**Partial Differential Equations, Integral Equations** — *Boundary value problems and integral operators for the bi-Laplacian in non-smooth domains*, by IRINA MITREA[†] and MARIUS MITREA, communicated on 8 February 2013.

*Dedicated to the memory of G. Fichera.*

ABSTRACT. — In this paper we explore the effectiveness of the classical method of layer potentials in the treatment of boundary value problems for the bi-Laplacian formulated in arbitrary Lipschitz domains, Lipschitz domains whose outward unit normal has small mean-oscillations, and domains of class $\mathscr{C}^1$.

KEY WORDS:   bi-Laplacian, multi-layers, Lipschitz domains, conormal derivative, bilinear form.

MATHEMATICS SUBJECT CLASSIFICATION:   Primary: 35J40, 35B30, 31B10, 35C15, 31A30, 31B30; Secondary: 35J05, 35J08.

## 1. INTRODUCTION

The discussion in this paper is largely motivated by the classical free-plate problem arising in the Kirchhoff-Love theory of thin plates. In the case of a domain $\Omega$ in the two dimensional setting, this problem reads as follows:

$$(1.1) \qquad \Delta^2 u = 0 \text{ in } \Omega, \text{ with } Mu \text{ and } Nu \text{ prescribed on } \partial\Omega,$$

where the boundary operators $M$, $N$ are defined by

$$(1.2) \qquad Mu := \eta\Delta u + (1-\eta)\frac{\partial^2 u}{\partial v^2},$$

$$Nu := \frac{\partial \Delta u}{\partial v} + (1-\eta)\frac{\partial^3 u}{\partial v \partial \tau^2},$$

where $\eta$ is the Poisson coefficient of the plate, and $v$, $\tau$ denote, respectively, the outward unit normal and unit tangent to $\partial\Omega$. See, e.g., [1], [3, (3.29)–(3.31), p. 679], [2, (10)–(11), p. 1237], [10, p. 6], [11], [18, Proposition 3.1], [20, (2.2)–(2.3), p. 24], [23, (2.12), p. 136], [4, pp. 420–423], [19] as well as the informative discussion in [17] where it is indicated that the above problem has been first

solved by Gustav Kirchhoff in a variational sense. Indeed, it is now folklore that, for boundary data in appropriate function spaces (and by imposing suitable bounds on the Poisson coefficient), the problem (1.1) has a variational solution $u$ in $W^{2,2}(\Omega)$, which is unique modulo polynomials of degree $\leq 1$.

One of our main goals is to study further regularity properties of such a solution, measured on Besov and Triebel-Lizorkin scales. We shall do so working in the higher dimensional setting and the starting point is to establish well-posedness results when the size of the solution is measured using the nontangential maximal operator. The final results are then obtained via interpolation and Fredholm theory.

The plan of the paper is as follows. In Section 2 we review a number of basic definitions, including Lipschitz and $\mathscr{C}^1$ domains, nontangential maximal functions and poitnwise traces on the boundary, smoothness scalar spaces of Lebesgue, Sobolev, and Besov type on Lipschitz surfaces as well as their vector valued counterparts consisting of Whitney arrays (satisfying certain first order differential compatibility conditions). This section also contains a brief review of smoothness spaces in Lipschitz domains (Besov, Triebel-Lizorkin, and weighted Sobolev spaces) as well as boundary trace results formulated for these spaces. The starting point in Section 3 is the consideration of a distinguished family of bilinear forms associated with $\Delta^2$ which have played a basic role in G. Verchota's work in [26]. In particular, we explain how Green's formula for $\Delta^2$ involving such bilinear forms naturally involves the higher-dimensional versions of operators $M$, $N$ from (1.2). In turn, the latter operators are used to define a family of conormal derivatives for the bi-Laplacian (see Proposition 3.2) which plays a key role in subsequent work.

Boundary operators of multi-layer type are introduced and studied in Section 4. Such operators fit into the general Calderón-Zygmund theory developed recently in [15] and a number of basic properties follow as a result of the latter. Our operators interface tightly with certain versions considered by G. Verchota and Z. Shen in [26], [24], and such connections are made transparent in Proposition 4.2 and Proposition 4.6. Section 5 is primarily devoted to establishing invertibility results for the (boundary-to-boundary versions of our) multi-layer operators. Such results constitute the key ingredient in the proof of well-posedness of boundary-value problems for the bi-Laplacian in Section 6. Here we treat the Dirichlet problem for $\Delta^2$ in Theorem 6.3, the regularity problem in Theorem 6.5, the Neumann problem for the bi-Laplacian with boundary data from the dual of Whitney-Lebesgue spaces in Theorem 6.6, the inhomogeneous Dirichlet problem for $\Delta^2$ with boundary data from Whitney-Besov spaces in Theorem 6.10, and the inhomogeneous Neumann problem for the bi-Laplacian with boundary data from duals of Whitney-Besov spaces in Theorem 6.16. Along the way, we prove a number of significant consequences and also discuss the sharpness of some of the aforementioned well-posedness results. Throughout this section, we work in the geometrical context of arbitrary Lipschitz domains, Lipschitz domains whose outward unit normal has small mean-oscillations, and domains of class $\mathscr{C}^1$. Naturally, all theorems involved are correspondingly nuanced depending on the strength of the geometrical hypotheses enforced in each case. This body of results

complements, sharpens and extends the work done by G. Verchota and Z. Shen in [26], [24].

In closing, we wish to acknowledge the lasting, influential role of the pioneering work of G. Fichera in the areas of elasticity theory and boundary integral methods for higher-order systems (cf. [5], [6], [7] to cite just a small fraction of his extended scientific output in this regard), to which the topic of the current paper is closely related.

## 2. FUNCTION SPACES ON LIPSCHITZ DOMAINS

With $\mathbb{N}$ denoting the collection of all (strictly) positive integers, we shall abbreviate $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, $\mathbb{N}_0^n$ may be regarded as the set of all multi-indices $\{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{N}_0, 1 \le i \le n\}$. As usual, for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we denote by $|\alpha| := \alpha_1 + \cdots + \alpha_n$ its length, and define $\alpha! := \alpha_1! \dots \alpha_n!$ (with the usual convention that $0! := 1$). Also, write $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ and, given $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, by $\beta \le \alpha$ it is understood that $\beta_j \le \alpha_j$ for each $j \in \{1, \dots, n\}$. Denote by $\{e_j\}_{1 \le j \le n}$ the standard orthonormal basis in $\mathbb{R}^n$. Whenever useful, we shall canonically identify these vectors with multi-indices from $\mathbb{N}_0^n$. Generally speaking, given a set $A$ (clear from context), for each $a, b \in A$ we let

$$(2.1) \qquad \delta_{ab} := \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

stand for the usual Kronecker symbol. Given an open subset $\mathcal{O}$ of $\mathbb{R}^n$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, we shall denote by $\mathscr{C}^k(\mathcal{O})$ the collection of all $k$-times continuously differentiable functions in $\mathcal{O}$, and by $\mathscr{C}_c^\infty(\mathcal{O})$ the collection of all indefinitely differentiable functions which are compactly supported in $\mathcal{O}$. In this connection, let us define $\mathscr{C}^\infty(\overline{\mathcal{O}}) := \{\varphi|_{\mathcal{O}} : \varphi \in \mathscr{C}^\infty(\mathbb{R}^n)\}$. Finally, we shall write $\mathcal{D}(\mathcal{O})$ and $\mathcal{D}'(\mathcal{O})$, respectively, for the space of test functions and distributions in $\mathcal{O}$.

We continue by recalling a basic definition.

DEFINITION 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open, bounded set. Then $\Omega$ is called a* bounded Lipschitz domain *(respectively,* bounded domain of class $\mathscr{C}^1$*) if for any $X_0 \in \partial\Omega$ there exist $r, h > 0$ and a coordinate system $(x_1, \dots, x_n) = (x', x_n)$ in $\mathbb{R}^n$ which is isometric to the canonical one and has origin at $X_0$, along with a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ which is Lipschitz (respectively, of class $\mathscr{C}^1$) and for which*

$$(2.2) \quad \Omega \cap \mathcal{C}(r, h) = \{X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < r \text{ and } \varphi(x') < x_n < h\},$$

*where $\mathcal{C}(r, h)$ denotes the open cylinder*

$$(2.3) \qquad \{X = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < r \text{ and } -h < x_n < h\} \subseteq \mathbb{R}^n.$$

*An* atlas *for $\partial\Omega$ is a finite collection of cylinders $\{\mathcal{C}_k(r_k, h_k)\}_{1 \le k \le N}$ (with associated Lipschitz maps $\{\varphi_k\}_{1 \le k \le N}$) covering $\partial\Omega$. Having fixed such an*

*atlas, the* Lipschitz character *of* $\Omega$ *is defined as the quartet consisting of numbers* $N$, $\max\{\|\nabla\varphi_k\|_{L^\infty(\mathbb{R}^{n-1})} : 1 \le k \le N\}$, $\min\{r_k : 1 \le k \le N\}$, *and* $\min\{h_k : 1 \le k \le N\}$.

As is well-known, for a bounded Lipschitz domain $\Omega$, the surface measure $\sigma$ is well-defined on $\partial\Omega$ and may be described as

$$(2.4) \qquad\qquad \sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega,$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. As a consequence of the classical Rademacher theorem, the outward pointing normal vector $v = (v_1, \ldots, v_n)$ to a given bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ exists at $\sigma$-almost every point on $\partial\Omega$. We shall extensively work with the tangential derivative operators $\partial_{\tau_{jk}}$ by

$$(2.5) \qquad\qquad \partial_{\tau_{jk}} := v_j\partial_k - v_k\partial_j, \quad j,k \in \{1, \ldots, n\}.$$

In this notation, the tangential gradient, $\nabla_{tan}f$, of a function $f$ on $\partial\Omega$ is given by

$$(2.6) \qquad\qquad \nabla_{tan}f := (v_k\partial_{\tau_{kj}}f)_{1 \le j \le n},$$

with the summation convention over repeated indices understood.

For a fixed parameter $\kappa > 0$ define next the nontangential maximal operator by setting, for any given function $u$ in $\Omega$,

$$(2.7) \quad \mathcal{N}u(X) := \sup\{|u(Y)| : Y \in \Omega \text{ s.t. } |X - Y| < (1 + \kappa)\operatorname{dist}(Y, \partial\Omega)\}.$$

Also, define the nontangential boundary trace of a function $u$ in $\Omega$ as

$$(2.8) \qquad\qquad u\big\lfloor_{\partial\Omega}(X) := \lim_{\substack{\Omega \ni Y \to X \\ |X-Y|<(1+\kappa)\operatorname{dist}(Y, \partial\Omega)}} u(Y), \quad X \in \partial\Omega,$$

whenever meaningful. Also, if $v = (v_j)_{1 \le j \le n}$ denotes the outward unit normal to $\Omega$ then for any function $u \in \mathscr{C}^1(\Omega)$ we define its normal derivative, $\partial_v u$, by the formula

$$(2.9) \qquad\qquad \partial_v u := \sum_{i=1}^n v_i(\partial_i u)\big\lfloor_{\partial\Omega},$$

whenever the boundary traces in the right-hand side are meaningful. Hence,

$$(2.10) \qquad\qquad \partial_v u = v \cdot ((\nabla u)\big\lfloor_{\partial\Omega}).$$

Going further, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, for each index $p \in (0, \infty]$ we shall denote by $L^p(\partial\Omega)$ the Lebesgue space of $\sigma$-measurable, $p$-th power integrable functions on $\partial\Omega$ (with respect to $\sigma$). That is,

$$(2.11) \qquad L^p(\partial\Omega) := \{f\,\sigma\text{-measurable on } \partial\Omega : \|f\|_{L^p(\partial\Omega)} < \infty\},$$

where

$$(2.12) \qquad \|f\|_{L^p(\partial\Omega)} := \left( \int_{\partial\Omega} |f|^p \, d\sigma \right)^{1/p}.$$

We shall also need Sobolev spaces of order one on the boundary of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$. Specifically, for each $p \in (1, \infty)$ we set

$$(2.13) \qquad L^p_1(\partial\Omega) := \{ f \in L^p(\partial\Omega) : \|f\|_{L^p_1(\partial\Omega)} < \infty \},$$

where

$$(2.14) \qquad \|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \|\nabla_{tan} f\|_{L^p(\partial\Omega)}.$$

Finally, for $1 < p < \infty$, $0 < q \leq \infty$, and $s \in (0,1)$, the Besov space $B^{p,q}_s(\partial\Omega)$ may be defined as

$$(2.15) \qquad B^{p,q}_s(\partial\Omega) := (L^p_1(\partial\Omega), L^p(\partial\Omega))_{s,q},$$

where $(\cdot, \cdot)_{s,q}$ denotes the real interpolation bracket. For future reference, let us also set

$$(2.16) \qquad L^p_{-1}(\partial\Omega) := (L^{p'}_1(\partial\Omega))^*, \quad 1/p + 1/p' = 1.$$

At this stage, we insert a brief discussion of smoothness spaces consisting of Whitney arrays on the boundary of a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$. Call a family of $n + 1$ functions from $L^1(\partial\Omega)$,

$$(2.17) \qquad \dot{f} = (f_0, f_1, \ldots, f_n),$$

a `Whitney array` provided the following compatibility conditions are satisfied:

$$(2.18) \qquad \partial_{\tau_{jk}} f_0 = v_j f_k - v_k f_j, \quad j, k \in \{1, \ldots, n\}.$$

Then the general recipe for constructing function spaces consisting of Whitney arrays is described next. Given a quasi-Banach space of functions $\mathscr{X} \subseteq L^1(\partial\Omega)$, set

$$(2.19) \quad WA(\mathscr{X}) := \{ \dot{f} = (f_0, f_1, \ldots, f_n) : f_j \in \mathscr{X}, 0 \leq j \leq n, \text{ satisfying } (2.18) \},$$

which we equip with the quasi-norm

$$(2.20) \qquad \|\dot{f}\|_{WA(\mathscr{X})} := \sum_{j=0}^{n} \|f_j\|_{\mathscr{X}}.$$

In this paper, we shall primarily work with three such scales of function spaces consisting of Whitney arrays, corresponding to $\mathscr{X}$ being one of the spaces $L^p(\partial\Omega)$, $L^p_1(\partial\Omega)$, and $B^{p,q}_s(\partial\Omega)$. The resulting Whitney array function spaces

constructed according to the recipe (2.19)–(2.20) for the choices $\mathscr{X} := L^p(\partial\Omega)$, $\mathscr{X} := L_1^p(\partial\Omega)$, and $\mathscr{X} := B_s^{p,q}(\partial\Omega)$ are going to be denoted by

$$(2.21) \qquad \dot{L}_{1,0}^p(\partial\Omega), \quad \dot{L}_{1,1}^p(\partial\Omega), \quad \text{and} \quad \dot{B}_{1,s}^{p,q}(\partial\Omega), \quad \text{respectively.}$$

We continue by recording a result pertaining to the nature of $\dot{L}_{1,0}^p(\partial\Omega)$ and $\dot{L}_{1,1}^p(\partial\Omega)$.

**PROPOSITION 2.2.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by $v = (v_j)_{1 \le j \le n}$ its outward unit normal. Then for every $p \in (1, \infty)$, the mapping*

$$(2.22) \qquad \Psi : \dot{L}_{1,0}^p(\partial\Omega) \to L_1^p(\partial\Omega) \times L^p(\partial\Omega),$$

*given by*

$$(2.23) \qquad \Psi(\dot{f}) := \left(f_0, -\sum_{j=1}^n v_j f_j\right), \quad \forall \dot{f} = (f_0, f_1, \ldots, f_n) \in \dot{L}_{1,0}^p(\partial\Omega),$$

*is an isomorphism, whose inverse $\Psi^{-1} : L_1^p(\partial\Omega) \times L^p(\partial\Omega) \to \dot{L}_{1,0}^p(\partial\Omega)$ may be described as*

$$(2.24) \qquad \Psi^{-1}(F, g) = \dot{f} := (f_0, f_1, \ldots, f_n) \text{ where } f_0 := F$$

$$\text{and } f_j := -v_j g + \sum_{k=1}^n v_k \partial_{\tau_{kj}} F \text{ for } 1 \le j \le n,$$

$$\text{for every } (F, g) \in L_1^p(\partial\Omega) \times L^p(\partial\Omega).$$

*Furthermore, if $v \in \mathscr{C}^1(\Omega)$ is a function with the property that*

$$(2.25) \qquad \mathcal{N}(v), \mathcal{N}(\nabla v) \in L^p(\partial\Omega), \quad \exists v\lfloor_{\partial\Omega} \text{ and } \exists (\nabla v)\lfloor_{\partial\Omega},$$

*then*

$$(2.26) \qquad \Psi^{-1}(v\lfloor_{\partial\Omega}, -\partial_v v) = (v\lfloor_{\partial\Omega}, (\nabla v)\lfloor_{\partial\Omega}).$$

**PROOF.** This is a version of a more general result proved in Proposition 3.3 in [15] corresponding to the case $m = 2$. $\qquad\square$

We shall also need to use the adjoint of the operator $\Psi$. Its main properties are summarized below.

