Rend. Lincei Mat. Appl. 24 (2013), 303–309
DOI 10.4171/RLM/655

Group Theory — *A note on subgroups of cofinite volume*, by MARTIN MOSKOWITZ, communicated on 9 November 2012.

ABSTRACT. — This note explores the relationship of closed cofinite volume subgroups H to lattices in Lie groups G, particularly when $G = G_{\mathbb{R}}$, the group of real points of an algebraic $\mathbb{Q}$-group. In some, but by no means all cases, H contains a lattice.

KEY WORDS: Lattice, cofinite volume subgroup, Zariski density, real points, linear algebraic $\mathbb{Q}$-group, arithmetic subgroup, rational form.

MATHEMATICS SUBJECT CLASSIFICATION: 20G25, 22E40.

Let G be a Lie group and H a closed subgroup with G/H of finite volume. A question I was asked some time ago by Michel Kervaire at a lecture I gave on this subject was whether all such H are lattices. In many cases, for example $G = \mathbb{R}^n$, or more generally when G is a simply connected nilpotent group G with center $Z(G)$, given a lattice, or in the latter case a log lattice, Γ, using the fact that exp is a global diffeomorphism, in the first case one can certainly run lines through some of the generators, or in the latter case 1-parameter groups through the generators of $\Gamma \cap Z(G)$ (which is a lattice in $Z(G)$, see e.g. Corollary 6 of [4]) to get a non discrete closed subgroup $H \supseteq \Gamma$. Hence H also has cofinite volume. In this way the question then becomes, must H contain a lattice?

In general, the answer is no. The most accessible examples illustrating this are groups G which have no lattices at all. Then, (assuming G had such an H) since any lattice in H is also a lattice in G, this can not occur.

EXAMPLE 1. *For example, let G be any 2-step simply connected nilpotent group without lattices (see Proposition 3.1.71 of [1]) and Z be its center. Then G/Z is abelian and so is $\mathbb{R}^n$ for some n. Take $A = H/Z$ to be any proper closed subgroup of G/Z with compact quotient. Then H is a proper closed subgroup of G with G/H compact and so of finite volume. But this cannot occur since G has no lattices.*

The significance of Example 1 is that according to the well known criterion of Malcev (see [10] Theorem 2.12) a simply connected nilpotent group G posseses a lattice if and only if its Lie algebra, $\mathfrak{g}$, has a rational form. That is, has a basis

The research on this paper was completed while the author was a visitor at the Universita di Roma, Tor Vergata, in May 2012. He warmly thanks the Mathematics Department for its hospitality and support.

under which all the structure constants are rational. The following proposition formulates this in group theoretic terms.

PROPOSITION 1. *Let G be a simply connected nilpotent group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has a rational form if and only if in some unipotent representation G is the real points of an algebraic group defined over $\mathbb{Q}$.*

PROOF. Suppose G is real points of an algebraic group defined over $\mathbb{Q}$. Since G is nilpotent, by [6] Proposition 19.2, G is the direct product, $G_u \times A$, where A is abelian and G_u is the unipotent radical. Therefore, to see that $\mathfrak{g}$ has a rational form, we may assume G itself is unipotent. Then the group, $X_{\mathbb{Q}}(G)$, of $\mathbb{Q}$ characters is trivial. By the Borel-Harish Chandra theorem [3] $G_{\mathbb{Z}}$ is a lattice in G and by Malcev's theorem (Theorem 2.12 of [10]) $\mathfrak{g}$ has a rational form.

Now suppose $\mathfrak{g}$ has a rational form. Then the proof of Theorem (2.3) pg. 218 of [5] dealing with faithful representations of a simply connected nilpotent group shows that when $\mathfrak{g}$ has a $\mathbb{Q}$-form the faithful representation goes to the real points of an algebraic $\mathbb{Q}$-group. □

Proposition 1 together with Example 1 suggests that for positive results (that is, where H must contain a lattice) one should look at groups G which are the real points of a linear algebraic group defined over $\mathbb{Q}$.

Before proceeding we formalize our terminology. A closed subgroup H of a connected Lie group G is said to be of *cofinite volume* if there is a (non trivial) finite G-invariant measure on G/H. If H is discrete it is called a *lattice*, usually written Γ. A closed subgroup H of a connected Lie group G is called *uniform* if G/H is compact. A result of Mostow (Theorem 3.1 in [10]) tells us, for a *solvable* group G, H has cofinite volume if and only if it is uniform.

