b \overset{2,3}{=\!=\!=} c \xleftarrow{1,3} d$$

we have $b+c=-e_2-e_3$, $c-d=e_1-e_3 \Rightarrow b+d=-e_1-e_2$, i.e. there is a red edge marked $(1,2)$ that connects b and d. Hence $G=\overline{T}$ is not a tree.

c2) If T has the form

$$0 \xrightarrow{1,2} b \overset{2,3}{=\!=\!=} c \overset{1,3}{-\!\!-\!\!-} d$$

we have $b=e_1-e_2$, $b+c=-e_2-e_3 \Rightarrow c=e_1-e_3$, i.e. there is a black edge marked $(1,3)$ that connects 0 and c. Hence $G=\overline{T}$ is not a tree.

c3) If T has the form:

$$0 \xleftarrow{1,2} b \overset{2,3}{=\!=\!=} c \xrightarrow{1,3} d$$

we have

$$\chi_T = \det\begin{pmatrix} t & -2\sqrt{\xi_1\xi_2} & 0 & 0 \\ -2\sqrt{\xi_1\xi_2} & t-\xi_1+\xi_2 & 2\sqrt{\xi_2\xi_3} & 0 \\ 0 & -2\sqrt{\xi_2\xi_3} & t-\xi_1+2\xi_2+\xi_3 & 2\sqrt{\xi_1\xi_3} \\ 0 & 0 & 2\sqrt{\xi_1\xi_3} & t-2\xi_1+2\xi_2+2\xi_3 \end{pmatrix}$$

We used the program Mathematica to compute χ_T and to verify that it is irreducible.

2) When T has the form as in figure (17):

a) When in T there are 3 red edges, then $G=\overline{T}$ has the form:

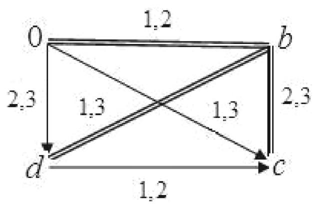


Figure 19

This figure can be obtained from figure (11) by exchanging the role of indices (i.e. the role of variables ξ_1, ξ_2, ξ_3). Hence χ_T is irreducible.

b) When in T there is only one red edge, by the symmetry property of T we may suppose that this red edge connects 0 and b.

b1) If T has the form:

$$\begin{tikzcd} 0 \arrow[r, equal, "1,2"] & b \arrow[r, "2,3"] \arrow[d, no head, "1,3"] & c \\ & d & \end{tikzcd}$$

in G there is a red edge marked $(1,3)$ that connects 0 and c. Hence $G = \overline{T}$ is not a tree.

b2) If T has the form:

$$\begin{tikzcd} 0 \arrow[r, equal, "1,2"] & b \arrow[r, no head, "2,3"] \arrow[d, "1,3"] & c \\ & d & \end{tikzcd}$$

we have $b = -e_1 - e_2$, $d - b = e_1 - e_3 \Rightarrow d = -e_2 - e_3$, hence in G there is a red edge marked $(2,3)$ that connects 0 and d. Hence $G = \overline{T}$ is not a tree.

b3) If T has the form:

$$\begin{tikzcd} 0 \arrow[r, equal, "1,2"] & b & c \arrow[l, "2,3"'] \\ & d \arrow[u, "1,3"'] & \end{tikzcd}$$

we have $b - c = e_2 - e_3$, $b - d = e_1 - e_3 \Rightarrow d - c = e_2 - e_1$, hence there is a black edge marked $(2,1)$ that connects c and d. Hence $G = \overline{T}$ is not a tree.