**PROPOSITION 2.3.** *Retain the same background hypotheses as in Proposition 2.2 and denote by $\Psi^*$ the adjoint of the operator $\Psi$ defined in (2.22)–(2.23). Then, for each $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$,*

$$(2.27) \qquad \Psi^* : L_{-1}^{p'}(\partial\Omega) \times L^{p'}(\partial\Omega) \to (\dot{L}_{1,0}^p(\partial\Omega))^* \quad \text{isomorphically.}$$

*Moreover, for each $(G, f) \in L^{p'}_{-1}(\partial\Omega) \times L^{p'}(\partial\Omega)$ one has*

$$(2.28) \qquad \Psi^*(G, f) = (G, -(v_j f)_{1 \leq j \leq n}),$$

*in the sense that*

$$(2.29) \qquad \langle \Psi^*(G, f), \dot{g} \rangle = \langle G, g_0 \rangle - \sum_{j=1}^{n} \int_{\partial\Omega} v_j f g_j \, d\sigma,$$

$$\text{for all } \dot{g} = (g_0, (g_j)_{1 \leq j \leq n}) \in \dot{L}^p_{1,0}(\partial\Omega).$$

*Furthermore, the inverse of $\Psi^*$ in (2.27) may be described as*

$$(2.30) \qquad (\Psi^*)^{-1}(\Lambda) = \left( \eta_0 - \sum_{j,k=1}^{n} \partial_{\tau_{kj}}(v_k \eta_j), \; -\sum_{j=1}^{n} v_j \eta_j \right)$$

*if the functional $\Lambda \in (\dot{L}^p_{1,0}(\partial\Omega))^*$ is given by paring against the $(n+1)$-tuple $\dot{\eta}$ where $\dot{\eta} = (\eta_0, (\eta_j)_{1 \leq j \leq n}) \in L^{p'}_{-1}(\partial\Omega) \times [L^{p'}(\partial\Omega)]^n$.*

PROOF. This follows by unraveling definitions, in a straightforward manner. □

Next, we shall succinctly recall the smoothness scales of spaces of Besov and Triebel-Lizorkin type on arbitrary open subsets of the Euclidean ambient. For the definitions and properties of the standard scales of Triebel-Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$ and Besov spaces $B_s^{p,q}(\mathbb{R}^n)$ (indexed by $s \in \mathbb{R}$ and $0 < p, q \leq \infty$), we refer the reader to, e.g., [25], [8], [9], [22]. Next, given an arbitrary open subset $\Omega$ of $\mathbb{R}^n$, denote by $f|_\Omega$ the restriction of a distribution $f$ in $\mathbb{R}^n$ to $\Omega$. For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ we then set

$$(2.31) \qquad F_s^{p,q}(\Omega) := \{u \text{ distribution in } \Omega : \exists v \in F_s^{p,q}(\mathbb{R}^n) \text{ s.t. } v|_\Omega = u\},$$
$$\|u\|_{F_s^{p,q}(\Omega)} := \inf\{\|v\|_{F_s^{p,q}(\mathbb{R}^n)} : v \in F_s^{p,q}(\mathbb{R}^n), v|_\Omega = u\},$$
$$\forall u \in F_s^{p,q}(\Omega),$$

and

$$(2.32) \qquad B_s^{p,q}(\Omega) := \{u \text{ distribution in } \Omega : \exists v \in B_s^{p,q}(\mathbb{R}^n) \text{ s.t. } v|_\Omega = u\},$$
$$\|u\|_{B_s^{p,q}(\Omega)} := \inf\{\|v\|_{B_s^{p,q}(\mathbb{R}^n)} : v \in B_s^{p,q}(\mathbb{R}^n), v|_\Omega = u\},$$
$$\forall u \in B_s^{p,q}(\Omega).$$

A detailed analysis of these scales in the setting of Lipschitz domains may be found in [15]. In particular, it has been shown here that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $1 < p < \infty$, $0 < q \leq \infty$, and $0 < s < 1$, then the following boundary trace operators are well-defined, linear, bounded and onto (in fact, in each case there is a linear and bounded right-inverse):

$$(2.33) \qquad \text{Tr}: F^{p,q}_{s+1/p}(\Omega) \to B^{p,p}_s(\partial\Omega),$$

$$(2.34) \qquad \text{Tr}: B^{p,q}_{s+1/p}(\Omega) \to B^{p,q}_s(\partial\Omega),$$

$$(2.35) \qquad F^{p,q}_{1+s+1/p}(\Omega) \ni u \mapsto (\text{Tr}\,u, \text{Tr}(\nabla u)) \in \dot{B}^{p,p}_{1,s}(\partial\Omega),$$

$$(2.36) \qquad B^{p,q}_{1+s+1/p}(\Omega) \ni u \mapsto (\text{Tr}\,u, \text{Tr}(\nabla u)) \in \dot{B}^{p,q}_{1,s}(\partial\Omega).$$

In the last part of this section we shall define certain weighted Sobolev spaces. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by $\rho$ the distance function to the boundary of $\Omega$. Then for each $p \in [1, \infty]$, $a \in (-1/p, 1 - 1/p)$, and $k \in \mathbb{N}_0$, introduce

$$(2.37) \quad W^{k,p}_a(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ locally integrable, and} \right.$$

$$\left. \|u\|_{W^{k,p}_a(\Omega)} := \sum_{|\alpha| \le k} \left( \int_\Omega |\partial^\alpha u(X)|^p \rho(X)^{ap}\, dX \right)^{1/p} < \infty \right\}.$$

When $a = 0$, we agree to drop it as a subscript. In particular, we set

$$(2.38) \qquad \mathring{W}^{k,p}(\Omega) := \text{the closure of } \mathscr{C}^\infty_c(\Omega) \text{ in } (W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$$

and, assuming that $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, define

$$(2.39) \qquad W^{-k,p}(\Omega) := (\mathring{W}^{k,p'}(\Omega))^*.$$

For any bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$ and any $k \in \mathbb{Z}$, $p \in (1, \infty)$, we then have (cf. [15])

$$(2.40) \qquad W^{k,p}(\Omega) = F^{p,2}_k(\Omega),$$

and

$$(2.41) \qquad \mathring{W}^{2,p}(\Omega) = \{u \in W^{2,p}(\Omega) : (\text{Tr}\,u, \text{Tr}(\nabla u)) = \dot{0}\}.$$

## 3. Bilinear forms associated to the bi-Laplacian

To set the stage, fix $n \in \mathbb{N}$ with $n \ge 2$ and, given an arbitrary number $\theta \in \mathbb{R}$, consider the coefficient tensor

$$(3.1) \qquad A_\theta := (A_{\alpha\beta}(\theta))_{|\alpha|=|\beta|=2},$$

with scalar entries, defined for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n_0$ with the property that $|\alpha| = |\beta| = 2$ by the formula

$$(3.2) \qquad A_{\alpha\beta}(\theta) := \frac{1}{1 + 2\theta + n\theta^2} \sum_{i,j=1}^{n} \left( \delta_{\beta(e_i+e_j)} + \theta\delta_{ij} \sum_{k=1}^{n} \delta_{\beta(2e_k)} \right) \times$$

$$\times \left( \delta_{\alpha(e_i+e_j)} + \theta\delta_{ij} \sum_{k=1}^{n} \delta_{\alpha(2e_k)} \right),$$

where the Kronecker symbols are defined as in (2.1). Next, consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and, in relation to the coefficient tensor from (3.1)–(3.2), for each $\theta \in \mathbb{R}$ introduce the bilinear form (as usual, with $\Delta := \partial_1^2 + \cdots + \partial_n^2$ denoting the Laplacian in $\mathbb{R}^n$)

$$(3.3) \quad \mathcal{B}_\theta(u,v) := \sum_{|\alpha|=|\beta|=2} \int_\Omega A_{\alpha\beta}(\theta)(\partial^\beta u)(X)(\partial^\alpha v)(X) \, dX$$

$$= \frac{1}{1 + 2\theta + n\theta^2} \sum_{i,j=1}^{n} \int_\Omega [(\partial_i\partial_j + \theta\delta_{ij}\Delta)u](X)[(\partial_i\partial_j + \theta\delta_{ij}\Delta)v](X) \, dX,$$

where $u$, $v$ are any two reasonably behaved (real-valued) functions in $\Omega$. See, e.g., [3, Lemma 3.4, p. 680], [10, p. 5], [20, (2.13), p. 25], [26, (10.2)]. Then one can readily verify that for each $\theta \in \mathbb{R}$ the bi-Laplacian may be written as

$$(3.4) \qquad\qquad \Delta^2 = \sum_{|\alpha|=|\beta|=2} \partial^\alpha A_{\alpha\beta}(\theta)\partial^\beta.$$

In particular, for each $\theta \in \mathbb{R}$ the bilinear form $\mathcal{B}_\theta(\cdot,\cdot)$ introduced in (3.3) satisfies

$$(3.5) \qquad\qquad \mathcal{B}_\theta(u,v) = \int_\Omega (\Delta^2 u)(X)v(X) \, dX, \quad \forall u,v \in \mathcal{C}_c^\infty(\Omega).$$

Indeed, it is easy to check that

$$(3.6) \qquad \frac{1}{1 + 2\theta + n\theta^2} \sum_{i,j=1}^{n} (\partial_i\partial_j + \theta\delta_{ij}\Delta)(\partial_i\partial_j + \theta\delta_{ij}\Delta) = \Delta^2.$$

Let us also note that $\Delta^2$ is strongly elliptic since, as a direct calculation based on (3.2) shows,

$$(3.7) \qquad\qquad \sum_{|\alpha|=|\beta|=2} A_{\alpha\beta}(\theta)\xi^{\alpha+\beta} = |\xi|^4, \quad \text{for each } \xi \in \mathbb{R}^n.$$

Going further, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with outward unit normal $v = (v_j)_{1 \le j \le n}$ and a function $u \in \mathcal{C}^2(\Omega)$, define its second-order normal

derivative, $\partial_\nu^2 u$, by the formula

$$(3.8) \qquad \partial_\nu^2 u := \sum_{i,j=1}^n \nu_i \nu_j (\partial_i \partial_j u)\big\lfloor_{\partial\Omega}.$$

Then, if $u \in \mathscr{C}^3(\Omega)$, for each $\theta \in \mathbb{R}$ set (in analogy with (1.2), following [26])

$$(3.9) \qquad N_\theta(u) := \partial_\nu(\Delta u) + \frac{1}{2(1 + 2\theta + n\theta^2)} \sum_{i,j=1}^n \partial_{\tau_{ij}} \left( \sum_{k=1}^n \nu_k \partial_{\tau_{ij}} \partial_k u \right),$$

$$M_\theta(u) := \frac{2\theta + n\theta^2}{1 + 2\theta + n\theta^2} \Delta u + \frac{1}{1 + 2\theta + n\theta^2} \sum_{j,k=1}^n \nu_j \nu_k \partial_j \partial_k u,$$

where all spacial partial derivatives of $u$ in the right-hand sides are understood as being restricted (either in a nontangential pointwise sense, or as tangential derivatives of such traces) to $\partial\Omega$. Simple algebraic manipulations show that the above operators may be alternatively expressed as

$$(3.10) \qquad N_\theta(u) = \partial_\nu(\Delta u) + \frac{1}{1 + 2\theta + n\theta^2} \sum_{i,j=1}^n \partial_{\tau_{ij}} \left( \sum_{k=1}^n \nu_k \nu_i \partial_j \partial_k u \right)$$

$$= \partial_\nu(\Delta u) + \frac{1}{1 + 2\theta + n\theta^2} \sum_{i,j=1}^n \partial_{\tau_{ij}} \left( \sum_{k=1}^n \nu_i \partial_{\tau_{kj}} \partial_k u \right),$$

and

$$(3.11) \qquad M_\theta(u) = \Delta u + \frac{1}{1 + 2\theta + n\theta^2} \sum_{j,k=1}^n \nu_j \partial_{\tau_{kj}} \partial_k u.$$

The relationship between the operators $N_\theta$, $M_\theta$ and the bilinear form $\mathcal{B}_\theta(\cdot,\cdot)$ is brought to prominence in the following result, describing a Green-type formula for the bi-Laplacian (cf. [3, Lemma 3.4, p. 680] and [20, (2.20), p. 26] for a proof in domains in $\mathbb{R}^2$, and [26, (10.2)] for a statement in the setting of biharmonic functions in domains in $\mathbb{R}^n$, $n \geq 2$).

PROPOSITION 3.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with outward unit normal $\nu = (\nu_j)_{1 \leq j \leq n}$ and surface measure $\sigma$. Let $\theta \in \mathbb{R}$ and recall the operators $N_\theta$ and $M_\theta$ introduced in (3.9), relative to this setting. Then for any $u, v \in \mathscr{C}^\infty(\overline{\Omega})$ there holds*

$$(3.12) \qquad \mathcal{B}_\theta(u,v) = \int_{\partial\Omega} \langle (M_\theta(u), N_\theta(u)), (\partial_\nu v, -v) \rangle \, d\sigma + \int_\Omega (\Delta^2 u) v \, dX,$$

*where $\langle \cdot, \cdot \rangle$ denotes the canonical pointwise scalar product between vector-valued functions.*

*In particular, if $v \in \mathscr{C}^{\infty}(\overline{\Omega})$ and $u$ is a reasonably behaved null-solution of the bi-Laplacian $\Delta^2$ in $\Omega$, then*

$$(3.13) \qquad \mathcal{B}_{\theta}(u, v) = \int_{\partial\Omega} \langle (M_{\theta}(u), N_{\theta}(u)), (\partial_{\nu}v, -v) \rangle \, d\sigma.$$

PROOF. Integrating by parts and using Einstein's convention of summation over repeated indices, we may write

$$(3.14) \qquad \int_{\Omega} (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot (\partial_i\partial_j + \theta\delta_{ij}\Delta)v \, dX$$

$$= -\int_{\Omega} \partial_i(\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_j v \, dX$$

$$+ \int_{\partial\Omega} \nu_i \cdot (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_j v \, d\sigma$$

$$- \theta\delta_{ij} \cdot \int_{\Omega} \partial_k(\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_k v \, dX$$

$$- \theta\delta_{ij} \cdot \int_{\partial\Omega} (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_{\nu}v \, d\sigma.$$

Integrating by parts one more time and using (3.6), identity (3.14) further implies

$$(3.15) \qquad \int_{\Omega} (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot (\partial_i\partial_j + \theta\delta_{ij}\Delta)v \, dX - (1 + 2\theta + n\theta^2)\int_{\Omega} (\Delta^2 u)v \, dX$$

$$= -\int_{\partial\Omega} \nu_j \cdot \partial_i(\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot v \, d\sigma$$

$$+ \int_{\partial\Omega} \nu_i \cdot (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_j v \, d\sigma$$

$$- \theta\delta_{ij} \cdot \int_{\partial\Omega} \nu_k \cdot \partial_k(\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot v \, d\sigma$$

$$- \theta\delta_{ij} \cdot \int_{\partial\Omega} (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_{\nu}v \, d\sigma.$$

Using that $\partial_j = \nu_r\nu_r\partial_j = \nu_r\partial_{\tau_{rj}} + \nu_j\partial_{\nu}$ in the second term in the right-hand side of (3.15) allows us to express this as

$$(3.16) \qquad \int_{\partial\Omega} \nu_i \cdot (\partial_i\partial_j + \theta\delta_{ij}\Delta)u \cdot \partial_j v \, d\sigma = \int_{\partial\Omega} (\partial_i\partial_j + \theta\delta_{ij}\Delta)u\nu_i\nu_j\partial_{\nu}v \, d\sigma$$

$$+ \int_{\partial\Omega} \partial_{\tau_{jr}}[\nu_i\nu_r(\partial_i\partial_j + \theta\delta_{ij}\Delta)u]v \, d\sigma.$$

In turn, this and (3.15) give

(3.17)
$$\int_\Omega (\partial_i \partial_j + \theta \delta_{ij} \Delta) u \cdot (\partial_i \partial_j + \theta \delta_{ij} \Delta) v \, dX$$

$$= (1 + 2\theta + n\theta^2) \int_\Omega (\Delta^2 u)(X) v(X) \, dX$$

$$+ \int_{\partial\Omega} \mathrm{I}(u) \cdot v \, d\sigma + \int_{\partial\Omega} \mathrm{II}(u) \cdot \partial_\nu v \, d\sigma,$$

where we have set

(3.18)
$$\mathrm{I}(u) := \partial_{\tau_{jr}}[v_i v_r (\partial_i \partial_j + \theta \delta_{ij} \Delta) u] - v_j \partial_i (\partial_i \partial_j + \theta \delta_{ij} \Delta) u$$
$$- \theta \delta_{ij} v_k \partial_k (\partial_i \partial_j + \theta \delta_{ij} \Delta) u,$$

and

(3.19)
$$\mathrm{II}(u) := \theta \delta_{ij} (\partial_i \partial_j + \theta \delta_{ij} \Delta) u + v_i v_j (\partial_i \partial_j + \theta \delta_{ij} \Delta) u.$$

Next, observe that

(3.20)
$$\partial_{\tau_{jr}}[v_i v_r \theta \delta_{ij} \Delta u] = \theta \partial_{\tau_{jr}}[v_j v_r \Delta u] = 0$$

by symmetry considerations, and that

(3.21)
$$\partial_{\tau_{jr}}[v_i v_r \partial_i \partial_j u] = \frac{1}{2} \{ \partial_{\tau_{jr}}[v_i v_r \partial_i \partial_j u] + \partial_{\tau_{rj}}[v_i v_j \partial_i \partial_r u] \}$$

$$= -\frac{1}{2} \partial_{\tau_{jr}}[v_i \partial_{\tau_{jr}} \partial_i u],$$

where the first identity in formula (3.21) follows from rewriting the expression $\partial_{\tau_{jr}}[v_i v_r \partial_i \partial_j u]$ as $\partial_{\tau_{rj}}[v_i v_j \partial_i \partial_r u]$ and the second one uses the definition of $\partial_{\tau_{jr}}$. Based on (3.18) and (3.20)–(3.21), straightforward algebraic manipulations yield

(3.22)
$$\mathrm{I}(u) = -(1 + 2\theta + n\theta^2) \partial_\nu \Delta u - \frac{1}{2} \partial_{\tau_{ij}}[v_k \partial_{\tau_{ij}} \partial_k u]$$

$$= -(1 + 2\theta + n\theta^2) N_\theta(u).$$

Also, a simple inspection of (3.19) reveals that

(3.23)
$$\mathrm{II}(u) = (2\theta + n\theta^2) \Delta u + \partial_\nu^2 u = (1 + 2\theta + n\theta^2) M_\theta(u).$$

At this stage, (3.13) follows form (3.17) and (3.22)–(3.23).     □

In our next proposition we identify the formula for the conormal derivative associated with the writing of the bi-Laplacian as in (3.4) for the tensor coefficient given in (3.1)–(3.2).