When there is a cofinite volume subgroup there is often an accompanying density result. The oldest of these concerns semisimple Lie groups without compact factors due to Borel (see Theorem 5.5 of [10]). More general density results have been proved in [7] (as well as by others). In [7] a density theorem relevant to this note holds when G is a connected subgroup of $\mathrm{GL}(n, \mathbb{R})$, with radical R, G/R has no compact factors and R acts on $\mathbb{R}^n$ with real eigenvalues. However, as the reader will see, here density is by no means the whole story.

1. UNIPOTENT GROUPS

We first turn to the real points G of a linear algebraic $\mathbb{Q}$ group which is unipotent and where H a closed subgroup of G of cofinite volume (here uniform). In general, H need not contain a lattice.

Let $\mathfrak{h}$ denote the Lie algebra of H. When $\dim \mathfrak{h} = 0$, H is discrete and therefore contains a lattice, namely itself. But if $\dim \mathfrak{h} > 0$, then H_0 is a non trivial Euclidean closed unipotent subgroup of G. By [7] H is Zariski dense in $G = G_{\mathbb{R}}$. Hence $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ and H_0 is normal in G. Thus H/H_0 is a lattice in G/H_0. Since this latter group is simply connected and nilpotent, its Lie algebra, $\mathfrak{g}/\mathfrak{h}$, has a rational form by Malcev's theorem. So $\mathfrak{h}$ is an ideal and both $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{h}$ have rational forms. The question of whether H contains a lattice then becomes, does $\mathfrak{h}$ itself have a rational form?

EXAMPLE 2. *We will now construct a pair* $(\mathfrak{g}, \mathfrak{h})$ *with the following properties:*

1. $\mathfrak{g}$ *is a* 2 *step nilpotent real Lie algebra with a* $\mathbb{Q}$*-structure.*
2. $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h} \subseteq \mathfrak{g}$.
3. $\mathfrak{h}$ *is an ideal and has no* $\mathbb{Q}$*-structure.*

Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$, *the latter is certainly an ideal and* $\mathfrak{g}/\mathfrak{h}$ *is commutative and so has a* $\mathbb{Q}$*-structure.*

To construct $(\mathfrak{g}, \mathfrak{h})$ *we begin with the standard construction of a* 2 *step real nilpotent Lie algebra,* $\mathfrak{h}$*, which has no rational form* (*see e.g. Proposition* 3.1.71 *of* [1]).

$$\mathfrak{h} = \bigoplus_{i=1}^{m} \mathbb{R}X_i \oplus \bigoplus_{i=1}^{3} \mathbb{R}Y_i,$$

where $[X_i, X_j] = \sum_{k=1}^{3} c_{i,j}^k Y_k$, *the* Y_k *are in the center of* $\mathfrak{h}$ *and* $c_{i,j}^k = -c_{j,i}^k$ *are algebraically independent over* $\mathbb{Q}$. *Then for* m *is sufficiently large,* $\mathfrak{h}$ *will not have a* $\mathbb{Q}$*-structure.*

To embed $\mathfrak{h}$ *in a* 2 *step nilpotent Lie algebra* $\mathfrak{g}$ *with a* $\mathbb{Q}$*-stricture, let*

$$\mathfrak{g} = \bigoplus_{i=1}^{m} (\mathbb{R}X_i' \oplus \mathbb{R}X_i) \oplus \bigoplus_{i=1}^{3} \mathbb{R}Y_i,$$

where the Y_k *are central and* $[X_i, X_j]$ *are defined as above. For each* $i \le m$, *let* α_i *and* β_i *be distinct non zero real numbers, and for* $i < j \le m$ *let*

$$[X_i', X_j'] = \sum_{k=1}^{3} a_{i,j}^k Y_k$$

and

$$[X_i', X_j] = \sum_{k=1}^{3} b_{i,j}^k Y_k,$$

where the $a_{i,j}^k$ *and* $b_{i,j}^k$ *are chosen so that* $[X_i' + \alpha_i X_i, X_j' + \alpha_j X_j]$ *and* $[X_i' + \beta_i X_i, X_j' + \beta_j X_j] \in \bigoplus_{k=1}^{3} \mathbb{Q} Y_k$. *Then* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{g}$.