10.16. $n \geq 4$. At this point we are assuming that we have $n \geq 4$ edges in a maximal tree T and n indices, each appearing twice. Thus given an index, say 1, it appears in two edges paired with at most two other indices, thus we can find another index, say 2 which is not in these two edges. Up to symmetry we may have six cases displayed in figures (20)–(25):

$$\begin{tikzcd} & & D \arrow[d, dashed, no head, "1,i"'] & \\ A \arrow[r, dashed, no head, "1,h"] & B \arrow[r, dashed, no head, "2,k"] & C \arrow[r, dashed, no head, "2,j"] & E \end{tikzcd}$$

Figure 20

$$\begin{tikzcd} A \arrow[r, dashed, no head, "1,h"] & B \arrow[r, dashed, no head, "2,k"] & C \arrow[r, dashed, no head, "2,i"] & D \arrow[r, dashed, no head, "1,j"] & E \end{tikzcd}$$

Figure 21

$$\begin{array}{ccccccc} & & & & D & & \\ & & & & \Big| {\scriptstyle 2,i} & & \\ A & \overset{1,h}{---} & B & \overset{1,k}{---} & C & \overset{2,j}{---} & E \end{array}$$

Figure 22

$$\begin{array}{ccccc} & & C & & \\ & & \Big| {\scriptstyle 2,i} & & \\ A & \overset{1,h}{---} & B & \overset{1,k}{---} & E \\ & & \Big| {\scriptstyle 2,j} & & \\ & & D & & \end{array}$$

Figure 23

$$A \overset{1,h}{---} B \overset{2,k}{---} C \overset{1,i}{---} D \overset{2,j}{---} E$$

Figure 24

$$A \overset{1,h}{---} B \overset{1,k}{---} C \overset{2,i}{---} D \overset{2,j}{---} E$$

Figure 25

When we put $\xi_1 = 0$ or $\xi_2 = 0$ we have 3 connected components in the graph, so by induction we deduce that, if the characteristic polynomial is not irreducible it can factor in at most 3 factors. We will perform a case analysis in order to produce two pairs of disjoint blocks which give under specialization $\xi_1 = \xi_2 = 0$ the same characteristic polynomials and we apply Lemma 10.3. In this way we will prove the irreducibility of χ_T in each case, displayed in figures (20)–(25).

10.16.1. Figure (20).

$$\begin{array}{ccccccc} & & & & D & & \\ & & & & \Big| {\scriptstyle 1,i} & & \\ A & \overset{1,h}{---} & B & \overset{2,k}{---} & C & \overset{2,j}{---} & E \end{array}$$

We have

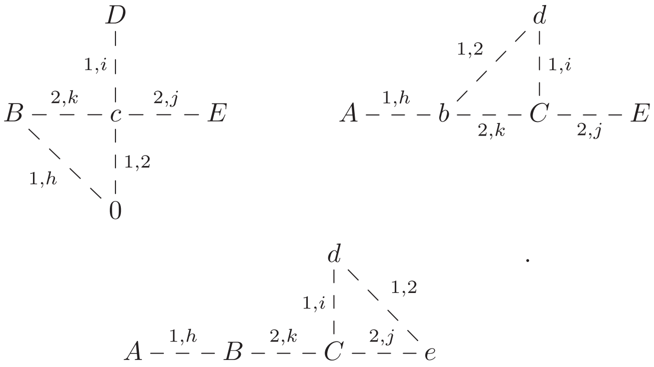
$$\chi_T \cong \chi_A \chi_{B \cup C \cup E} \chi_D \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A \cup B} \chi_{C \cup D} \chi_E \text{ mod. } \xi_2 = 0, \tag{60}$$

Suppose that χ_T is not irreducible, then there is an irreducible factor U congruent to either χ_A or χ_D or finally $\chi_A \chi_D$ modulo $\xi_1 = 0$.

Then U is congruent to χ_E or $\chi_{A\cup B}$ or $\chi_{C\cup D}$ modulo $\xi_2 = 0$.

We now specialize $\xi_1 = \xi_2 = 0$ and apply Lemma 10.3 and we have several possibilities of two blocks giving the same characteristic polynomial. Of these possibilities some are excluded by the parity condition of the indices 1, 2 in the path joining them.

We then see that we are left with the ones listed which all produce an extra edge contradicting the assumption that $G = \overline{T}$ is a tree.