PROPOSITION 3.2.   *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with outward unit normal $v = (v_j)_{1 \le j \le n}$ and surface measure $\sigma$. Pick $\theta \in \mathbb{R}$ and recall the operators $N_\theta$ and $M_\theta$ from (3.9), corresponding to this setting. Then for any reasonably well-behaved biharmonic function $u$ in $\Omega$ there holds:*

$$(3.24) \qquad \text{coefficient tensor } A_\theta \text{ as in } (3.1)-(3.2) \Rightarrow \partial_v^{A_\theta} u := \{(\partial_v^{A_\theta} u)_r\}_{0 \le r \le n}$$

$$\text{where } (\partial_v^{A_\theta} u)_0 = -N_\theta(u) \text{ and } (\partial_v^{A_\theta} u)_r = v_r M_\theta(u) \text{ for } 1 \le r \le n.$$

PROOF. Let $u$ be as in the statement of the proposition and pick an arbitrary function $v \in \mathscr{C}^\infty(\overline{\Omega})$. Based on successive integrations by parts we may compute

$$(3.25) \qquad \int_{\partial\Omega} \langle \partial_v^{A_\theta} u, (\mathrm{Tr}\, v, \mathrm{Tr}(\nabla v)) \rangle \, d\sigma$$

$$= \mathcal{B}_\theta(u, v) = \int_{\partial\Omega} [M_\theta(u) \partial_v v - N_\theta(u) v] \, d\sigma$$

$$= \int_{\partial\Omega} \langle (-N_\theta(u), v_1 M_\theta(u), \dots, v_n M_\theta(u)), (\mathrm{Tr}\, v, \mathrm{Tr}(\nabla v)) \rangle \, d\sigma.$$

Therefore, (3.24) follows.                                                                       $\square$

It is useful to record the explicit expressions of the components of the conormal. Indeed, making use of the first formula in (3.10) and the second formula in (3.9), it follows that the components of $\partial_v^{A_\theta} u$ described in (3.24) are (using the usual summation convention over repeated indices):

$$(3.26) \qquad (\partial_v^{A_\theta} u)_0 = -\partial_v(\Delta u) - c_n(\theta) \cdot \partial_{\tau_{ij}}(v_\ell v_i \partial_j \partial_\ell u)$$

and

$$(3.27) \quad (\partial_v^{A_\theta} u)_r = (1 - c_n(\theta)) \cdot v_r \Delta u + c_n(\theta) \cdot v_r v_j v_\ell \partial_j \partial_\ell u \quad \text{for each } r \in \{1, \dots, n\},$$

where

$$(3.28) \qquad\qquad\qquad c_n(\theta) := \frac{1}{1 + 2\theta + n\theta^2},$$

again, with the understanding that all derivatives in the right-hand sides are restricted to the boundary.

## 4. BOUNDARY LAYER POTENTIALS

Let $E$ be the canonical fundamental solution for $\Delta^2$ in $\mathbb{R}^n$ given at each $X \in \mathbb{R}^n \backslash \{0\}$ by

$$(4.1) \qquad E(X) := \begin{cases} \dfrac{1}{2(n-4)(n-2)\omega_{n-1}} |X|^{4-n} & \text{if } n \in \mathbb{N}\setminus\{1,2,4\}, \\[2ex] -\dfrac{1}{4\omega_3} \log|X| & \text{if } n = 4, \\[2ex] -\dfrac{1}{8\pi} |X|^2 (1 - \log|X|) & \text{if } n = 2, \end{cases}$$

where $\omega_{n-1}$ denotes the surface area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. In particular,

$$(4.2) \qquad (\Delta E)(X) := \begin{cases} \dfrac{1}{(2-n)\omega_{n-1}} |X|^{2-n} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \log|X| & \text{if } n = 2. \end{cases}$$

In relation to the latter, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with outward unit normal $v$ and surface measure $\sigma$, recall that the harmonic double and single layer operators are, respectively, given by

$$(4.3) \qquad \mathcal{D}_\Delta f(X) := \int_{\partial\Omega} \partial_{v(Y)}[(\Delta E)(X - Y)] f(Y)\, d\sigma(Y), \quad X \in \mathbb{R}^n\setminus\partial\Omega,$$

and

$$(4.4) \qquad \mathcal{S}_\Delta f(X) := \int_{\partial\Omega} (\Delta E)(X - Y) f(Y)\, d\sigma(Y), \quad X \in \mathbb{R}^n\setminus\partial\Omega.$$

Also, the boundary version of $\mathcal{D}_\Delta$ is

$$(4.5) \qquad \mathcal{K}_\Delta f(X) := \lim_{\varepsilon \to 0^+} \int_{\partial\Omega\setminus B(X,\varepsilon)} \partial_{v(Y)}[(\Delta E)(X - Y)] f(Y)\, d\sigma(Y), \quad X \in \partial\Omega.$$

Based on definitions, if $f \in L^p(\partial\Omega)$ for some $p \in (1, \infty)$ and $\ell \in \{1, \ldots, n\}$, then the following identities may be readily verified for each $X \in \mathbb{R}^n\setminus\partial\Omega$:

$$(4.6) \qquad \partial_\ell(\mathcal{D}_\Delta f)(X) = -\sum_{i=1}^{n} \partial_i \mathcal{S}_\Delta(\partial_{\tau_{i\ell}} f)(X),$$

$$(4.7) \qquad \partial_\ell(\mathcal{S}_\Delta f)(X) = -(\mathcal{S}_\Delta(\partial_{\tau_{\ell i}}(v_i f)))(X) - (\mathcal{D}_\Delta(v_\ell f))(X).$$

We are now prepared to make the following basic definition.

DEFINITION 4.1.  *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal $v = (v_j)_{1 \leq j \leq n}$ and surface measure $\sigma$. Also, fix $\theta \in \mathbb{R}$ and recall $c_n(\theta)$ from (3.28). In this context, define the action of the biharmonic double multi-layer $\mathcal{D}_\theta$ on each Whitney array $\dot{f} = (f_0, f_1, \ldots, f_n)$, by*

$$(4.8) \qquad \dot{\mathcal{D}}_\theta \dot{f}(X) := -\frac{1}{2} \sum_{|\alpha|=|\beta|=2} \alpha! A_{\beta\alpha}(\theta) \sum_{\substack{i,j \text{ so that} \\ e_i+e_j=\alpha}} \int_{\partial\Omega} v_j(Y)$$

$$\times \{(\partial^\beta E)(X-Y)f_i(Y) + (\partial^{\beta+e_i}E)(X-Y)f_0(Y)\} \, d\sigma(Y)$$

for $X \in \mathbb{R}^n \backslash \partial\Omega$. In addition, denote by $\dot{K}_\theta$ the boundary biharmonic double multi-layer, whose action on an arbitrary Whitney array $\dot{f} = (f_0, f_1, \ldots, f_n)$ such that $\dot{f} \in \dot{L}^p_{1,0}(\partial\Omega)$ is the Whitney array

$$(4.9) \qquad \dot{K}_\theta \dot{f} := ((\dot{K}_\theta \dot{f})_0, (\dot{K}_\theta \dot{f})_1, \ldots, (\dot{K}_\theta \dot{f})_n)$$

where, for $\sigma$ a.e. $X \in \partial\Omega$,

$$(4.10) \qquad (\dot{K}_\theta \dot{f})_0(X) := \lim_{\varepsilon\to 0^+} \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \partial_{v(Y)}[(\Delta E)(X-Y)]f_0(Y)\, d\sigma(Y)$$

$$- \int_{\partial\Omega} (\Delta E)(X-Y) \sum_{k=1}^n v_k(Y)f_k(Y)\, d\sigma(Y)$$

$$+ c_n(\theta) \cdot \int_{\partial\Omega} \sum_{j,k=1}^n \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)]f_j(Y)\, d\sigma(Y),$$

while for each $\ell \in \{1, \ldots, n\}$,

$$(4.11) \quad (\dot{K}_\theta \dot{f})_\ell(X) := \lim_{\varepsilon\to 0^+} \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \left\{ \partial_{v(Y)}[(\Delta E)(X-Y)]f_\ell(Y) \right.$$

$$+ \sum_{i=1}^n \partial_{\tau_{i\ell}(Y)}[(\Delta E)(X-Y)]f_i(Y)$$

$$+ c_n(\theta) \times$$

$$\left. \times \sum_{j,k=1}^n \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X-Y)]f_j(Y) \right\} d\sigma(Y).$$

Finally, denote by $\dot{K}_\theta^*$ the adjoint of the operator $\dot{K}_\theta$ considered above.

It should be noted that the operators introduced in the above definition are constructed according to the general recipes from Definitions 4.2–4.3 in [15], implemented for the writing of $\Delta^2$ as in (3.4), corresponding to the tensor coefficient $A_\theta = (A_{\alpha\beta}(\theta))_{|\alpha|=|\beta|=2}$ from (3.2) (and with $E$ as in (4.1)). Indeed, formula (4.164)

from Theorem 4.6 in [15] shows that, in the context of the above definition, whenever $1 < p < \infty$ we have (with $I$ denoting the identity operator)

$$(4.12) \qquad (\dot{\mathcal{D}}_\theta \dot{f}\rfloor_{\partial\Omega}, (\nabla\dot{\mathcal{D}}_\theta \dot{f})\rfloor_{\partial\Omega}) = \left(\frac{1}{2}I + \dot{K}_\theta\right)\dot{f}, \quad \forall \dot{f} \in \dot{L}^p_{1,0}(\partial\Omega).$$

This may be used to identify a concrete formula for $\dot{K}_\theta$, and the fact that formulas (4.9)–(4.11) are natural may be seen by combining (4.12) with (4.25) and (4.39) (proved later). For further reference let us also note here the estimates

$$(4.13) \qquad \|\mathcal{N}(\dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\nabla\dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)}$$
$$\leq C\|\dot{f}\|_{\dot{L}^p_{1,0}(\partial\Omega)} \quad \text{for each } \dot{f} \in \dot{L}^p_{1,0}(\partial\Omega),$$

and

$$(4.14) \qquad \|\mathcal{N}(\nabla^2 \dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)} \leq C\|\dot{f}\|_{\dot{L}^p_{1,1}(\partial\Omega)} \quad \forall \dot{f} \in \dot{L}^p_{1,1}(\partial\Omega),$$

which are particular cases of Theorem 4.2 in [15]. In concert, (4.12)–(4.14) also show that for each $p \in (1, \infty)$ the operator

$$(4.15) \qquad \dot{K}_\theta \text{ is well-defined and bounded on } \dot{L}^p_{1,0}(\partial\Omega) \text{ and on } \dot{L}^p_{1,1}(\partial\Omega).$$

In fact, based on this and interpolation (cf. [15]), we also have that

$$(4.16) \qquad \dot{K}_\theta \text{ is well-defined and bounded on } \dot{B}^{p,q}_{1,s}(\partial\Omega)$$

whenever $1 < p, q < \infty$, and $s \in (0, 1)$.

In addition to the operator $\dot{\mathcal{D}}_\theta$ defined above, with $N_\theta$, $M_\theta$ as in (3.9), consider the integral operator acting on each pair $(F, g) \in L^p_1(\partial\Omega) \oplus L^p(\partial\Omega)$, where the index $p \in (1, \infty)$, according to the formula

$$(4.17) \qquad \tilde{\mathcal{D}}_\theta(F, g)(X) := \int_{\partial\Omega} \{M_\theta[E(X - \cdot)](Y)g(Y)$$
$$+ N_\theta[E(X - \cdot)](Y)F(Y)\}\, d\sigma(Y),$$

at each $X \in \mathbb{R}^n\backslash\partial\Omega$. The goal now is to elaborate on the relationship between the operators $\dot{\mathcal{D}}_\theta$ and $\tilde{\mathcal{D}}_\theta$ just introduced. In this vein, it helps to recall the isomorphism $\Psi$ described in Proposition 2.2.

PROPOSITION 4.2. *Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $\theta \in \mathbb{R}$. Then*

$$(4.18) \qquad \dot{\mathcal{D}}_\theta = \tilde{\mathcal{D}}_\theta \circ \Psi \quad \text{in } \mathbb{R}^n\backslash\partial\Omega,$$

*when both operators are acting on arbitrary Whitney arrays from $\dot{L}^p_{1,0}(\partial\Omega)$ with $p \in (1, \infty)$.*

PROOF. Thanks to Definition 4.1 and a density argument, it suffices to show that the two operators from (4.18) act identically on Whitney arrays of the form

$$(4.19) \qquad \dot{f} = (\operatorname{Tr} v, \operatorname{Tr}(\nabla v)), \quad v \in \mathscr{C}^\infty(\overline{\Omega}).$$

Assume that this is the case, i.e.,

$$(4.20) \qquad \dot{f} = (f_0, f_1, \ldots, f_n) = (v|_{\partial\Omega}, (\partial_1 v)|_{\partial\Omega}, \ldots, (\partial_n v)|_{\partial\Omega}),$$

and introduce

$$(4.21) \qquad F := f_0 \in L_1^p(\partial\Omega) \quad \text{and} \quad g := -\sum_{j=1}^n v_j f_j \in L^p(\partial\Omega),$$

where $v = (v_j)_{1 \leq j \leq n}$ denotes the outward unit normal to $\partial\Omega$. Hence,

$$(4.22) \qquad \Psi(\dot{f}) = (F, g) = (v|_{\partial\Omega}, -\partial_v v).$$

Then, based on (4.17), Proposition 3.1, Green's representation formula for the bi-Laplacian (cf. [15, Proposition 4.2] for a general result of this nature), and (4.19), for every point $X \in \mathbb{R}^n \setminus \partial\Omega$ we may write

$$(4.23) \quad v(X) - \tilde{\mathcal{D}}_\theta(F, g)(X) = v(X) - \int_{\partial\Omega} \{-M_\theta[E(X - \cdot)](Y)\partial_v v(Y)$$
$$+ N_\theta[E(X - \cdot)](Y)v(Y)\}\, d\sigma(Y)$$
$$= \mathcal{B}_\theta(E(X - \cdot), v) = v(X) - \dot{\mathcal{D}}_\theta((\operatorname{Tr} v, \operatorname{Tr}(\nabla v)))(X)$$
$$= v(X) - (\dot{\mathcal{D}}_\theta \dot{f})(X).$$

As such, in light of (4.22) we conclude that (4.18) holds.                    □

Next we take a closer look at the action of the biharmonic double multi-layer, originally introduced in Definition 4.1, on Whitney arrays.

PROPOSITION 4.3. *Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ with outward unit normal $v = (v_j)_{1 \leq j \leq n}$ and surface measure $\sigma$. Also, fix a number $\theta \in \mathbb{R}$ and set*

$$(4.24) \qquad c_n(\theta) := \frac{1}{1 + 2\theta + n\theta^2},$$

*Then the action of the double multi-layer $\dot{\mathcal{D}}_\theta$ introduced in Definition 4.1 on a Whitney array $\dot{f} = (f_0, f_1, \ldots, f_n)$ from $\dot{L}_{1,0}^p(\partial\Omega)$, with $1 < p < \infty$, may be*

*described as*

$$(4.25) \qquad (\dot{\mathcal{D}}_\theta \dot{f})(X) = \int_{\partial\Omega} \partial_{\nu(Y)}[(\Delta E)(X - Y)] f_0(Y) \, d\sigma(Y)$$

$$- \int_{\partial\Omega} (\Delta E)(X - Y) \sum_{k=1}^{n} \nu_k(Y) f_k(Y) \, d\sigma(Y)$$

$$+ c_n(\theta) \cdot \int_{\partial\Omega} \sum_{j,k=1}^{n} \partial_{\tau_{kj}(Y)}[(\partial_k E)(X - Y)] f_j(Y) \, d\sigma(Y),$$

*for each* $X \in \mathbb{R}^n \backslash \partial\Omega$.

In particular, using the notation introduced in (4.3)–(4.4),

$$(4.26) \qquad (\dot{\mathcal{D}}_\theta \dot{f})(X) = (\mathcal{D}_\Delta f_0)(X) - \mathcal{S}_\Delta \left( \sum_{k=1}^{n} \nu_k f_k \right)(X)$$

$$+ c_n(\theta) \cdot \int_{\partial\Omega} \sum_{j,k=1}^{n} \partial_{\tau_{kj}(Y)}[(\partial_k E)(X - Y)] f_j(Y) \, d\sigma(Y),$$

*for each* $X \in \mathbb{R}^n \backslash \partial\Omega$.