Moreover, these conditions give a $\mathbb{Q}$*-structure on the* 2 *step nilpotent Lie algebra,* $\mathfrak{g}$ *with* $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{h}$. *This is because for* $i < j$ *there exist* $a_{i,j}^k$ *and* $b_{i,j}^k$ *so that*

$$\begin{bmatrix} a_{i,j}^1 & b_{i,j}^1 & -b_{j,i}^1 \\ a_{i,j}^2 & b_{i,j}^2 & -b_{j,i}^2 \\ a_{i,j}^3 & b_{i,j}^3 & -b_{j,i}^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ \alpha_i & \beta_i \\ \alpha_j & \beta_j \end{bmatrix} = \begin{bmatrix} \alpha_i\alpha_j c_{i,j}^1 + x_{i,j}^1 & \beta_i\beta_j c_{i,j}^1 \\ \alpha_i\alpha_j c_{i,j}^2 + x_{i,j}^2 & \beta_i\beta_j c_{i,j}^2 \\ \alpha_i\alpha_j c_{i,j}^3 + x_{i,j}^3 & \beta_i\beta_j c_{i,j}^3 \end{bmatrix},$$

where $x_{i,j}^k \in \mathbb{Q}$ are chosen so that the columns of the right side are linearly independent.

Applying Proposition 1 this construction supplies numerous examples of 2-step unipotent $\mathbb{Q}$-groups, where $\mathfrak{h}$ doesn't have a rational form and hence H (which here is connected) cannot contain a lattice. This means if we want a positive result for the real points of a unipotent algebraic $\mathbb{Q}$-group G, we are forced to take G abelian. Of course, such a G is simply connected and is isomorphic to some $\mathbb{R}^n$ and here H contains a lattice. Henceforth our standing assumption will be that the unipotent radical G_u of G is *abelian*.

2. Semisimple groups

We next turn to *semisimple groups without compact factors* where the situation is positive. The Weyl-Chevalley root space decomposition guarantees $\mathrm{Ad}(G)$ is represented as the real points of a linear algebraic group defined over $\mathbb{Q}$.

Proposition 2. *Let G be a connected semisimple group without compact factors and represented as the real points of a linear algebraic group defined over $\mathbb{Q}$. Then any closed subgroup H of cofinite volume contains a lattice.*

Proof. First consider the case when $G = G_{\mathbb{R}}$ is simple. We may assume H is non discrete; otherwise we would be done. Since H is closed it is a Lie subgroup of $G_{\mathbb{R}}$. Let $\mathfrak{h} \neq (0)$ be its Lie algebra. Since $\mathfrak{h}$ is invariant under $\mathrm{Ad}(H)$, the Borel density theorem (see [10], or [7]) tells us $\mathfrak{h}$ is invariant under $\mathrm{Ad}(G)$; that is it is an ideal. Therefore $\mathfrak{h} = \mathfrak{g}$ and so H is open in G. Therefore $HG_{\mathbb{Z}}$ is open and therefore closed in G. By [3], $G_{\mathbb{R}}/G_{\mathbb{Z}}$ has finite volume. Hence so does $HG_{\mathbb{Z}}/G_{\mathbb{Z}}$ and therefore also $H/H \cap G_{\mathbb{Z}}$. Thus H contains the lattice $H \cap G_{\mathbb{Z}}$. We consider this to be the inductive case, $n = 1$.

Now let G be semisimple and without compact factors and H a closed subgroup with G/H of finite volume. Then, just as before, $\mathfrak{h}$ is an ideal. If the Lie algebra $\mathfrak{g}$ of G is the direct sum of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, then $\mathfrak{h}$ is the direct sum of some subset of these and H is a product of the corresponding G_i. There are two possibilities. Either this subset is proper, or $\mathfrak{h} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$. In the first case we get our conclusion that H contains a lattice by induction. If the Lie algebras of G and H coincide, then H is an open subgroup of G and since G is connected $H = G = G_{\mathbb{R}}$ and, again by [3], H contains the lattice $G_{\mathbb{Z}}$. □

Lemma 1. *Let $G = X \cdot Y$, where X and Y are algebraic subgroups of the real points of an algebraic group, G, and $[X, Y] = (1)$. If $X \cap Y$ is finite, then multiplication, m, gives a finite sheeted covering from the direct product $X \times Y \to G$. If H is closed in G and has cofinite volume, then $m^{-1}(H)$ is a closed subgroup of $X \times Y$ also of cofinite volume. If Γ is a lattice in $X \times Y$, its image is a lattice in G.*

Proof. Consider the multiplication map $m : X \times Y \to G$. Because $[X, Y] = (1)$ this continuous surjective map is a homomorphism. It has finite kernel since