□

10.16.2. Figure (21).

$$A \overset{1,h}{-\!-\!-} B \overset{2,k}{-\!-\!-} C \overset{2,i}{-\!-\!-} D \overset{1,j}{-\!-\!-} E$$

$$\chi_T \cong \chi_A \chi_{B\cup C\cup D} \chi_E \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A\cup B} \chi_C \chi_{D\cup E} \text{ mod. } \xi_2 = 0 \tag{61}$$

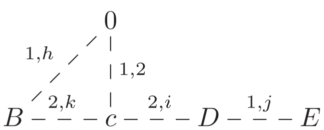
Suppose that χ_T is not irreducible, then there is an irreducible factor U such that U is congruent, modulo $\xi_1 = 0$ to χ_A or χ_E or finally $\chi_A \chi_E$.

Then U is congruent, modulo $\xi_2 = 0$ to either χ_C or $\chi_{A\cup B}$ or $\chi_{D\cup E}$. We reason as in previous cases, specializing $\xi_1 = \xi_2 = 0$ we deduce that there are four possible applications of Lemma 10.3 for the blocks A, E and the blocks C, B, D. We exclude those for which an index 1, 2 in the path connecting them occurs only once and the other 0 or 2. We then are left with the cases:

$$\chi_A \cong \chi_C, \quad \chi_C \cong \chi_E, \quad \text{mod } \xi_1 = \xi_2 = 0 \tag{62}$$

By symmetry we need to consider only the first.

Assume thus that $\chi_A \cong \chi_C$ modulo $\xi_1 = \xi_2 = 0$, by Lemma 10.3 we have $|C| = |A| = 1$, $C = \{c\}$, $A = \{0\}$, $c = \tau_{\pm e_1 \pm e_2}(0)$, $c = \pm(e_1 - e_2)$, $-e_1 - e_2$. Hence there is an edge marked $(1, 2)$ connecting 0 and c.



and $G = \overline{T}$ is not a tree. □

10.16.3. Figure (23).

$$\begin{array}{ccccc} & & C & & \\ & & \Big| {\scriptstyle 2,i} & & \\ A & \overset{1,h}{- - -} & B & \overset{1,k}{- - -} & E \\ & & \Big| {\scriptstyle 2,j} & & \\ & & D & & \end{array}$$

We have:

$$\chi_T \cong \chi_A \chi_{C \cup B \cup D} \chi_E \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A \cup B \cup E} \chi_C \chi_D \text{ mod. } \xi_2 = 0. \tag{63}$$

If χ_T is not irreducible by considering a suitable irreducible factor U and by a simple analysis we get the following subcases:

$$\chi_A \cong \chi_C, \quad \chi_A \cong \chi_D, \quad \chi_E \cong \chi_D, \quad \chi_E \cong \chi_C \quad \text{mod. } \xi_1 = \xi_2 = 0.$$

By the symmetry of the tree in figure (23), we need consider only the first case. We get easily by Lemma 10.3 $|A| = |C| = 1$, $A = \{0\}$, $C = \{c\}$, $c = \pm(e_1 - e_2)$, $-e_1 - e_2$. So 0, c are connected by an edge and $G = \bar{T}$ is not a tree.

$$\begin{array}{ccccc} & & E & & \\ & & \Big| {\scriptstyle 1,k} & & \\ 0 & \overset{1,h}{- - -} & B & \overset{2,j}{- - -} & D \\ & {\scriptstyle 1,2} \searrow & \Big| {\scriptstyle 2,i} & & \\ & & c & & \end{array}$$

□

10.16.4. Figure (22).

$$\begin{array}{ccccccc} & & & & D & & \\ & & & & \Big| {\scriptstyle 2,i} & & \\ A & \overset{1,h}{- - -} & B & \overset{1,k}{- - -} & C & \overset{2,j}{- - -} & E \end{array}$$

We have:

$$\chi_T|_{\xi_1=0} \cong \chi_A \chi_B \chi_{C \cup D \cup E} \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A \cup B \cup C} \chi_D \chi_E \text{ mod. } \xi_2 = 0, \tag{64}$$

Suppose that χ_T is not irreducible. The usual reasoning gives an irreducible factor U so that $U \cong \chi_A, \chi_B, \chi_A \chi_B$ modulo $\xi_1 = 0$.

We may have $U \cong \chi_D, \chi_E, \chi_D \chi_E, \chi_{D \cup E}$ modulo $\xi_2 = 0$.