PROOF. For every $\dot{f} = (f_0, f_1, \dots, f_n) \in \dot{L}_{1,0}^p(\partial\Omega)$, based on (4.18), (2.22)–(2.23), and (4.68), at every $X \in \mathbb{R}^n \backslash \partial\Omega$ we may write (using the summation convention over repeated indices)

$$(4.27) \quad (\dot{\mathcal{D}}_\theta \dot{f})(X) = \tilde{\mathcal{D}}_\theta(f_0, -\nu_i f_i)(X)$$

$$= \int_{\partial\Omega} \{ -(\Delta E)(X - Y)$$

$$+ c_n(\theta) \cdot \nu_j(Y) \partial_{\tau_{kj}(Y)}[(\partial_k E)(X - Y)] \} \nu_i(Y) f_i(Y) \, d\sigma(Y)$$

$$+ \int_{\partial\Omega} \{ \partial_{\nu(Y)}[(\Delta E)(X - Y)]$$

$$- c_n(\theta) \cdot \partial_{\tau_{ij}(Y)}(\nu_i(Y) \partial_{\tau_{kj}(Y)}[(\partial_k E)(X - Y)]) \} f_0(Y) \, d\sigma(Y)$$

$$= \int_{\partial\Omega} \{ \partial_{\nu(Y)}[(\Delta E)(X - Y)] f_0(Y) - (\Delta E)(X - Y) \nu_i(Y) f_i(Y)$$

$$+ c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_k E)(X - Y)] \times$$

$$\times (\nu_i \partial_{\tau_{ij}} f_0 + \nu_j \nu_i f_i)(Y) \} \, d\sigma(Y),$$

thanks to (3.10)–(3.11) and an integration by parts on the boundary. Now, the claim made in (4.25) follows from (4.27) after observing that

$$(4.28) \qquad \nu_i \partial_{\tau_{ij}} f_0 + \nu_j \nu_i f_i = \nu_i(\nu_i f_j - \nu_j f_i) + \nu_j \nu_i f_i = f_j,$$

by the compatibility conditions satisfied by the components of the Whitney array $\dot{f}$. $\qquad\square$

The next order of business is to study the mapping properties for the conormal derivative of the biharmonic double multi-layer. Our first result in this regard is the following theorem (we shall return to this topic later, after completing a necessary detour).

**THEOREM 4.4.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $\theta \in \mathbb{R}$. Also, assume that $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$. Recall $\dot{\mathcal{D}}_\theta$ introduced in Definition 4.1 and the conormal $\partial_\nu^{A_\theta}$ from Proposition 3.2. Then the operator*

$$(4.29) \qquad \partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta : \dot{L}_{1,1}^p(\partial\Omega) \to (\dot{L}_{1,0}^{p'}(\partial\Omega))^*$$

*is well-defined, linear and bounded. Moreover, this operator further extends as a linear and bounded mapping in the context*

$$(4.30) \qquad \partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta : \dot{L}_{1,0}^p(\partial\Omega) \to (\dot{L}_{1,1}^{p'}(\partial\Omega))^*.$$

**PROOF.** For each $\dot{f} \in \dot{L}_{1,1}^p(\partial\Omega)$ we know from (4.13)–(4.14) that the double multi-layer satisfies the nontangential maximal function estimate

$$(4.31) \quad \|\mathcal{N}(\dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\nabla \dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\nabla^2 \dot{\mathcal{D}}_\theta \dot{f})\|_{L^p(\partial\Omega)} \le C\|\dot{f}\|_{\dot{L}_{1,1}^p(\partial\Omega)},$$

for some finite constant $C > 0$ independent of $\dot{f}$. Moreover, one can show that

$$(4.32) \qquad \partial^\gamma \dot{\mathcal{D}}_\theta \dot{f}\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \ \forall \dot{f} \in \dot{L}_{1,1}^p(\partial\Omega)$$
$$\text{whenever } \gamma \in \mathbb{N}_0^n \text{ satisfies } |\gamma| \le 2.$$

However, the conormal entails up to three derivatives on $\dot{\mathcal{D}}_\theta$. Indeed, as seen from (3.26)–(3.28), the components of $\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f}$ are given by (here and elsewhere the usual summation convention over repeated indices is used)

$$(4.33) \qquad (\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f})_0 = -\partial_\nu(\Delta \dot{\mathcal{D}}_\theta \dot{f}) - c_n(\theta) \cdot \partial_{\tau_{ij}}(\nu_\ell \nu_i \partial_j \partial_\ell \dot{\mathcal{D}}_\theta \dot{f}),$$

and, for $1 \le r \le n$,

$$(4.34) \qquad (\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f})_r = (1 - c_n(\theta)) \cdot \nu_r \Delta \dot{\mathcal{D}}_\theta \dot{f} + c_n(\theta) \cdot \nu_r \nu_j \nu_\ell \partial_j \partial_\ell \dot{\mathcal{D}}_\theta \dot{f}$$

with the understanding that all derivatives in the right-hand sides are restricted to the boundary and that $c_n(\theta)$ is as in (3.28). Note that, thanks to (4.31) and (4.32), the map

$$(4.35) \qquad \dot{L}_{1,1}^p(\partial\Omega) \ni \dot{f} \mapsto \partial_j \partial_\ell \dot{\mathcal{D}}_\theta \dot{f}\big|_{\partial\Omega} \in L^p(\partial\Omega)$$

is well-defined, linear and bounded, for every $\ell, j \in \{1, \ldots, n\}$. As such, the mapping

$$(4.36) \qquad \dot{L}^p_{1,1}(\partial\Omega) \ni \dot{f} \mapsto \partial_{\tau_{ij}}(\nu_\ell \nu_i \partial_j \partial_\ell \dot{\mathcal{D}}_\theta \dot{f}) \in L^p_{-1}(\partial\Omega)$$

is also well-defined, linear and bounded, for every $i, j, \ell \in \{1, \ldots, n\}$.

We propose to take a closer look at the structure of the derivatives of the biharmonic double multi-layer operator. In a first stage, fix an arbitrary Whitney array $\dot{f} = (f_0, f_1, \ldots, f_n) \in \dot{L}^p_{1,0}(\partial\Omega)$ then for every $\ell \in \{1, \ldots, n\}$ and $X \in \mathbb{R}^n \backslash \partial\Omega$ compute

$$(4.37) \qquad \partial_\ell(\dot{\mathcal{D}}_\theta \dot{f})(X) = \partial_\ell(\mathcal{D}_\Delta f_0)(X) - \partial_\ell \mathcal{S}_\Delta(\nu_i f_i)(X)$$
$$+ c_n(\theta) \cdot \int_{\partial\Omega} \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X - Y)] f_j(Y) \, d\sigma(Y),$$

where $c_n(\theta)$ is as in (4.24). Upon observing that, for every $X \in \mathbb{R}^n \backslash \partial\Omega$, identity (4.6) and the compatibility conditions satisfied by the components of the Whitney array $\dot{f}$ allow us to write

$$(4.38) \qquad \partial_\ell(\mathcal{D}_\Delta f_0)(X) - \partial_\ell \mathcal{S}_\Delta(\nu_i f_i)(X)$$
$$= -\partial_i \mathcal{S}_\Delta(\partial_{\tau_{i\ell}} f_0)(X) - \partial_\ell \mathcal{S}_\Delta(\nu_i f_i)(X)$$
$$= -\partial_i \mathcal{S}_\Delta(\nu_i f_\ell)(X) + \partial_i \mathcal{S}_\Delta(\nu_\ell f_i)(X) - \partial_\ell \mathcal{S}_\Delta(\nu_i f_i)(X)$$
$$= (\mathcal{D}_\Delta f_\ell)(X) + \int_{\partial\Omega} \partial_{\tau_{i\ell}(Y)}[(\Delta E)(X - Y)] f_i(Y) \, d\sigma(Y)$$
$$= (\mathcal{D}_\Delta f_\ell)(X) + \mathcal{S}_\Delta(\partial_{\tau_{\ell i}} f_i)(X),$$

we deduce from (4.37) that, at each point $X \in \mathbb{R}^n \backslash \partial\Omega$,

$$(4.39) \quad \partial_\ell(\dot{\mathcal{D}}_\theta \dot{f})(X) = (\mathcal{D}_\Delta f_\ell)(X) + \mathcal{S}_\Delta(\partial_{\tau_{\ell i}} f_i)(X)$$
$$+ c_n(\theta) \cdot \int_{\partial\Omega} \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X - Y)] f_j(Y) \, d\sigma(Y)$$
$$= \int_{\partial\Omega} \{\partial_{\nu(Y)}[(\Delta E)(X - Y)] f_\ell(Y) + \partial_{\tau_{i\ell}(Y)}[(\Delta E)(X - Y)] f_i(Y)$$
$$+ c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X - Y)] f_j(Y)\} \, d\sigma(Y).$$

In the case when the array $\dot{f} = (f_0, f_1, \ldots, f_n)$ actually belongs to the Whitney-Sobolev space $\dot{L}^p_{1,1}(\partial\Omega)$, we may integrate by parts on the boundary in (4.37) in order to write, for every $\ell \in \{1, \ldots, n\}$,

$$(4.40) \qquad \partial_\ell(\dot{\mathcal{D}}_\theta \dot{f})(X) = (\mathcal{D}_\Delta f_\ell)(X)$$
$$+ \int_{\partial\Omega} \{(\Delta E)(X - Y)(\partial_{\tau_{\ell i}} f_i)(Y)$$
$$+ c_n(\theta) \cdot (\partial_\ell \partial_k E)(X - Y)(\partial_{\tau_{ik}} f_i)(Y)\} \, d\sigma(Y),$$

for each $X \in \mathbb{R}^n \backslash \partial\Omega$. In this scenario, we may take one extra derivative while still retaining control of the finiteness of the $L^p$-norm of the nontangential maximal function. Concretely, for each $j, \ell \in \{1, \ldots, n\}$ we obtain (with the help of (4.6))

$$(4.41) \quad \partial_j \partial_\ell (\dot{\mathcal{D}}_\theta \dot{f})(X) = -\partial_i \mathcal{S}_\Delta (\partial_{\tau_{ij}} f_\ell)(X)$$
$$+ \int_{\partial\Omega} \{ (\partial_j \Delta E)(X - Y)(\partial_{\tau_{\ell i}} f_i)(Y)$$
$$+ c_n(\theta) \cdot (\partial_j \partial_\ell \partial_k E)(X - Y)(\partial_{\tau_{ik}} f_i)(Y) \} \, d\sigma(Y),$$

at each $X \in \mathbb{R}^n \backslash \partial\Omega$, whenever $\dot{f} = (f_0, f_1, \ldots, f_n)$ belongs to the Whitney-Sobolev space $\dot{L}_{1,1}^p(\partial\Omega)$. Concisely, for every $\dot{f} = (f_0, f_1, \ldots, f_n) \in \dot{L}_{1,1}^p(\partial\Omega)$ we have

$$(4.42) \quad \partial_j \partial_\ell \dot{\mathcal{D}}_\theta \dot{f} = -\partial_i \mathcal{S}_\Delta (\partial_{\tau_{ij}} f_\ell) + \partial_j \mathcal{S}_\Delta (\partial_{\tau_{\ell i}} f_i)$$
$$+ c_n(\theta) \cdot \int_{\partial\Omega} (\partial_j \partial_\ell \partial_k E)(\cdot - Y)(\partial_{\tau_{ik}} f_i)(Y) \, d\sigma(Y) \quad \text{in } \mathbb{R}^n \backslash \partial\Omega.$$

In particular, summing up over $j = \ell$ and using (4.24) yields

$$(4.43) \qquad \Delta \dot{\mathcal{D}}_\theta \dot{f} = \frac{1}{1 + 2\theta + n\theta^2} \partial_k \mathcal{S}_\Delta (\partial_{\tau_{ik}} f_i) \quad \text{in } \mathbb{R}^n \backslash \partial\Omega,$$
$$\forall \dot{f} = (f_0, f_1, \ldots, f_n) \in \dot{L}_{1,1}^p(\partial\Omega),$$

and, further,

$$(4.44) (\partial_\nu \Delta \dot{\mathcal{D}}_\theta \dot{f})(X) = \frac{1}{1 + 2\theta + n\theta^2} \times$$
$$\times \partial_{\tau_{jk}(X)} \left( \lim_{\varepsilon \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X - Y| > \varepsilon}} (\partial_j \Delta E)(X - Y)(\partial_{\tau_{ik}} f_i)(Y) \, d\sigma(Y) \right)$$

in $\mathbb{R}^n \backslash \partial\Omega$, for every $\dot{f} = (f_0, f_1, \ldots, f_n) \in \dot{L}_{1,1}^p(\partial\Omega)$. Consequently,

$$(4.45) \qquad \dot{L}_{1,1}^p(\partial\Omega) \ni \dot{f} \mapsto \partial_\nu \Delta \dot{\mathcal{D}}_\theta \dot{f} \in L_{-1}^p(\partial\Omega)$$

is a well-defined, linear and bounded mapping.

In summary, from (4.33)–(4.34) and (4.35), (4.36), (4.45) we deduce that the mapping

$$(4.46) \quad \dot{L}_{1,1}^p(\partial\Omega) \ni \dot{f} \mapsto ((\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f})_0, (\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f})_{1 \leq r \leq n}) \in L_{-1}^p(\partial\Omega) \oplus [L^p(\partial\Omega)]^n$$

is well-defined, linear and bounded. Furthermore, it is clear from (4.34) that

$$(4.47) \qquad (\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta \dot{f})_{1 \leq r \leq n} = (-\nu_1 f, \ldots, -\nu_n f) \quad \text{for each } \dot{f} \in \dot{L}_{1,1}^p(\partial\Omega),$$

where

$$(4.48) \quad f := -\frac{2\theta + n\theta^2}{1 + 2\theta + n\theta^2}\Delta\dot{\mathcal{D}}_\theta\dot{f} - \frac{1}{1 + 2\theta + n\theta^2}\nu_j\nu_\ell\partial_j\partial_\ell\dot{\mathcal{D}}_\theta\dot{f} \in L^p(\partial\Omega).$$

At this stage, the fact that the operator $\partial_\nu^{A_\theta}\dot{\mathcal{D}}_\theta$ is well-defined, linear and bounded in the context of (4.29) follows from (4.46)–(4.48) and Proposition 2.3. Lastly, that this operator further extends as a linear and bounded mapping in the context of (4.30), follows from what we have proved so far, the fact that $\partial_\nu^{A_\theta}\dot{\mathcal{D}}_\theta$ coincides with its own transpose (which is the case for any conormal derivative of any double multi-layer associated with a symmetric higher-order elliptic operator; cf. Proposition 5.17 in [15]), and duality.                                                     □

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $p \in (1, \infty)$. The bi-Laplacian single multi-layer operator $\dot{\mathcal{S}}$ acts on an arbitrary functional $\Lambda \in (\dot{L}_{1,0}^p(\partial\Omega))^*$ according to the formula (with $E$ as in (4.1))

$$(4.49) \quad (\dot{\mathcal{S}}\Lambda)(X) := \langle(E(X - \cdot)|_{\partial\Omega}, -(\nabla E)(X - \cdot)|_{\partial\Omega}), \Lambda\rangle \quad \text{for each } X \in \mathbb{R}^n\backslash\partial\Omega,$$

where the expression in round parentheses is regarded as a Whitney array in $\dot{L}_{1,0}^p(\partial\Omega)$. In a similar fashion, let us also consider the boundary version of (4.49) defined, for each $\Lambda \in (\dot{L}_{1,0}^p(\partial\Omega))^*$ by

$$(4.50) \quad (\dot{S}\Lambda)(X) := \langle(E(X - \cdot)|_{\partial\Omega}, -(\nabla E)(X - \cdot)|_{\partial\Omega}), \Lambda\rangle, \quad X \in \partial\Omega.$$

**PROPOSITION 4.5.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and assume that $p \in (1, \infty)$. Fix $\theta \in \mathbb{R}$ and recall the conormal $\partial_\nu^{A_\theta}$ from Proposition 3.2. Also, let $\dot{S}$ stand for the bi-Laplacian single multi-layer operator from (4.49). Then*

$$(4.51) \quad \partial_\nu^{A_\theta}\dot{S} : (\dot{L}_{1,0}^p(\partial\Omega))^* \to (\dot{L}_{1,0}^p(\partial\Omega))^*$$

*is a well-defined, linear and bounded operator.*

*As a corollary, for each $p \in (1, \infty)$,*

$$(4.52) \quad \partial_\nu^{A_\theta}\dot{S} = -\frac{1}{2}I + \dot{K}_\theta^* \quad \text{as operators on } (\dot{L}_{1,0}^p(\partial\Omega))^*.$$

**PROOF.** As usual, let $p'$ denote the Hölder conjugate exponent of $p$. As far as the claim pertaining to the operator in (4.51) is concerned, the crux of the matter is establishing the estimate

$$(4.53) \quad \|\partial_\nu\Delta\dot{S}\Lambda\|_{L_{-1}^{p'}(\partial\Omega)} \leq C\|\Lambda\|_{(\dot{L}_{1,0}^p(\partial\Omega))^*}$$

for some finite constant $C > 0$ independent of $\Lambda \in (\dot{L}_{1,0}^p(\partial\Omega))^*$. Once this has been done, the same type of reasoning as in Theorem 4.4 (which also uses the nontangential maximal function estimates for generic single multi-layers proved

in Proposition 5.1 in [15]) may then be used to complete the proof of the boundedness of the operator in (4.51).