$X \cap Y$ is finite. The statement concerning H is clear. We prove the image of Γ is a lattice by first showing $m(\Gamma)$ is closed. Consider a sequence from $m(\Gamma)$ converging to a point in G. Because the preimage of every element is finite, one of these must repeat infinitely often. This means there is some constant subsequence which also converges to this point and so the limit is in the image. Now lattices in connected Lie groups are finitely generated (see e.g. pg. 3 of [8]) and so in particular are countable and, of course, the same is true for a Lie group with a finite number of connected components. Since X and Y each has a finite number of connected components (see [11]), so does $X \times Y$. Hence Γ is countable. The Open Mapping theorem (Corollary 0.47 [1]) then tells us that m restricted to Γ is an open map onto its image and so $m(\Gamma)$ is discrete. Finally, the image has cofinite volume in G because Γ has cofinite volume in $X \times Y$ and this measure can be pushed forward. $\square$

Now suppose G is a connected semisimple group perhaps with some compact factors and represented as the real points of a linear algebraic group defined over $\mathbb{Q}$. Then $G = S \cdot K$, where S has no compact factors, K is compact, each is an algebraic group over $\mathbb{Q}$ and $[K, S] = (1)$. Since $S \cap K$ is discrete and K is compact $S \cap K$ is finite. By Lemma 1 we may assume G is the direct product of S and K.

Let π_S be the projection of G onto S. Then $\pi_S(H)^-$ is a cofinite volume subgroup of S. By Proposition 2 either $\pi_S(H)^-$ is discrete and so is a lattice, or contains a lattice of S. Thus in either case $\pi_S(H)^-$ contains a lattice Γ of S. Since K is compact, $\Gamma \times (1)$ is a lattice in $S \times K = G$ and is contained in H. Hence,

PROPOSITION 3. *Let G be any semisimple group represented as the real points of a linear algebraic group defined over $\mathbb{Q}$ and H be a closed subgroup of G with G/H of finite volume. Then $H \supseteq \Gamma$ for some lattice Γ.*

3. MIXED GROUPS

Finally we turn to mixed groups, that is, those with both a reductive part and a unipotent radical. By the Mostow decomposition theorem [9], $G = G_u \times R$ (semidirect product), where R is a maximal reductive of G.

The next example, which is the first one to try, shows in general that in mixed groups H fails to contain a lattice even though G is the real points of an algebraic $\mathbb{Q}$-group and G_u is abelian.

EXAMPLE 3. *Let G be the affine group of $\mathbb{R}^n$, that is the semidirect product* $\mathrm{GL}(n, \mathbb{R})$ *with $\mathbb{R}^n$. G has a faithful representation as a linear algebraic group defined over $\mathbb{Q}$ and is its group of real points. Since* $\mathrm{GL}(n, \mathbb{R})$ *distorts volume on $\mathbb{R}^n$, G is non unimodular. As is well known* (*see Proposition* 2.4.2 *of* [1]), *if G contains a lattice it must be unimodular. Hence here G has no lattices at all. But of course G has subgroups H of cofinite volume. For instance, $H = \mathbb{R}^\times \cdot \mathrm{SL}(n, \mathbb{Z}) \cdot \mathbb{R}^n$. H contains no lattice as G has none.*

The problem here is that the reductive part distorts Haar measure on the unipotent radical. This suggests that in looking for positive results in mixed groups, the reductive part R should be semisimple since then this cannot happen.

Theorem 1. *Let $G = G_u \times R$ (semidirect product) be connected component of the real points of an algebraic $\mathbb{Q}$-group, where R is semisimple, and G_u be abelian. If H is a closed subgroup of G of cofinite volume, then H contains a lattice of G. Similarly, if G is reductive and the connected component of the real points of an algebraic $\mathbb{Q}$-group, the same conclusion holds.*

Proof. $X_{\mathbb{Q}}(G)$ is trivial since $X_{\mathbb{Q}}(G_u) = (1)$ and R has no characters at all. Hence by [3], $G_{\mathbb{Z}}$ is a lattice in G. To prove the theorem it suffices to show $HG_{\mathbb{Z}}$ is closed in G. For then $HG_{\mathbb{Z}}/H$ has finite volume (see Theorem 1.13 of [10]) and therefore also (Lemma 1.7 of [10]) so does $H/H \cap G_{\mathbb{Z}}$. Thus H would contain the lattice, $H \cap G_{\mathbb{Z}}$.

By Corollaire 7.13 of [2], $(G_u)_{\mathbb{Z}} R_{\mathbb{Z}}$ has finite index in $G_{\mathbb{Z}}$ so we can replace $G_{\mathbb{Z}}$, as above, by $(G_u)_{\mathbb{Z}} R_{\mathbb{Z}}$. Let π_R be the projection of G onto R and π_u the projection of G onto G_u. Then $\pi_R(H)^-$ and $\pi_u(H)^-$ are closed subgroups of cofinite volume in R and G_u respectively. By Propositions 3 and our assumption that G_u is abelian respectively, $\pi_R(H)^- \cap R_{\mathbb{Z}}$ is a lattice in R (or $\pi_R(H)^-$ is discrete, that is $\pi_R(H)$ is discrete) and $\pi_u(H)^- \cap (G_u)_{\mathbb{Z}}$ is a lattice in G_u (or $\pi_u(H)^-$ is discrete, that is $\pi_u(H)$ is discrete).