Arguing as in the previous case we only have the possibility

$$\chi_B \cong \chi_D, \quad \chi_B \cong \chi_E \quad \text{modulo } \xi_1 = \xi_2 = 0.$$

By symmetry we need to consider only the first case. We get by Lemma 10.3 $B = \{b\}$, $D = \{d\}$, and d, b are joined by an edge $\pm(e_1 - e_2)$, $-e_1 - e_2$.

$$\begin{tikzcd}
 & & d \arrow[dl, dash, dashed, "1,2"'] \arrow[d, dash, dashed, "2,i"] & \\
A \arrow[r, dash, dashed, "1,h"] & b \arrow[r, dash, dashed, "1,k"'] & C \arrow[r, dash, dashed, "2,j"] & E
\end{tikzcd}$$

and $G = \overline{T}$ is not a tree. □

10.16.5. Figure (24), (25). We treat these two cases together.

$$\begin{aligned}
&\mathrm{I)} \quad A \stackrel{1,h}{---} B \stackrel{2,k}{---} C \stackrel{1,i}{---} D \stackrel{2,j}{---} E\\
&\mathrm{II)} \quad A \stackrel{1,h}{---} B \stackrel{1,k}{---} C \stackrel{2,i}{---} D \stackrel{2,j}{---} E
\end{aligned}$$

PROOF. I) We have:

$$\chi_T \cong \chi_A \chi_{B\cup C} \chi_{D\cup E} \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A\cup B} \chi_{C\cup D} \chi_E \text{ mod. } \xi_2 = 0. \tag{65}$$

Inspecting (65), by a simple analysis we get the following possibilities:

If χ_T is not irreducible it has a factor U congruent, modulo $\xi_1 = 0$ to i) χ_A or ii) $\chi_{B\cup C}$ or $\chi_{D\cup E}$. If $U \cong \chi_A$ modulo $\xi_1 = 0$ we must have $U \cong \chi_E$ modulo $\xi_2 = 0$ and

$$\chi_A \cong \chi_E \text{ mod. } \xi_1 = \xi_2 = 0. \tag{66}$$

Otherwise we have that U is congruent to $\chi_{A\cup B}$ or $\chi_{C\cup D}$ modulo $\xi_2 = 0$.

$$\chi_A \cong \chi_C, \quad \chi_C \cong \chi_E, \quad \chi_B \cong \chi_D, \quad \chi_D \cong \chi_A, \quad \chi_E \cong \chi_B \quad \text{mod. } \xi_1 = \xi_2 = 0. \tag{67}$$

The last two can be excluded by parity of occurrences of 1, 2 in their path. The first two are symmetric. Therefore we are left to consider three cases $\chi_A \cong \chi_E$, $\chi_A \cong \chi_C$, $\chi_B \cong \chi_D$.

If we are in case $\chi_A \cong \chi_C$, $\chi_B \cong \chi_D$ by Lemma 10.3 we get $|A| = |C| = 1$, $A = \{0\}$, $C = \{c\}$ (resp. $|B| = |C| = 1$, $A = \{b\}$, $C = \{c\}$) are joined by an edge $\pm(e_1 - e_2)$, $-e_1 - e_2$ and $G = \overline{T}$ is not a tree.

If we have $\chi_A \cong \chi_E$ always by Lemma 10.3 we get $|A| = |E| = 1$, $A = \{0\}$, $E = \{e\}$, $e \in \{\pm 2(e_1 - e_2), -2e_1, -2e_2\}$.

$$\mathrm{I)} \quad 0 \stackrel{1,h}{---} B \stackrel{2,k}{---} C \stackrel{1,i}{---} D \stackrel{2,j}{---} e \tag{68}$$

II)

$$\chi_T \cong \chi_A \chi_B \chi_{C\cup D\cup E} \text{ mod. } \xi_1 = 0, \quad \chi_T \cong \chi_{A\cup B\cup C} \chi_D \chi_E \text{ mod. } \xi_2 = 0, \tag{69}$$

If χ_T is not irreducible, one easily sees that there is a factor U congruent modulo $\xi_1 = 0$ to χ_A or χ_B or finally $\chi_A \chi_B$. Then U modulo $\xi_2 = 0$ is congruent either to