Given an arbitrary $\Lambda \in (\dot{L}_{1,0}^{p}(\partial\Omega))^{*}$, a reasoning based on the Hahn-Banach theorem leads to the conclusion that there exists an $(n+1)$-tuple of functions $(g_0, g_1, \ldots, g_n) \in [L^{p'}(\partial\Omega)]^{n+1}$ with the property that

$$(4.54) \qquad \sum_{j=0}^{n} \|g_j\|_{L^{p'}(\partial\Omega)} \le \|\Lambda\|_{(\dot{L}_{1,0}^{p}(\partial\Omega))^{*}},$$

and such that, for each $X \in \mathbb{R}^n \backslash \partial\Omega$,

$$(4.55) \qquad \dot{\mathcal{S}}\Lambda(X) = \int_{\partial\Omega} E(X - Y)g_0(Y)\, d\sigma(Y)$$

$$- \sum_{j=1}^{n} \int_{\partial\Omega} (\partial_j E)(X - Y)g_j(Y)\, d\sigma(Y).$$

As such, we may write

$$(4.56) \qquad \Delta\dot{\mathcal{S}}\Lambda = \mathcal{S}_\Delta g_0 - \sum_{j=1}^{n} \partial_j \mathcal{S}_\Delta g_j \quad \text{in } \Omega,$$

where $\mathcal{S}_\Delta$ is the harmonic single layer in $\Omega$ (cf. (4.4)). To proceed, recall that if $S_\Delta$ denotes the boundary harmonic single layer, then the identity

$$(4.57) \qquad \partial_\nu \mathcal{D}_\Delta g = \frac{1}{2} \sum_{i,k=1}^{n} \partial_{\tau_{ki}} S_\Delta(\partial_{\tau_{ki}} g),$$

is valid for any $g \in L^p(\partial\Omega)$. Consequently, based on (4.7), (4.56), and (4.57), we may compute

$$(4.58) \qquad \partial_\nu \Delta\dot{\mathcal{S}}\Lambda = \partial_\nu \mathcal{S}_\Delta g_0 + \sum_{i,j=1}^{n} \partial_\nu \mathcal{S}_\Delta(\partial_{\tau_{ji}}(v_i g_j)) + \sum_{j=1}^{n} \partial_\nu \mathcal{D}_\Delta(v_j g_j)$$

$$= \left(-\frac{1}{2}I + K_\Delta^*\right)g_0 + \sum_{i,j=1}^{n} \left(-\frac{1}{2}I + K_\Delta^*\right)(\partial_{\tau_{ji}}(v_i g_j))$$

$$+ \frac{1}{2} \sum_{i,j,k=1}^{n} \partial_{\tau_{ki}} S_\Delta(\partial_{\tau_{ki}}(v_i g_j)),$$

where $K_\Delta^*$ is the adjoint of the boundary harmonic double layer from (4.5). Since the operators

$$K_\Delta^* : L_{-1}^{p'}(\partial\Omega) \to L_{-1}^{p'}(\partial\Omega),$$

(4.59)
$$K_\Delta^* : L^{p'}(\partial\Omega) \to L^{p'}(\partial\Omega),$$

$$S_\Delta : L_{-1}^{p'}(\partial\Omega) \to L^{p'}(\partial\Omega),$$

are bounded, estimate (4.53) now follows from (4.58) and (4.54). This completes the proof of the boundedness of the operator in (4.51). With this in hand, identity (4.52) follows from the jump-formula for generic conormal derivatives of single multi-layers on Besov spaces (cf. (5.152) in [15]), and a density argument. □

Moving on, assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, with outward unit normal $v = (v_j)_{1 \le j \le n}$ and surface measure $\sigma$. Also, fix $\theta \in \mathbb{R}$ and $1 < p < \infty$ and recall $c_n(\theta)$ from (4.24). In this setting, consider the $2 \times 2$ matrix-valued singular integral operator

(4.60)        $$\tilde{K}_\theta : L_1^p(\partial\Omega) \oplus L^p(\partial\Omega) \to L_1^p(\partial\Omega) \oplus L^p(\partial\Omega),$$

(4.61)                 $$\tilde{K}_\theta := \begin{pmatrix} \mathcal{R}_\theta^{11} & \mathcal{R}_\theta^{12} \\ \mathcal{R}_\theta^{21} & \mathcal{R}_\theta^{22} \end{pmatrix},$$

where the entries in the above matrix,

$$\mathcal{R}_\theta^{11} : L_1^p(\partial\Omega) \to L_1^p(\partial\Omega),$$

(4.62)
$$\mathcal{R}_\theta^{12} : L^p(\partial\Omega) \to L_1^p(\partial\Omega),$$

$$\mathcal{R}_\theta^{21} : L_1^p(\partial\Omega) \to L^p(\partial\Omega),$$

$$\mathcal{R}_\theta^{22} : L^p(\partial\Omega) \to L^p(\partial\Omega),$$

(are the principal-value singular integral operators given at $\sigma$-a.e. $X \in \partial\Omega$ by

(4.63)    $$(\mathcal{R}_\theta^{11} F)(X) := \lim_{\varepsilon \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} \{\partial_{v(Y)}[(\Delta E)(X-Y)]F(Y)$$

$$+ c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)] \times$$
$$\times v_i(Y)(\partial_{\tau_{ij}} F)(Y)\} d\sigma(Y),$$

(4.64)     $$(\mathcal{R}_\theta^{12} g)(X) := \lim_{\varepsilon \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} \{(\Delta E)(X-Y)g(Y)$$

$$- c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)] \times$$
$$\times v_j(Y)g(Y)\} d\sigma(Y),$$

(4.65)     $$(\mathcal{R}_\theta^{21} F)(X) := \lim_{\varepsilon \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} v_\ell(X)\{-(\partial_i \Delta E)(X-Y)(\partial_{\tau_{\ell i}} F)(Y)$$

$$- c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X-Y)] \times$$
$$\times v_i(Y)(\partial_{\tau_{ij}} F)(Y)\} d\sigma(Y),$$

and

$$(4.66) \quad (\mathcal{R}_\theta^{22} g)(X) := \lim_{\varepsilon \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} v_\ell(X)\{-(\partial_\ell \Delta E)(X-Y)g(Y)$$

$$+ c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X-Y)] \times$$

$$\times v_j(Y)g(Y)\} \, d\sigma(Y),$$

for each $F \in L_1^p(\partial\Omega)$ and each $g \in L^p(\partial\Omega)$. Here the summation convention over repeated indices has been used.

PROPOSITION 4.6. *Retain the same setting as above, and recall the definition of the boundary biharmonic double multi-layer operator $\dot{K}_\theta$ on $\partial\Omega$ from Definition 4.1. Also, recall the mapping $\Psi$ from Proposition 2.2. Then, for each $p \in (1, \infty)$, the following diagram is commutative:*

$$
\begin{array}{ccc}
L_1^p(\partial\Omega) \oplus L^p(\partial\Omega) & \xrightarrow{\tilde{K}_\theta} & L_1^p(\partial\Omega) \oplus L^p(\partial\Omega) \\
\Big\uparrow{\scriptstyle \Psi} & & \Big\uparrow{\scriptstyle \Psi} \\
\dot{L}_{1,0}^p(\partial\Omega) & \xrightarrow[\dot{K}_\theta]{} & \dot{L}_{1,0}^p(\partial\Omega)
\end{array}
$$

PROOF. Fix an arbitrary pair of functions, $(F, g) \in L_1^p(\partial\Omega) \oplus L^p(\partial\Omega)$, along with an arbitrary point $X \in \mathbb{R}^n \backslash \partial\Omega$. Based on (4.17) and (3.10)–(3.11) we may write, in a manner analogous to (4.27) (using the summation convention over repeated indices),

$$(4.67) \quad \tilde{\mathcal{D}}_\theta(F, g)(X) = \int_{\partial\Omega} \{(\Delta E)(X-Y)$$

$$- c_n(\theta) \cdot v_j(Y) \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)]\} g(Y) \, d\sigma(Y)$$

$$+ \int_{\partial\Omega} \{\partial_{v(Y)}[(\Delta E)(X-Y)] - c_n(\theta) \times$$

$$\times \partial_{\tau_{ij}(Y)}(v_i(Y) \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)])\} F(Y) \, d\sigma(Y),$$

where we have also integrated by parts on the boundary and $c_n(\theta)$ is as in (4.24). Hence, for every pair of functions $(F, g) \in L_1^p(\partial\Omega) \oplus L^p(\partial\Omega)$ we have

$$(4.68) \quad \tilde{\mathcal{D}}_\theta(F, g)(X) = (\mathcal{D}_\Delta F)(X) + (\mathcal{S}_\Delta g)(X)$$

$$+ c_n(\theta) \cdot \int_{\partial\Omega} \partial_{\tau_{kj}(Y)}[(\partial_k E)(X-Y)] \times$$

$$\times (v_i \partial_{\tau_{ij}} F - v_j g)(Y) \, d\sigma(Y),$$

at every $X \in \mathbb{R}^n \backslash \partial\Omega$. Consequently, for every number $\ell \in \{1, \ldots, n\}$, at each point $X \in \mathbb{R}^n \backslash \partial\Omega$ we may write

(4.69)     $\partial_\ell(\tilde{\mathcal{D}}_\theta(F,g))(X)$

$$= \int_{\partial\Omega} \{(\partial_i \Delta E)(X-Y)(\partial_{\tau_{\ell i}}F)(Y) + (\partial_\ell \Delta E)(X-Y)g(Y)$$

$$+ c_n(\theta) \cdot \partial_{\tau_{kj}(Y)}[(\partial_\ell \partial_k E)(X-Y)](\nu_i \partial_{\tau_{ij}}F - \nu_j g)(Y)\} \, d\sigma(Y),$$

based on (4.68) and (4.6). From (4.68)–(4.69), on the one hand, and (4.60)–(4.66), on the other hand, we deduce by also making use of general jump-formulas for layer potentials of Calderón-Zygmund type (cf., e.g., (2.530) in [15]), that (cf. [26, (14.2) on p. 253])

(4.70)     $$(\tilde{\mathcal{D}}_\theta(F,g)\lfloor_{\partial\Omega}, -\partial_\nu \tilde{\mathcal{D}}_\theta(F,g)) = \left(\frac{1}{2}I + \tilde{K}_\theta\right)(F,g),$$

for each $F \in L_1^p(\partial\Omega)$ and each $g \in L^p(\partial\Omega)$

As such, for every $\dot{f} \in \dot{L}_{1,0}^p(\partial\Omega)$ we may compute

(4.71)     $$\left(\frac{1}{2}I + \tilde{K}_\theta\right)\Psi(\dot{f}) = ([\tilde{\mathcal{D}}_\theta \circ \Psi(\dot{f})]\lfloor_{\partial\Omega}, -\partial_\nu[\tilde{\mathcal{D}}_\theta \circ \Psi(\dot{f})])$$

$$= (\dot{\mathcal{D}}_\theta\dot{f}\lfloor_{\partial\Omega}, -\partial_\nu\dot{\mathcal{D}}_\theta\dot{f})$$

$$= \Psi((\mathrm{Tr}\,\dot{\mathcal{D}}_\theta\dot{f}, \mathrm{Tr}(\nabla\dot{\mathcal{D}}_\theta\dot{f})))$$

$$= \Psi\left(\left(\frac{1}{2}I + \dot{K}_\theta\right)\dot{f}\right),$$

where the first equality is (4.70) written for $(F,g) := \Psi(\dot{f})$, the second equality has been established in Proposition 4.2, the third equality makes use of (2.26), and the fourth equality is a consequence of the jump-formula for the double multi-layer (cf. Theorem 4.6 in [15]). Now the claim about the commutativity of the diagram in the statement of the proposition readily follows from (4.71).     $\square$

We conclude this section by recording a couple of useful operator identities involving the multi-layers considered earlier. Specifically, whenever $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $\theta \in \mathbb{R}$, $1 < p, q < \infty$, and $s \in (0,1)$, we have

(4.72)     $$\partial_\nu^{A_\theta}\dot{\mathcal{D}}_\theta \circ \dot{S} = \left(\frac{1}{2}I + \dot{K}_\theta^*\right) \circ \left(-\frac{1}{2}I + \dot{K}_\theta^*\right) \quad \text{on } (\dot{B}_{1,s}^{p,q}(\partial\Omega))^*,$$

and

(4.73)     $$\dot{S} \circ \partial_\nu^{A_\theta}\dot{\mathcal{D}}_\theta = \left(\frac{1}{2}I + \dot{K}_\theta\right) \circ \left(-\frac{1}{2}I + \dot{K}_\theta\right) \quad \text{on } \dot{B}_{1,s}^{p,q}(\partial\Omega).$$

Both are particular cases of similar identities valid for multi-layers associated with generic higher-order operators (cf. (5.176)–(5.177) in [15]). In the same geo-metric context if $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$ then for each $s \in (0,1)$ we

also have

$$(4.74) \qquad \dot{S}\dot{K}_\theta^* = \dot{K}_\theta \dot{S}$$

as linear, bounded operators from the space $(\dot{B}_{1,s}^{p;q}(\partial\Omega))^*$ into $\dot{B}_{1,1-s}^{p';q'}(\partial\Omega)$. See Proposition 5.15 in [15].

## 5. INVERTIBILITY RESULTS

The first goal here is to state and prove a basic invertibility result, extending work in [26] and [24]. In preparation, let $\mathcal{P}(\mathbb{R}^n)$ stand for the space of all polynomials of degree $\leq 1$ in $\mathbb{R}^n$, and set

$$(5.1) \qquad \mathcal{P}(\Omega) := \{P|_\Omega : P \in \mathcal{P}(\mathbb{R}^n)\},$$

$$\dot{\mathcal{P}}(\partial\Omega) := \{\dot{P} := (P|_{\partial\Omega}, (\nabla P)|_{\partial\Omega}) : P \in \mathcal{P}(\Omega)\}.$$

Our first main invertibility result reads as follows.

THEOREM 5.1. *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Also, recall the boundary bi-harmonic double multi-layer operator $\dot{K}_\theta$ on $\partial\Omega$ from Definition 4.1. Then there exists $\varepsilon > 0$ with the property that*

$$(5.2) \qquad \frac{1}{2}I + \dot{K}_\theta : \dot{L}_{1,0}^p(\partial\Omega) \to \dot{L}_{1,0}^p(\partial\Omega) \text{ is an isomorphism}$$

$$\text{whenever } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (2 - \varepsilon, \infty) \text{ if } n \in \{2, 3\},$$

*and*

$$(5.3) \qquad \frac{1}{2}I + \dot{K}_\theta : \dot{L}_{1,1}^p(\partial\Omega) \to \dot{L}_{1,1}^p(\partial\Omega) \text{ is an isomorphism}$$

$$\text{whenever } p \in \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (1, 2 + \varepsilon) \text{ if } n \in \{2, 3\}.$$

*In addition, the inverses of the isomorphisms in (5.2) and (5.3) act in a compatible manner on the intersection of their domains. Furthermore,*

$$(5.4) \qquad -\frac{1}{2}I + \dot{K}_\theta : \dot{L}_{1,0}^p(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \to \dot{L}_{1,0}^p(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$$

$$\text{is an isomorphism whenever } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (2 - \varepsilon, \infty) \text{ if } n \in \{2, 3\},$$

*and*

$$(5.5) \qquad -\frac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,1}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \to \dot{L}^p_{1,1}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$$

*is an isomorphism whenever* $p \in \left(\dfrac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right)$ *if* $n \geq 4$,

*and whenever* $p \in (1, 2 + \varepsilon)$ *if* $n \in \{2, 3\}$,

*and once again the inverses of the isomorphisms in (5.4) and (5.5) act in a compatible manner on the intersection of their domains.*

PROOF. Given $\varepsilon \in (0, 1)$, consider the open intervals

$$(5.6) \qquad I_\varepsilon := \begin{cases} \left(\frac{2(n-1)}{n+1+\varepsilon}, \frac{2(n-1)}{n-1-\varepsilon}\right) & \text{if } n \geq 4, \\ \left(1, \frac{2(n-1)}{n-1-\varepsilon}\right) & \text{if } n \in \{2, 3\}, \end{cases}$$

and

$$(5.7) \qquad I'_\varepsilon := \begin{cases} \left(\frac{2(n-1)}{n-1+\varepsilon}, \frac{2(n-1)}{n-3-\varepsilon}\right) & \text{if } n \geq 4, \\ \left(\frac{2(n-1)}{n-1+\varepsilon}, \infty\right) & \text{if } n \in \{2, 3\}. \end{cases}$$

Hence, for any $p, p' \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ we have

$$(5.8) \qquad p \in I'_\varepsilon \Leftrightarrow p' \in I_\varepsilon.$$

The starting point is the result asserting that there exists $\varepsilon \in (0, 1)$ such that, with $\tilde{K}_\theta$ as in (4.60)–(4.66), the operators

$$(5.9) \qquad \pm\frac{1}{2}I + \tilde{K}_\theta : L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \to L^p_1(\partial\Omega) \oplus L^p(\partial\Omega)$$

are Fredholm with index zero whenever $p \in I'_\varepsilon$.