If either of these discrete projections occur we would be done. For suppose $\pi_u(H)$ is discrete. Then the inverse image, $\pi_u^{-1}(\pi_u(H)) = RH$, is closed in G. Therefore $RH/H = R/R \cap H$ has finite volume and so by Proposition 3 $R \cap H$ contains a lattice of R. Hence $\pi_u(H) \cdot R \cap H$ is a lattice in G contained in H. Similarly, if $\pi_R(H)$ is discrete, reversing the roles of G_u and R and using the fact that G_u is abelian we conclude $G_u \cap H \cdot \pi_R(H)$ is also a lattice in G contained in H.

Thus we can assume $\pi_R(H)^- \cap R_{\mathbb{Z}}$ is a lattice in R and $\pi_u(H)^- \cap (G_u)_{\mathbb{Z}}$ is a lattice in G_u. Since these conditions are respectively equivalent to $\pi_R(H)^- \cdot R_{\mathbb{Z}} = R^*$ is closed in R and $\pi_u(H)^- \cdot (G_u)_{\mathbb{Z}} = G_u^*$ is closed in G_u. Hence their direct product, $G_u^* \times R^*$, is closed in G. To show $H \cdot (G_u)_{\mathbb{Z}} \cdot R_{\mathbb{Z}}$ is closed in G it is sufficient to show that it is closed in $G_u^* \times R^*$. Let h_n, $(g_u)_n$ and r_n be sequences in H, $(G_u)_{\mathbb{Z}}$ and $R_{\mathbb{Z}}$ respectively and suppose $h_n (g_u)_n r_n$ converges to (x^*, y^*), where $x^* \in G_u^*$ and $y^* \in R^*$. Writing $h_n = x_n y_n = (x_n, y_n)$, then for all n, $x_n \in \pi_u(H)$ and $y_n \in \pi_R(H)$. Because G_u is normal, $x_n y_n (g_u)_n r_n = x_n (g_u)_n z_n r_n$, where $z_n = y_n^{-1}$. Hence $x_n (g_u)_n \in G_u$ and $z_n r_n \in R$. It follows that $x_n (g_u)_n$ converges to x^* in G_u^* and $z_n r_n$ converges to y^* in R^*. Hence $h_n (g_u)_n r_n$ converges to a point of $G_u^* \times R^*$.

When G is reductive as above $G = Z(G)_0 \times [G, G]$, where $[G, G]$ is semisimple. Since $Z(G)$ is also algebraic $\mathbb{Q}$-group, by [11] $[Z(G) : Z(G)_0]$ is finite. The argument above works here as well. $\square$

CUNY Graduate Center

References

[1] H. Abbaspour - M. Moskowitz, *Basic Lie Theory*, 2007, World Scientific Publishing Co., Singapore.

[2] A. Borel, *Introduction aux Groupes Arithmetiques*, Hermann, Paris, 1969.

[3] A. Borel - Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Annals of Math., 75 (1962), 485–535.

[4] F. Greenleaf - M. Moskowitz - L. Rothschild, *Automorphisms, orbits and homogeneous spaces of non-connected Lie Groups*, Math. Annalen, 212 (1974), 145–155.

[5] G. P. Hochschild, *The Structure of Lie Groups*, Holden Day, San Francisco, London, Amsterdam, 1965.

[6] J. E. Humphries, *Linear Algebraic Groups*, Springer-Verlag, New York, Berlin, Heidelberg, 1998.

[7] R. Mosak - M. Moskowitz, *Zariski density in Lie groups*, Israel J. of Math., 52 (1985), 1–14.

[8] R. Mosak - M. Moskowitz, *Stabilizers of lattices in Lie groups*, J. of Lie Theory, Vol 4 (1994), 1–16.

[9] G. D. Mostow, *Fully Reducible Subgroups of Algebraic Groups*, Amer J. of Math., 78 (1956), 200–221.

[10] M. S. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergb. Band 68, Springer Verlag, New York, Heidelberg, 1972.

[11] H. Whitney, *Elementary structure of real algebraic varieties*, Ann. of Math., 66 (1957), 545–556.

Received 28 May 2012,
and in revised form 27 October 2012.

martin.moskowitz@gmail.com