χ_D or χ_E or $\chi_D\chi_E$. Applying Lemma 10.3 a priori there are 4 possibilities that a block A, B specializes to a block D, E, but in that Lemma we also have the parity of 1, 2 in a path joining the two blocks must be the same hence we only have two cases.

i) $\chi_B \cong \chi_D$ modulo $\xi_1 = \xi_2 = 0$, and $|B| = |D| = 1$, $B = \{b\}$, $D = \{d\}$, and d, b are joined by an edge $\pm(e_1 - e_2)$, $-e_1 - e_2$. In this case we contradict the fact that $G = T$ is a tree.

ii) $\chi_A = \chi_E$ modulo $\xi_1 = \xi_2 = 0$ and $|A| = |E| = 1$, $A = \{0\}$, $E = \{e\}$. By inspection since $e \neq 0$ we must have $e = \pm(2e_1 - 2e_2), -2e_1, -2e_2$. All indices in the path from 0 to e appear twice.

$$\text{II)}\quad 0 \overset{1,h}{—} B \overset{1,k}{—} C \overset{2,i}{—} D \overset{2,j}{—} e \tag{70}$$

We now have to exclude in both cases the second possibility (68), (70).

I) Start from the first case. If $k = h$ we have

$$0 \overset{1,h}{—} B \overset{2,h}{—} C \overset{1,i}{—} D \overset{2,j}{—} e$$

If $i \neq j$ we must have that i appears in one of the blocks B, C, D. For instance if i is in D we have

$$0 \overset{1,h}{—} B \overset{2,h}{—} C \overset{1,i}{—} D_1 \overset{s,i}{—} D_2 \overset{2,j}{—} e$$

we apply the previous analysis to the pair h, i and deduce that $|D_2 \cup e| = 1$ a contradiction.

$$\begin{array}{ccccccccc} & & & & & & D' & & \\ & & & & & & \Big| {\scriptstyle u,i} & & \\ 0 & \overset{1,h}{—} & B & \overset{2,h}{—} & C & \overset{1,i}{—} & D & \overset{2,j}{—} & e \end{array}$$

is like Picture (20) for indices 2, i.

The other cases are similar to this or to the previous case of (24). If $i = j$ we have

$$0 \overset{1,h}{—} B \overset{2,h}{—} C \overset{1,i}{—} D \overset{2,i}{—} e.$$

We apply the previous analysis to the pair h, i deducing $e = \pm 2(e_i - e_h), -2e_i$, $-2e_h$ clearly a contradiction since we already have $e = \pm(2e_1 - 2e_2), -2e_1, -2e_2$.

If $k \neq h$ consider the positions of k. If $k \in B \cup C$

$$0 \overset{1,h}{—} B_1 \overset{s,k}{—} B_2 \overset{1,k}{—} C \overset{2,i}{—} D \overset{2,j}{—} e$$
$$0 \overset{1,h}{—} B \overset{1,k}{—} C_1 \overset{s,k}{—} C_2 \overset{2,i}{—} D \overset{2,j}{—} e$$

by the previous discussion applied to k, 2 we have that $|0 \cup B_1| = 1$ or $|0 \cup B| = 1$ a contradiction. If $k \in D$

$$0 \overset{1,h}{—} B \overset{1,k}{—} C \overset{2,i}{—} D_1 \overset{s,k}{—} D_2 \overset{2,j}{—} e$$

we are in the previous case of (24) for the indices k, 2 deducing $|0 \cup B| = 1$ again a contradiction.

II). We now finish the second case. If $k = h$ we have

$$0 \overset{1,h}{-\!-\!-} B \overset{1,h}{-\!-\!-} C \overset{2,i}{-\!-\!-} D \overset{2,j}{-\!-\!-} e$$

we are in the same situation but for the pair h, 2. We deduce that $e = -2e_2$. Now if $i = j$ we are in the same situation for the pair h, j (or 1, j) and deduce that $e = \pm 2(e_1 - e_j), -2e_1, -2e_j$ a contradiction. If $i \neq j$ we must have that i appears in one of the blocks B, C, D. For instance if i is in D we have

$$0 \overset{1,h}{-\!-\!-} B \overset{1,h}{-\!-\!-} C \overset{2,i}{-\!-\!-} D_1 \overset{s,i}{-\!-\!-} D_2 \overset{2,j}{-\!-\!-} e$$

we apply the previous analysis to the pair 1, i and deduce that $|D_2 \cup e| = 1$ a contradiction. The other cases are similar to this or to the previous case of (24).