This result (which uses the fact that $\theta > -\frac{1}{n}$) has been established first when $p \in (2 - \varepsilon, 2 + \varepsilon)$ by G. Verchota in [26], and the extension to the larger range $p \in I'_\varepsilon$ is due to Z. Shen in [24]. Moreover, it has been established in [24] that, for some $\varepsilon \in (0, 1)$,

$$(5.10) \qquad \frac{1}{2}I + \tilde{K}_\theta : L^p_1(\partial\Omega) \oplus L^p(\partial\Omega) \to L^p_1(\partial\Omega) \oplus L^p(\partial\Omega)$$

is an isomorphism whenever $p \in I'_\varepsilon$.

Granted this, the invertibility of the operator in (5.2) then follows from (5.10), Proposition 4.6, and Proposition 2.2. Concerning the operator in (5.4), in a first stage the same circle of ideas give, based on (5.9), that

(5.11) $\qquad$ the operator $-\dfrac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,0}(\partial\Omega) \to \dot{L}^p_{1,0}(\partial\Omega)$

$\qquad\qquad$ is Fredholm with index zero for each $p \in I'_\varepsilon$.

In turn, thanks to (5.11) and the fact that $2\dot{K}_\theta$ reproduces functions in $\dot{\mathcal{P}}(\partial\Omega)$ (cf. Proposition 4.5 in [15]), we also have that

(5.12) $\qquad -\dfrac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \to \dot{L}^p_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$

$\qquad\qquad$ is Fredholm with index zero whenever $p \in I'_\varepsilon$.

Given that the embedding $\dot{B}^{2,2}_{1,1/2}(\partial\Omega) \hookrightarrow \dot{L}^p_{1,0}(\partial\Omega)$ is well-defined, continuous and with dense range, for each $p \in I_\varepsilon$ provided that $\varepsilon > 0$ is small enough, we deduce from the invertibility of the operator $-\frac{1}{2}I + \dot{K}_\theta$ both on $\dot{B}^{2,2}_{1,1/2}(\partial\Omega)$ and on $\dot{B}^{2,2}_{1,1/2}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$ (itself, a consequence of variational arguments; see Corollary 6.1 in [15] and (6.174) in [15]), (5.12), plus a little functional analysis (cf. Lemma 6.6 in [15]) that

(5.13) $\qquad -\dfrac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \to \dot{L}^p_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$

$\qquad\qquad$ is an injective operator for each $p \in I'_\varepsilon$.

In passing, let us also note that the same type of reasoning as above gives

(5.14) $\qquad \left\{ \dot{f} \in \dot{L}^p_{1,0}(\partial\Omega) : \left(-\dfrac{1}{2}I + \dot{K}_\theta\right)\dot{f} = 0 \right\} = \dot{\mathcal{P}}(\partial\Omega), \quad \forall p \in I'_\varepsilon.$

This is going to be useful later on.

$\qquad$ Let us now consider the claims made in (5.3) and (5.5). To this end, we first note that, thanks to (5.9), Proposition 4.6, and Proposition 2.2, we have that

(5.15) $\qquad \pm\dfrac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,0}(\partial\Omega) \to \dot{L}^p_{1,0}(\partial\Omega)$

$\qquad\qquad$ are Fredholm with index zero for each $p \in I'_\varepsilon$.

Thus, by duality,

(5.16) $\qquad \pm\dfrac{1}{2}I + \dot{K}^*_\theta : (\dot{L}^p_{1,0}(\partial\Omega))^* \to (\dot{L}^p_{1,0}(\partial\Omega))^*$

$\qquad\qquad$ are Fredholm with index zero for each $p \in I'_\varepsilon$.

Let us also remark that, as seen from (5.15), the composition

(5.17) $\qquad \left(\dfrac{1}{2}I + \dot{K}_\theta\right) \circ \left(-\dfrac{1}{2}I + \dot{K}_\theta\right) : \dot{L}^p_{1,0}(\partial\Omega) \to \dot{L}^p_{1,0}(\partial\Omega)$

$\qquad\qquad$ is Fredholm with index zero for each $p \in I'_\varepsilon$,

hence by duality,

$$(5.18) \qquad \left(\tfrac{1}{2}I + \dot{K}_\theta^*\right) \circ \left(-\tfrac{1}{2}I + \dot{K}_\theta^*\right) : (\dot{L}_{1,0}^p(\partial\Omega))^* \to (\dot{L}_{1,0}^p(\partial\Omega))^*$$

is Fredholm with index zero for each $p \in I_\varepsilon'$.

To proceed, recall that $\dot{S}$ denotes the boundary version of the biharmonic single multi-layer introduced in (4.50). Hence, if $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, then the boundedness results for general single multi-layer type operators from Theorem 5.1 in [15] ensure that

$$(5.19) \qquad \dot{S} : (\dot{L}_{1,1}^{p'}(\partial\Omega))^* \to \dot{L}_{1,0}^p(\partial\Omega) \text{ boundedly, and}$$

$$(5.20) \qquad \dot{S} : (\dot{L}_{1,0}^p(\partial\Omega))^* \to \dot{L}_{1,1}^{p'}(\partial\Omega) \text{ boundedly.}$$

Based on these and Theorem 4.4 we may therefore conclude that, for each index $p \in (1, \infty)$, the operators

$$(5.21) \qquad \partial_\nu^{A_\theta}\dot{D}_\theta \circ \dot{S} : (\dot{L}_{1,0}^p(\partial\Omega))^* \to (\dot{L}_{1,0}^p(\partial\Omega))^*,$$

$$(5.22) \qquad \partial_\nu^{A_\theta}\dot{D}_\theta \circ \dot{S} : (\dot{L}_{1,1}^p(\partial\Omega))^* \to (\dot{L}_{1,1}^p(\partial\Omega))^*,$$

$$(5.23) \qquad \dot{S} \circ \partial_\nu^{A_\theta}\dot{D}_\theta : \dot{L}_{1,0}^p(\partial\Omega) \to \dot{L}_{1,0}^p(\partial\Omega),$$

$$(5.24) \qquad \dot{S} \circ \partial_\nu^{A_\theta}\dot{D}_\theta : \dot{L}_{1,1}^p(\partial\Omega) \to \dot{L}_{1,1}^p(\partial\Omega),$$

are well-defined, linear and bounded (where, as usual, $1/p + 1/p' = 1$). Having established these boundedness results, we may then conclude from (5.21), (5.18), formula (4.72), and density arguments, that

$$(5.25) \qquad \partial_\nu^{A_\theta}\dot{D}_\theta \circ \dot{S} : (\dot{L}_{1,0}^p(\partial\Omega))^* \to (\dot{L}_{1,0}^p(\partial\Omega))^*$$

is Fredholm with index zero for each $p \in I_\varepsilon'$.

In turn, (5.25), (5.20), and (4.29) readily imply that

$$(5.26) \qquad \text{the operator } \partial_\nu^{A_\theta}\dot{D}_\theta : \dot{L}_{1,1}^{p'}(\partial\Omega) \to (\dot{L}_{1,0}^p(\partial\Omega))^* \text{ has closed range,}$$

of finite codimension whenever $p \in I_\varepsilon'$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

With this in hand and availing ourselves of the fact that the operators in (4.29) and (4.30) are adjoint to one another, it follows from (5.26) and duality that

$$(5.27) \qquad \text{the operator } \partial_\nu^{A_\theta}\dot{D}_\theta : \dot{L}_{1,0}^p(\partial\Omega) \to (\dot{L}_{1,1}^{p'}(\partial\Omega))^*$$

has finite dimensional kernel, if $p \in I_\varepsilon'$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Next we claim that

(5.28) $\qquad \forall p' \in I_\varepsilon \quad \exists q \in I'_\varepsilon \quad$ such that $\quad \dot{L}^{p'}_{1,1}(\partial\Omega) \hookrightarrow \dot{L}^q_{1,0}(\partial\Omega).$

To justify this claim, assume first that $n \geq 4$ and note that this forces $p' \in (1, n-1)$ for any $p' \in I_\varepsilon$. As such, the embedding $\dot{L}^{p'}_{1,1}(\partial\Omega) \hookrightarrow \dot{L}^q_{1,0}(\partial\Omega)$ holds whenever $q := \left(\frac{1}{p'} - \frac{1}{n-1}\right)^{-1}$. On the other hand, it may be verified without difficulty that

(5.29) $$\left\{ \left(\frac{1}{p'} - \frac{1}{n-1}\right)^{-1} : p' \in I_\varepsilon \right\} = I'_\varepsilon.$$

This, of course, proves the claim in (5.28) when $n \geq 4$. When $n = 2$, the embedding in (5.28) holds for any $p', q \in (1, \infty)$, while when $n = 3$ is obviously true whenever indices $p' \in [2, \infty)$ and $q \in (1, \infty)$. Finally, in the remaining case, i.e., for $n = 3$ and $p' \in (1, 2)$, we may take the index $q := \left(\frac{1}{p'} - \frac{1}{2}\right)^{-1} \in (2, \infty) \subseteq I'_\varepsilon$. This finishes the proof of (5.28).

Moving on, we may then deduce from (5.27), (5.28), and (5.26), that the operator

(5.30) $$\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta : \dot{L}^{p'}_{1,1}(\partial\Omega) \rightarrow (\dot{L}^p_{1,0}(\partial\Omega))^*$$

has both closed range, of finite codimension, and finite dimensional kernel whenever $p \in I'_\varepsilon$ with $\frac{1}{p} + \frac{1}{p'} = 1$. In other words,

(5.31) $$\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta : \dot{L}^{p'}_{1,1}(\partial\Omega) \rightarrow (\dot{L}^p_{1,0}(\partial\Omega))^* \text{ is Fredholm}$$

$$\text{provided } p \in I'_\varepsilon \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

In particular, the operator $\partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta$ has, in the above context, a quasi-inverse. In concrete terms, this means that whenever $p \in I'_\varepsilon$ and $\frac{1}{p} + \frac{1}{p'} = 1$, there exist a Fredholm operator $R : (\dot{L}^p_{1,0}(\partial\Omega))^* \rightarrow \dot{L}^{p'}_{1,1}(\partial\Omega)$, and a linear compact operator Comp mapping $\dot{L}^{p'}_{1,1}(\partial\Omega)$ into some Banach space $\mathcal{X}$, with the property that

(5.32) $$R \circ \partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta = I + \text{Comp on } \dot{L}^{p'}_{1,1}(\partial\Omega).$$

Composing the Fredholm operator in (5.25) to the left with the Fredholm operator $R$ just considered, and keeping in mind that the class of Fredholm operators is closed under composition as well as additive compact perturbations, we arrive at the conclusion that

(5.33) $$\dot{S} : (\dot{L}^p_{1,0}(\partial\Omega))^* \rightarrow \dot{L}^{p'}_{1,1}(\partial\Omega) \text{ is a Fredholm operator}$$

$$\text{whenever } p \in I'_\varepsilon \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

In light of the self-adjointness of the single multi-layer (cf. (5.22) in [15]), we may take the dual of (5.33) in order to also obtain that

$$(5.34) \qquad \dot{S} : (\dot{L}_{1,1}^{p'}(\partial\Omega))^* \rightarrow \dot{L}_{1,0}^{p}(\partial\Omega) \text{ is a Fredholm operator}$$

$$\text{whenever } p \in I_\varepsilon' \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

At this stage, taking the composition of the Fredholm operators in (5.31) and (5.33) leads to the conclusion that

$$(5.35) \qquad \dot{S} \circ \partial_\nu^{A_\theta} \dot{\mathcal{D}}_\theta : \dot{L}_{1,1}^{p'}(\partial\Omega) \rightarrow \dot{L}_{1,1}^{p'}(\partial\Omega) \text{ is a Fredholm operator}$$

$$\text{whenever } p \in I_\varepsilon' \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

Granted this, from identity (4.73) and a density argument we deduce that

$$(5.36) \qquad \left(\frac{1}{2}I + \dot{K}_\theta\right) \circ \left(-\frac{1}{2}I + \dot{K}_\theta\right) : \dot{L}_{1,1}^{p'}(\partial\Omega) \rightarrow \dot{L}_{1,1}^{p'}(\partial\Omega)$$

$$\text{is a Freholm operator whenever } p \in I_\varepsilon' \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

In turn, this readily implies that the operators $\pm\frac{1}{2}I + \dot{K}_\theta$ (which commute with one another) have both closed ranges of finite codimension and finite dimensional kernels, thus, ultimately,

$$(5.37) \qquad \pm\frac{1}{2}I + \dot{K}_\theta : \dot{L}_{1,1}^{p'}(\partial\Omega) \rightarrow \dot{L}_{1,1}^{p'}(\partial\Omega) \text{ are Fredholm operators}$$

$$\text{whenever } p \in I_\varepsilon' \text{ and } \frac{1}{p} + \frac{1}{p'} = 1.$$

Going further, we make use of (4.74) (and the same type of boundedness and density results as before) in order to obtain the following intertwining identity

$$(5.38) \qquad \dot{S} \circ \left(\pm\frac{1}{2}I + \dot{K}_\theta^*\right) = \left(\pm\frac{1}{2}I + \dot{K}_\theta\right) \circ \dot{S} \quad \text{on } (\dot{L}_{1,0}^{p}(\partial\Omega))^*,$$

in which $\dot{S}$ is as in (5.33), $\dot{K}_\theta$ acts on $\dot{L}_{1,1}^{p'}(\partial\Omega)$, and $\dot{K}_\theta^*$ acts on $(\dot{L}_{1,0}^{p}(\partial\Omega))^*$. From (5.38), (5.37), (5.33), (5.16), and the additivity law for the Fredholm index, we eventually obtain (cf. also (5.8)) that

$$(5.39) \qquad \text{the operators } \pm\frac{1}{2}I + \dot{K}_\theta : \dot{L}_{1,1}^{p'}(\partial\Omega) \rightarrow \dot{L}_{1,1}^{p'}(\partial\Omega)$$

$$\text{are Fredholm with index zero if } p' \in I_\varepsilon.$$

In particular,

$$(5.40) \qquad \pm\frac{1}{2}I + \dot{K}_\theta \text{ are Fredholm with index zero}$$

$$\text{on } \dot{L}^{p'}_{1,1}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \text{ whenever } p' \in I_\varepsilon.$$

Using this, the embedding in (5.28), and the injectivity of the operator in (5.2) it follows that the operator in (5.3) is also injective, thus ultimately invertible by (5.39). This takes care of the claim made in (5.3). Finally, the same type of reasoning, based on the embedding (5.28) and the injectivity of the operator in (5.4), shows that the operator in (5.5) is also injective, thus ultimately invertible by (5.40).

Let us now prove that the inverses of the isomorphisms in (5.2) and (5.3) act in a compatible manner on the intersection of their domains. With this goal in mind, assume that

$$(5.41) \qquad \dot{f}_0 \in \dot{L}^{p_0}_{1,0}(\partial\Omega) \text{ with } p_0 \in I'_\varepsilon \text{ and } \dot{f}_1 \in \dot{L}^{p_1}_{1,1}(\partial\Omega) \text{ with } p_1 \in I_\varepsilon$$

$$\text{are such that } \left(\frac{1}{2}I + \dot{K}_\theta\right)\dot{f}_0 = \left(\frac{1}{2}I + \dot{K}_\theta\right)\dot{f}_1.$$

By (5.28), there exists $q \in I'_\varepsilon$ with the property that $\dot{L}^{p_1}_{1,1}(\partial\Omega) \hookrightarrow \dot{L}^q_{1,0}(\partial\Omega)$, hence if we now set $p := \min\{p_0, q\}$ then

$$(5.42) \qquad p \in I'_\varepsilon \quad \text{and} \quad \dot{L}^{p_0}_{1,0}(\partial\Omega) \cap \dot{L}^{p_1}_{1,1}(\partial\Omega) \hookrightarrow \dot{L}^p_{1,0}(\partial\Omega).$$

From (5.41)–(5.42) and the fact that $\frac{1}{2}I + \dot{K}_\theta$ is invertible on $\dot{L}^p_{1,0}(\partial\Omega)$, it follows that $\dot{f}_0 = \dot{f}_1$, as wanted. Finally, the compatibility of the inverses of the isomorphisms in (5.4) and (5.5) on the intersection of their domains is established analogously, completing the proof. □

We now proceed to record several significant consequences of Theorem 5.1 (and its proof).

COROLLARY 5.2. *Let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, be a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Also, recall the boundary biharmonic double multi-layer operator $\dot{K}_\theta$ on $\partial\Omega$ from Definition 4.1. Then there exists $\varepsilon > 0$ with the property that*

$$(5.43) \qquad \left\{\dot{f} \in \dot{L}^p_{1,1}(\partial\Omega) : \left(-\frac{1}{2}I + \dot{K}_\theta\right)\dot{f} = 0\right\} = \dot{\mathcal{P}}(\partial\Omega)$$

*whenever*

$$(5.44) \qquad p \in \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and } p \in (1, 2 + \varepsilon) \text{ if } n \in \{2, 3\}.$$

PROOF. This is a consequence of the formula in (5.14) and the embedding result
recorded in (5.28).                                                                     □

Further invertibility results for multi-layers, complementing those established
in Theorem 5.1, are discussed below.