If $k \neq h$ consider the positions of k. If $k \in B \cup C$

$$0 \overset{1,h}{-\!-\!-} B_1 \overset{s,k}{-\!-\!-} B_2 \overset{1,k}{-\!-\!-} C \overset{2,i}{-\!-\!-} D \overset{2,j}{-\!-\!-} e$$
$$0 \overset{1,h}{-\!-\!-} B \overset{1,k}{-\!-\!-} C_1 \overset{s,k}{-\!-\!-} C_2 \overset{2,i}{-\!-\!-} D \overset{2,j}{-\!-\!-} e$$

we apply the previous discussion to k, 2 and have that $|0 \cup B_1| = 1$ or $|0 \cup B| = 1$ a contradiction. If $k \in D$

$$0 \overset{1,h}{-\!-\!-} B \overset{1,k}{-\!-\!-} C \overset{2,i}{-\!-\!-} D_1 \overset{s,k}{-\!-\!-} D_2 \overset{2,j}{-\!-\!-} e$$

we are in the previous case of (24) for the indices k, 2 and again have $|0 \cup B| = 1$ a contradiction. □

References

[1] V. I. Arnol'd, *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*], Springer-Verlag, New York, second edition, 1988. Translated from the Russian by Joseph Szücs [József M. Szűcs].

[2] V. I. Arnold, *Hamiltonian systems*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 1994.

[3] M. Artin, *Algebra, second edition*, Pearson, New York, 2010.

[4] D. Bambusi - B. Grébert, *Birkhoff normal form for partial differential equations with tame modulus*, Duke Math. J., 135 n. 3:507–567, 2006.

[5] George D. Birkhoff, *Dynamical systems*, With an addendum by Jurgen Moser. American Mathematical Society Colloquium Publications, Vol. IX. American Mathematical Society, Providence, R.I., 1966.

[6] J. Colliander - M. Keel - G. Staffilani - H. Takaoka - T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math., 181(1):39–113, 2010.

[7] B. Grébert - Thomann L., *KAM for the quantum harmonic oscillator*, Communications in Mathematical Physics, 307(2):383–427, 2011.

[8] Helmut Hofer - Eduard Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 1994.

[9] W. Magnus - A. Karrass - D. Solitar, *Combinatorial group theory: presentations of groups in terms of generators and relations*, Dover Publications, New York, 1976.

[10] Henri Poincaré, *New methods of celestial mechanics. Vol.* 3, volume 13 of *History of Modern Physics and Astronomy*, American Institute of Physics, New York, 1993. Integral invariants and asymptotic properties of certain solutions, Translated from the French, Revised reprint of the 1967 English translation, With endnotes by G. A. Merman, Edited and with an introduction by Daniel L. Goroff.

[11] C. Procesi - M. Procesi, *A KAM algorithm for the resonant non-linear Schrödinger equation*, arXiv:1211.4242, 2012.

[12] C. Procesi - M. Procesi, *A normal form for the Schrödinger equation with analytic non-linearities*, Communications in Mathematical Physics, 312(2):501–557, 2012.

Received 14 June 2012,
and in revised form 12 January 2013.

Michela Procesi
Universita' di Roma La Sapienza
Piazzale Aldo Moro 5
00185 Roma, Italy
mprocesi@mat.uniroma1.it

Claudio Procesi
Universita' di Roma La Sapienza
Piazzale Aldo Moro 5
00185 Roma, Italy
procesi@mat.uniroma1.it

Bich Van Nguyen
Universita' di Roma La Sapienza
Piazzale Aldo Moro 5
00185 Roma, Italy
nguyen@mat.uniroma1.it