COROLLARY 5.3. *Suppose that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz do-
main with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. As before, let $\dot{K}_\theta$ denote
the boundary biharmonic double multi-layer operator on $\partial\Omega$ considered in Defini-
tion 4.1. Also, let $\dot{S}$ denote the boundary version of the biharmonic single multi-
layer associated with $L = \Delta^2$ as in (4.49). Then there exists $\varepsilon > 0$ with the property
that*

$$(5.45) \qquad \frac{1}{2}I + \dot{K}_\theta^* : (L_{1,0}^p(\partial\Omega))^* \to (L_{1,0}^p(\partial\Omega))^* \text{ is an isomorphism}$$

$$\text{whenever } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (2 - \varepsilon, \infty) \text{ if } n \in \{2, 3\},$$

*and*

$$(5.46) \qquad \frac{1}{2}I + \dot{K}_\theta^* : (L_{1,1}^p(\partial\Omega))^* \to (L_{1,1}^p(\partial\Omega))^* \text{ is an isomorphism}$$

$$\text{whenever } p \in \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (1, 2 + \varepsilon) \text{ if } n \in \{2, 3\}.$$

*In addition, the inverses of the isomorphisms in (5.45) and (5.46) act in a compatible
manner on the intersection of their domains. Moreover,*

$$(5.47) \qquad -\frac{1}{2}I + \dot{K}_\theta^* \text{ is an isomorphism on } (L_{1,0}^p(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^*$$

$$\text{whenever } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ for } n \geq 4,$$

$$\text{and whenever } p \in (2 - \varepsilon, \infty) \text{ for } n \in \{2, 3\},$$

*and*

$$(5.48) \qquad -\frac{1}{2}I + \dot{K}_\theta^* \text{ is an isomorphism on } (L_{1,1}^p(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^*$$

$$\text{whenever } p \in \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right) \text{ for } n \geq 4,$$

$$\text{and whenever } p \in (1, 2 + \varepsilon) \text{ for } n \in \{2, 3\}.$$

*In addition, the inverses of the isomorphisms in* (5.47) *and* (5.48) *act in a compatible fashion on the intersection of their domains.*

*Finally, if $n \geq 3$ and $n \neq 4$, then also*

$$(5.49) \qquad \dot{S} : (\dot{L}^p_{1,0}(\partial\Omega))^* \to \dot{L}^{p'}_{1,1}(\partial\Omega) \text{ is an isomorphism}$$

$$\text{provided } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 5,$$

$$\text{and provided } p \in (2 - \varepsilon, \infty) \text{ if } n = 3,$$

*and*

$$(5.50) \qquad \dot{S} : (\dot{L}^{p'}_{1,1}(\partial\Omega))^* \to \dot{L}^p_{1,0}(\partial\Omega) \text{ is an isomorphism}$$

$$\text{provided } p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 5,$$

$$\text{and provided } p \in (2 - \varepsilon, \infty) \text{ if } n = 3,$$

*and the inverses of the isomorphisms in* (5.49) *and* (5.50) *are compatible on the intersection of their domains.*

PROOF. The invertibility (and compatibility) claims concerning the boundary biharmonic double multi-layer operator are direct consequences of Theorem 5.1 and duality. As regards (5.49)–(5.50), these follow from (5.33)–(5.34), Theorem 6.6 in [15] when $n > 4$, and [26, Theorem 17.5] when $n = 3$, by reasoning as before. $\qquad \square$

## 6. BOUNDARY VALUE PROBLEMS

The work in this section concerns the existence, uniqueness, integral representation in terms of the multi-layers introduced in this paper, and regularity (measured on the Besov scale), of the solution of the Dirichlet and Neumann problems for the bi-Laplacian with boundary data from Whitney-Lebesgue and Whitney-Besov spaces, as well as their duals, in Lipschitz domains. In particular, this completes and refines work in [26] and [24]. To get started, we make the following definition.

DEFINITION 6.1. *Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ and fix $p \in (1, \infty)$. In this context, we say that* `property` $G_p$ `holds` *provided for each $X \in \Omega$ there exists a function $G(X, \cdot) \in \mathscr{C}^\infty(\Omega \backslash \{X\})$ satisfying (with $\delta_X$ denoting the Dirac distribution with mass at $X$)*

$$(6.1) \qquad \begin{cases} \Delta^2_Y G(X, Y) = \delta_X(Y), \\ G(X, \cdot)|_{\partial\Omega} = 0, \quad (\nabla_Y G(X, \cdot))|_{\partial\Omega} = 0, \\ \mathcal{N}(\nabla^2_Y G(X, \cdot)) \in L^p(\partial\Omega), \end{cases}$$

*where the nontangential maximal operator $\mathcal{N}$ is considered with respect to non-tangential approach regions truncated at height much smaller than the distance from $X$ to $\partial\Omega$.*

For example, if $\theta \in \mathbb{R}$ is such that

$$(6.2) \qquad \frac{1}{2}I + \dot{K}_\theta : \dot{L}^p_{1,1}(\partial\Omega) \to \dot{L}^p_{1,1}(\partial\Omega) \quad \text{is invertible,}$$

a Green function with the properties stipulated in (6.1) may be constructed by considering, for each $X, Y \in \Omega$ with $X \neq Y$,

$$(6.3) \quad G(X, Y) := E(X - Y)$$

$$- \dot{\mathcal{D}}_\theta\left[\left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1}(\operatorname{Tr}E(X - \cdot), \operatorname{Tr}(\nabla E(X - \cdot)))\right](Y),$$

where $E$ is the fundamental solution of the operator $\Delta^2$ from (4.1). To see that this is indeed the case, note that since for each point $X \in \Omega$ fixed we have $(\operatorname{Tr}E(X - \cdot), \operatorname{Tr}(\nabla E(X - \cdot))) \in \dot{L}^p_{1,1}(\partial\Omega)$, and using (6.2) and nontangential maximal function estimates for the double multi-layer we obtain

$$(6.4) \quad \mathcal{N}\left(\nabla^2\dot{\mathcal{D}}_\theta\left[\left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1}((\operatorname{Tr}E(X - \cdot), \operatorname{Tr}(\nabla E(X - \cdot))))\right]\right) \in L^p(\partial\Omega)$$

hence, ultimately,

$$(6.5) \qquad\qquad \mathcal{N}(\nabla^2_Y G(X, \cdot)) \in L^p(\partial\Omega)$$

if $G$ is as in (6.3). Furthermore, (6.3) and (4.12) ensure that the middle condition in (6.1) holds as well. Finally, the first condition in (6.1) is clear from the design of $G$.

The significance of the condition introduced in Definition 6.1 is most apparent from the following uniqueness result (which is a particular case of Theorem 6.18 in [15]).

THEOREM 6.2. *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, and that property $G_{p'}$ holds. If $u$ is a solution of the homogeneous Dirichlet boundary value problem*

$$(6.6) \qquad\qquad \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega), \\ (u\lfloor_{\partial\Omega}, (\nabla u)\lfloor_{\partial\Omega}) = \dot{0}, \end{cases}$$

*then necessarily $u \equiv 0$ in $\Omega$.*

As far as a genuine well-posedness result for the Dirichlet problem for the bi-Laplacian in Lipschitz domains is concerned, we have the following.

THEOREM 6.3. *Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with connected boundary. Then there exists $\varepsilon > 0$ such that property $G_p$ holds*

$$(6.7) \qquad \text{whenever } p \in \left( \frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon \right) \text{ if } n \geq 4,$$

$$\text{and whenever } p \in (1, 2 + \varepsilon) \text{ if } n \in \{2, 3\}.$$

*As a consequence, there exists $\varepsilon > 0$ such that if*

$$(6.8) \qquad p \in \left( 2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon \right) \text{ if } n \geq 4,$$

$$\text{and } p \in (2 - \varepsilon, \infty) \text{ if } n \in \{2, 3\},$$

*then Dirichlet boundary value problem for the bi-Laplacian with data from Whitney-Lebesgue spaces,*

$$(6.9) \qquad \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega), \\ (u\lfloor_{\partial\Omega}, (\nabla u)\lfloor_{\partial\Omega}) = \dot{f} \in \dot{L}^p_{1,0}(\partial\Omega), \end{cases}$$

*has a unique solution which, for every $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$, may be represented as*

$$(6.10) \qquad u(X) = \dot{\mathcal{D}}_\theta \left[ \left( \frac{1}{2} I + \dot{K}_\theta \right)^{-1} \dot{f} \right](X), \quad \forall X \in \Omega.$$

*In particular, the solution of (6.9) satisfies*

$$(6.11) \qquad \|u\|_{B^{p,p \vee 2}_{1+1/p}(\Omega)} \leq C \|\dot{f}\|_{\dot{L}^p_{1,0}(\partial\Omega)}$$

*for some finite constant $C = C(\Omega, p, \theta, n) > 0$ where, generally speaking, $a \vee b := \max\{a, b\}$.*

PROOF. The fact that property $G_p$ holds for $p$ as in (6.7) is seen from Theorem 5.1 and the discussion in (6.2)–(6.5). As such, the uniqueness part in the well-posedness of the boundary problem (6.9) follows from what we have just proved and Theorem 6.2. Next, $u$ in (6.10) is well-defined in light of (5.2), and solves (6.9) thanks to the biharmonicity of the double multi-layer, (4.13), and (4.12). Finally, that $u$ satisfies (6.11) follows from the integral representation formula (6.10) and the mapping properties of the double multi-layer (cf. (4.85) in [15]). $\qquad \square$

Our next result deals with the role of multi-layer potentials in the solvability of the so-called regularity problem for the bi-Laplacian in Lipschitz domains. As a preamble, we first recall the following estimate of Hardy-type.

LEMMA 6.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and suppose that $u$ is a biharmonic function in $\Omega$ which satisfies $\mathcal{N}(\nabla u) \in L^p(\partial\Omega)$ for some*

$p \in (0, n-1)$. *Then*

$$(6.12) \qquad \mathcal{N}u \in L^{p^*}(\partial\Omega) \quad where \ p^* := \left(\frac{1}{p} - \frac{1}{n-1}\right)^{-1}.$$

See [16, Lemma 11.9] for a proof of a more general result of this nature. Here is the well-posedness result advertised earlier, which refines earlier work in [26], and [24].

THEOREM 6.5. *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. As before, let $\dot{\mathcal{D}}_\theta$ and $\dot{K}_\theta$ denote the biharmonic double multi-layer operators (relative to $\Omega$) introduced in Definition 4.1.*

*Then there exists $\varepsilon > 0$ with the property that whenever $p \in (1, \infty)$ satisfies*

$$(6.13) \qquad p \in \left(\frac{2(n-1)}{n+1} - \varepsilon, 2 + \varepsilon\right) \ if \ n \geq 4,$$

$$and \ p \in (1, 2 + \varepsilon) \ if \ n \in \{2, 3\},$$

*the Dirichlet boundary value problem for the bi-Laplacian with data from Whitney-Sobolev spaces,*

$$(6.14) \qquad \begin{cases} \Delta^2 u = 0 & in \ \Omega, \\ \mathcal{N}(\nabla^2 u) \in L^p(\partial\Omega), \\ (u\lfloor_{\partial\Omega}, (\nabla u)\lfloor_{\partial\Omega}) = \dot{f} \in \dot{L}^p_{1,1}(\partial\Omega), \end{cases}$$

*has a unique solution, which actually admits the integral representation formula*

$$(6.15) \qquad u(X) = \dot{\mathcal{D}}_\theta\left[\left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1} \dot{f}\right](X), \quad \forall X \in \Omega.$$

*In particular, the solution of (6.14) satisfies*

$$(6.16) \qquad \|u\|_{B^{p,p\vee 2}_{2+1/p}(\Omega)} \leq C\|\dot{f}\|_{\dot{L}^p_{1,0}(\partial\Omega)}$$

*for some finite constant $C = C(\Omega, p, \theta, n) > 0$.*

PROOF. Let $\varepsilon > 0$ be as in Theorem 5.1. That the function $u$ given by (6.15) is well-defined whenever $p$ is as in (6.13) follows from the invertibility result recorded in (5.3). Also, the fact that this $u$ actually solves (6.14) is clear from the biharmonicity of the double multi-layer, (4.14), and (4.12). As regards uniqueness, suppose that $u$ solves the homogeneous version of the boundary problem (6.14) for some $p$ as in (6.13). Given the nature of the conclusion we seek, there is no loss of generality in assuming that the exponent $p$ also satisfies $p < n - 1$. Granted this, if $p^*$ is defined as in (6.12) then (much as it was the case in the proof of Theorem 5.1) $p^*$ satisfies the conditions listed in (6.8). Furthermore,

Lemma 6.4 applied to $\nabla u$ ensures that $\mathcal{N}(\nabla u) \in L^{p^*}(\partial\Omega)$, since we are assuming that $\mathcal{N}(\nabla^2 u) \in L^p(\partial\Omega)$ to begin with. As such, the uniqueness result established in the second part of Theorem 6.3 applies and yields that $u \equiv 0$ in $\Omega$, as wanted. Finally, that $u$ satisfies (6.16) follows from the integral representation formula (6.15) and mapping properties for the double multi-layer. $\qquad\square$

Next, we shall formulate and solve the Neumann problem for the bi-Laplacian with boundary data from the dual of Whitney-Lebesgue spaces. This parallels work in [26] and [24] where a different formulation is emphasized.

THEOREM 6.6. *Suppose that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. As before, let $\dot{K}_\theta$ denote the biharmonic double multi-layer operator (relative to $\partial\Omega$) introduced in Definition 4.1. Finally, recall the biharmonic single multi-layer $\dot{S}$ from (4.49) and the conormal derivative $\partial_v^{A_\theta}$ from Proposition 3.2. Then there exists $\varepsilon > 0$ with the property that whenever $p \in (1, \infty)$ satisfies*

$$(6.17) \qquad p \in \left(2 - \varepsilon, \frac{2(n-1)}{n-3} + \varepsilon\right) \text{ if } n \geq 4,$$

$$\text{and } p \in (2 - \varepsilon, \infty) \text{ if } n \in \{2, 3\},$$

*the Neumann boundary value problem for the bi-Laplacian with data from duals of Whitney-Lebesgue spaces,*

$$(6.18) \qquad \begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \mathcal{N}(\nabla^2 u) \in L^{p'}(\partial\Omega), \\ \partial_v^{A_\theta} u = \Lambda \in (\dot{L}_{1,0}^p(\partial\Omega))^* \end{cases}$$

*where $1/p + 1/p' = 1$ and the boundary data satisfies the necessary compatibility condition*

$$(6.19) \qquad \langle \Lambda, \dot{P} \rangle = 0 \quad \text{for each } \dot{P} \in \dot{\mathcal{P}}(\partial\Omega),$$

*is well-posed (with uniqueness understood modulo polynomials of degree $\leq 1$). Moreover, a solution may be given by the integral formula*

$$(6.20) \qquad u(X) = \dot{S}\left[\pi^* \left(-\frac{1}{2}I + \dot{K}_\theta^*\right)^{-1} \tilde{\Lambda}\right](X), \quad \forall X \in \Omega,$$

*where*

$$(6.21) \qquad \tilde{\Lambda} \in (\dot{L}_{1,0}^p(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^*$$

*is defined by setting*

$$(6.22) \qquad \langle \tilde{\Lambda}, [\dot{f}] \rangle := \langle \Lambda, \dot{f} \rangle, \quad \forall \dot{f} \in \dot{L}_{1,0}^p(\partial\Omega),$$

with $[\dot{f}]$ denoting the equivalence class of the Whitney array $\dot{f} \in \dot{L}^{p}_{1,0}(\partial\Omega)$ in the quotient space $\dot{L}^{p}_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$, and

$$(6.23) \qquad \pi^* : (\dot{L}^{p}_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^* \to (\dot{L}^{p}_{1,0}(\partial\Omega))^*$$

is the adjoint of the canonical projection

$$(6.24) \qquad \pi : \dot{L}^{p}_{1,0}(\partial\Omega) \to \dot{L}^{p}_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega),$$

taking a given arbitrary Whitney array $\dot{f} \in \dot{L}^{p}_{1,0}(\partial\Omega)$ into its equivalency class $[\dot{f}] \in \dot{L}^{p}_{1,0}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$.

PROOF. That the compatibility condition (6.19) is necessary is clear from integrations by parts and degree considerations. As regards existence, let $\varepsilon > 0$ be as in Corollary 5.3. Then the function $u$ given by (6.20) is well-defined whenever $p$ is as in (6.17) follows from the invertibility result recorded in (5.47). By the biharmonicity of the single multi-layer, the nontangential maximal function estimates for this operator, and (4.52), one may check that $u$ solves (6.18). Thus, as far as the well-posedness of the problem (6.18) is concerned, there remains to establish uniqueness (in the sense specified in the statement of the theorem). To this end, assume that $u$ is a solution of (6.18) with $\Lambda = 0$, and set

$$(6.25) \qquad \dot{f} := (u\big|_{\partial\Omega}, (\nabla u)\big|_{\partial\Omega}) \in \dot{L}^{p'}_{1,1}(\partial\Omega).$$

Keeping in mind that $\partial_{\nu}^{A_\theta} u = 0$, Green's formula gives

$$(6.26) \qquad u = \dot{\mathcal{D}}_\theta \dot{f} - \dot{\mathcal{S}}(\partial_{\nu}^{A_\theta} u) = \dot{\mathcal{D}}_\theta \dot{f} \quad \text{in } \Omega.$$

Taking the first-order nontangential boundary trace of both sides of (6.26) and using (4.12) then yields

$$(6.27) \qquad \dot{f} = \left(\frac{1}{2}I + \dot{K}_\theta\right)\dot{f},$$

which ultimately shows that $\left(-\frac{1}{2}I + \dot{K}_\theta\right)\dot{f} = 0$. From this and Corollary 5.2 we deduce that there exists $P \in \mathcal{P}(\Omega)$ such that $\dot{f} = \dot{P}$. Returning with this back in (6.26) and making use of the fact that the double multi-layer reproduces polynomials of degree $\leq 1$, finally gives that $u = \dot{\mathcal{D}}_\theta \dot{P} = P$ in $\Omega$, as desired. □

Our next goal is to explain how the invertibility results for the biharmonic layer potentials, as well as the well-posedness results for the various boundary problems for the bi-Laplacian, improve (in the sense that the range of exponents involved becomes larger) under additional regularity assumptions on the Lipschitz domain in question. This requires some preparations and we start by recalling that, given two quasi-Banach spaces $\mathcal{X}, \mathcal{Y}$, the space of all bounded linear operators mapping $\mathcal{X}$ into $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{X} \to \mathcal{Y})$. This becomes a quasi-Banach

itself when equipped with the canonical operator norm

$$(6.28) \quad \|T\|_{\mathcal{L}(\mathcal{X} \to \mathcal{Y})} := \sup\{\|Tx\|_{\mathcal{Y}} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} \le 1\}, \quad \text{for each } T \in \mathcal{L}(\mathcal{X} \to \mathcal{Y}).$$

Moreover, let us also define

$$(6.29) \quad \text{Comp}(\mathcal{X} \to \mathcal{Y}) := \begin{cases} \text{the space of all linear compact operators} \\ \text{from } \mathcal{X} \text{ into } \mathcal{Y}, \end{cases}$$

and note that $\text{Comp}(\mathcal{X}, \mathcal{Y})$ is a closed subspace of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. Finally, abbreviate

$$(6.30) \quad \mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X} \to \mathcal{X}), \quad \text{Comp}(\mathcal{X}) := \text{Comp}(\mathcal{X} \to \mathcal{X}).$$

Finally, given a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, we shall denote by $\text{BMO}(\partial\Omega)$ the John-Nirenberg space of functions with bounded mean oscillations on $\partial\Omega$ (naturally regarded as a space of homogeneous type, in the sense of Coifman-Weiss). In the same setting, we shall let $\text{VMO}(\partial\Omega)$ stand for the Sarason space of functions with vanishing mean oscillations on $\partial\Omega$.

The following is a particular case of a much more general result proved in [12, Theorem 4.36].

**THEOREM 6.7.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Denote by $\sigma$ and $v$, respectively, the surface measure and outward unit normal on $\partial\Omega$. Also, fix an arbitrary $p \in (1, \infty)$. Then for every $\varepsilon > 0$ the following holds. Given a function $k$ satisfying*

$$(6.31) \quad k : \mathbb{R}^n \backslash \{0\} \to \mathbb{R} \text{ is smooth, even, and homogeneous of degree } -n$$

*to which one associates the principal-value singular integral operator*

$$(6.32) \quad Tf(X) := \lim_{\eta \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y| > \eta}} \langle X - Y, v(Y) \rangle k(X - Y) f(Y) \, d\sigma(Y)$$

$$\text{whenever } X \in \partial\Omega,$$

*there exists $\delta > 0$, depending only on $\varepsilon$, the geometric characteristics of $\Omega$, $n$, $p$ and $\|k|_{S^{n-1}}\|_{C^N}$ (where the integer $N = N(n)$ is sufficiently large) with the property that*

$$(6.33) \quad \text{dist}(v, \text{VMO}(\partial\Omega)) < \delta \Rightarrow \begin{cases} T \text{ is well-defined} \\ \text{belongs to } \mathcal{L}(L^p(\partial\Omega)) \text{ and} \\ \text{dist}(T, \text{Comp}(L^p(\partial\Omega))) < \varepsilon, \end{cases}$$

*where the distance in the left-hand side is measured in $\text{BMO}(\partial\Omega)$, and the distance in the right-hand side is measured in $\mathcal{L}(L^p(\partial\Omega))$.*

*In particular, under the same background hypotheses, for every index $p \in (1, \infty)$ one has*

$$(6.34) \quad v \in \text{VMO}(\partial\Omega) \Rightarrow T : L^p(\partial\Omega) \to L^p(\partial\Omega) \text{ is compact.}$$

*Finally, the same claims remain valid when made for the operator*

$$(6.35) \quad T^{\#}f(X) := \lim_{\eta \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\eta}} \langle X - Y, \nu(X) \rangle \times$$

$$\times k(X - Y)f(Y)\, d\sigma(Y), \quad X \in \partial\Omega,$$

*with $k$ as in* (6.31), *as well as for the operator*

$$(6.36) \quad \tilde{T}f(X) := \lim_{\eta \to 0^+} \int_{\substack{Y \in \partial\Omega \\ |X-Y|>\eta}} (\nu(X) - \nu(Y))\tilde{k}(X - Y)f(Y)\, d\sigma(Y), \quad X \in \partial\Omega,$$

*this time provided that*

$$(6.37) \qquad \text{the function } \tilde{k} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \text{ is smooth, odd}$$
$$\text{and homogeneous of degree } 1 - n.$$

The following theorem augments earlier work in this section (compare with Theorem 5.1, Corollary 5.3, Theorem 6.3, Theorem 6.5, and Theorem 6.6).

**THEOREM 6.8.** *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Then given any $p \in (1, \infty)$ there exists $\varepsilon > 0$, depending only on $p$, the Lipschitz character of $\Omega$, $n$, and $\theta$, with the property that if the outward unit normal $\nu$ to $\Omega$ satisfies*

$$(6.38) \quad \limsup_{r \to 0^+} \left\{ \sup_{X \in \partial\Omega} \fint_{B(X,r) \cap \partial\Omega} \fint_{B(X,r) \cap \partial\Omega} |\nu(Y) - \nu(Z)|\, d\sigma(Y)\, d\sigma(Z) \right\} < \varepsilon,$$

*the following claims are true:*

 (i) *all invertibility results from Theorem 5.1 and Corollary 5.3 hold for the given $p$;*
(ii) *the well-posedness results from Theorem 6.3, Theorem 6.5, and Theorem 6.6 hold for the given $p$.*

*As a consequence, all results mentioned above actually hold for any integrability index $p \in (1, \infty)$ if*

$$(6.39) \qquad\qquad\qquad \nu \in \mathrm{VMO}(\partial\Omega)$$

*hence, in particular, if $\Omega$ is a $\mathscr{C}^1$ domain.*

PROOF. The crux of the matter is establishing that

$$(6.40) \qquad \pm\frac{1}{2}I + \dot{K}_\theta = R_0 + R_1 \quad \text{as operators on } \dot{L}^p_{1,0}(\partial\Omega),$$

where $R_0, R_1 \in \mathcal{L}(\dot{L}^p_{1,0}(\partial\Omega))$ satisfy

(6.41) $\qquad\qquad R_0$ is an invertible operator on $\dot{L}^p_{1,0}(\partial\Omega)$,

and

(6.42) $\qquad\qquad \mathrm{dist}(R_1, \mathrm{Comp}(\dot{L}^p_{1,0}(\partial\Omega))) < \|R_0\|_{\mathcal{L}(\dot{L}^p_{1,0}(\partial\Omega))}$,

where the distance in the left-hand side is taken in $\mathcal{L}(\dot{L}^p_{1,0}(\partial\Omega))$. The significance of the decomposition in (6.40) is that, granted (6.41)–(6.42), this readily implies that

(6.43) $\quad \pm\dfrac{1}{2}I + \dot{K}_\theta$ is a Fredholm operator with index zero on $\dot{L}^p_{1,0}(\partial\Omega)$.

With this in hand, earlier arguments then lead to the same type of invertibility results as in (5.2), (5.4) for the given $p$. In turn, the same type of analysis as in the proof of Theorem 5.1 then permits us to also establish analogous invertibility results to those stated in (5.3) and (5.5). Once these results are available, it is straightforward to complete the proof of the claim made in part (i) of the statement of the theorem. Then the claim made in part (ii) of the statement of the theorem becomes a consequence of the invertibility results from part (i), by reasoning as before.

Turning to the justification of the claims made in (6.40)–(6.42), there are two basic aspects we wish to emphasize. First, with equivalence constants depending only on the Lipschitz character of $\Omega$,

(6.44) $\quad \mathrm{dist}(v, \mathrm{VMO}(\partial\Omega))$

$$\approx \limsup_{r\to 0^+}\left\{\sup_{X\in\partial\Omega} \fint_{B(X,r)\cap\partial\Omega} \fint_{B(X,r)\cap\partial\Omega} |v(Y) - v(Z)|\, d\sigma(Y)\, d\sigma(Z)\right\},$$

where the distance in the left-hand side is measured in $\mathrm{BMO}(\partial\Omega)$. A proof of this claim may be found in [12], [14]. Hence, the smallness of the infinitesimal mean oscillation of the unit normal (defined as the limit in the left-hand side of (6.38)) forces the distance from the unit normal $v \in L^\infty(\partial\Omega)$ to the closed subspace $\mathrm{VMO}(\partial\Omega)$, measured in $\mathrm{BMO}(\partial\Omega)$, to be appropriately small. In turn, this opens the door for the close-to-compact criteria described in Theorem 6.7 to apply.

In the implementation of the aforementioned close-to-compact criteria, we find it useful to revert from the operator $\dot{K}_\theta$, considered on $\dot{L}^p_{1,0}(\partial\Omega)$, to the operator $\tilde{K}_\theta$ introduced in (4.60)–(4.66), considered on $L^p_1(\partial\Omega) \oplus L^p(\partial\Omega)$. That this is permissible is ensured by the intertwining result proved in Proposition 4.6, keeping in mind the invertibility of the mapping $\Psi$ established in Proposition 2.2. Thus, the goal becomes identifying various expressions from the makeup of the integral kernel of $\tilde{K}_\theta$ which have the desired algebraic structure (indicated in Theorem 6.7).

According to the arguments in [26, §11], there are four types of integral operators on $L^p(\partial\Omega)$ whose kernels must be shown to have the algebraic structure described in Theorem 6.7, namely:

$$(6.45) \qquad\qquad \partial_{v(Y)}[(\Delta E)(X - Y)],$$

$$(6.46) \qquad\qquad v_i(Y)v_j(Y)v_k(Y)(\partial_i\partial_j\partial_k E)(X - Y),$$

$$(6.47) \qquad\qquad \partial_{\tau_{ij}(Y)}\partial_{\tau_{k\ell}(Y)}\partial_{v(Y)}[E(X - Y)] := I_1 + I_2 + I_3 + I_4$$

where

$$(6.48) \qquad \begin{aligned} I_1 &:= v_i(Y)v_k(Y)v_r(Y)(\partial_j\partial_\ell\partial_r E)(X - Y) \\ I_2 &:= -v_i(Y)v_\ell(Y)v_r(Y)(\partial_j\partial_k\partial_r E)(X - Y) \\ I_3 &:= -v_j(Y)v_k(Y)v_r(Y)(\partial_i\partial_\ell\partial_r E)(X - Y) \\ I_4 &:= +v_j(Y)v_\ell(Y)v_r(Y)(\partial_i\partial_k\partial_r E)(X - Y), \end{aligned}$$

and

$$(6.49) \qquad v_i(Y)v_j(Y)\partial_{\tau_{k\ell}(Y)}[(\partial_i\partial_j E)(X - Y)] - \frac{1}{2}\partial_{\tau_{k\ell}(Y)}[(\Delta E)(X - Y)].$$

Concerning the kernel in (6.45), observe that (with $c_n$ denoting a dimensional constant)

$$(6.50) \qquad\qquad \partial_{v(Y)}[(\Delta E)(X - Y)] = c_n\frac{\langle v(Y), Y - X\rangle}{|X - Y|^n}$$

and this kernel gives rise to a principal-value singular integral operator $T$ of the type described in (6.32) with $k(X) := c_n|X|^{-n}$. Such a function is as in (6.31), so this integral operator satisfies (6.33).

Regarding the kernel in (6.46) we first note that, for each triplet of numbers $a, b, c \in \{1, \ldots, n\}$ and each point $X = (x_1, \ldots, x_n) \in \mathbb{R}^n\backslash\{0\}$,

$$(\partial_a\partial_b\partial_c E)(X) = \frac{c_n}{|X|^n}\left[\delta_{bc}x_a + \delta_{ac}x_b + \delta_{ab}x_c - n\frac{x_ax_bx_c}{|X|^2}\right].$$

Based on this, we may then compute

$$(6.51) \qquad v_i(Y)v_j(Y)v_k(Y)(\partial_i\partial_j\partial_k E)(X - Y)$$

$$= c_n\frac{\langle v(Y), X - Y\rangle}{|X - Y|^n}\left[3 - n\frac{v_i(Y)v_j(Y)(x_i - y_i)(x_j - y_j)}{|X - Y|^2}\right].$$

As such, this kernel gives rise to a principal-value singular integral operator $T$ of the form

$$(6.52) \qquad\qquad T = T_0 + \sum_{i,j=1}^n T_{ij} \circ M_{v_iv_j},$$

where, generally speaking, $M_\eta$ denotes the multiplication by the function $\eta$, and $T_0$, $T_{ij}$, $i, j \in \{1, \ldots, n\}$ are principal-value singular integral operators with kernels

$$(6.53) \qquad 3c_n \frac{\langle v(Y), X - Y \rangle}{|X - Y|^n}$$

and

$$(6.54) \qquad -nc_n \frac{\langle v(Y), X - Y \rangle}{|X - Y|^n} \frac{(x_i - y_i)(x_j - y_j)}{|X - Y|^2}, \quad i, j \in \{1, \ldots, n\},$$

respectively. Since $M_{v_i v_j}$ is a bounded operator on $L^p(\partial\Omega)$, and since the functions

$$(6.55) \quad k_0(X) := 3c_n|X|^{-n} \quad \text{and} \quad k_{ij}(X) := -nc_n x_i x_j |X|^{-n-2}, \quad i, j \in \{1, \ldots, n\},$$

are as in (6.31), the principal-value singular integral operator associated with the kernel (6.46) also satisfies (6.33). Finally, a similar (tedious, but straightforward) analysis shows that the principal-value singular integral operators associated with the kernels from (6.47)–(6.48) and (6.49) fit in the class of operators treated in Theorem 6.7 as well, and this finishes the proof of the theorem. $\qquad \square$

In the theorem below, the multi-layers $\dot{K}_\theta$ and $\dot{S}$ are associated with the bi-Laplacian, $\Delta^2$, as before (cf. Definition 4.1 and (4.50)).

**THEOREM 6.9.** *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Then there exists $\varepsilon > 0$ with the property that the operators*

$$(6.56) \qquad \frac{1}{2}I + \dot{K}_\theta : \dot{B}^{p,q}_{1,s}(\partial\Omega) \to \dot{B}^{p,q}_{1,s}(\partial\Omega),$$

$$(6.57) \qquad -\frac{1}{2}I + \dot{K}_\theta : \dot{B}^{p,q}_{1,s}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega) \to \dot{B}^{p,q}_{1,s}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega),$$

*are isomorphisms whenever $0 < q \leq \infty$ and the indices $p \in (1, \infty)$ and $s \in (0, 1)$ satisfy*

$$(6.58) \qquad \begin{aligned} \frac{n-3-\varepsilon}{2} &< \frac{n-1}{p} - s < \frac{n-1+\varepsilon}{2} \quad \text{when } n \geq 4, \\ 0 &< \frac{1}{p} - \left(\frac{1-\varepsilon}{2}\right)s < \frac{1+\varepsilon}{2} \quad \text{when } n \in \{2, 3\}. \end{aligned}$$

*Moreover, if $p$, $p'$, $q$, $q'$, $s$ satisfy $1 \leq q, q' \leq \infty$, $p, p' \in (1, \infty)$, $s \in (0, 1)$, as well as $1/p + 1/p' = 1/q + 1/q' = 1$ and (6.58), then the operators*

$$(6.59) \qquad \frac{1}{2}I + \dot{K}_\theta^* : (\dot{B}_{1,s}^{p,q}(\partial\Omega))^* \to (\dot{B}_{1,s}^{p,q}(\partial\Omega))^*,$$

$$(6.60) \qquad -\frac{1}{2}I + \dot{K}_\theta^* : (\dot{B}_{1,s}^{p,q}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^* \to (\dot{B}_{1,s}^{p,q}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^*,$$

$$(6.61) \qquad \dot{S} : (\dot{B}_{1,1-s}^{p',q'}(\partial\Omega))^* \to \dot{B}_{1,s}^{p,q}(\partial\Omega) \quad \text{if } n \geq 3 \text{ and } n \neq 4,$$

*are also isomorphisms. Finally, given any $p \in (1, \infty)$, $q \in (0, \infty]$, $s \in (0, 1)$ there exists $\varepsilon > 0$, depending only on $p$, the Lipschitz character of $\Omega$, $n$, and $\theta$, with the property that if the outward unit normal $\nu$ to $\Omega$ satisfies (6.38) then all operators in (6.56)–(6.61) are invertible (assuming $q \geq 1$ in (6.59)–(6.60) and $1/p + 1/p' = 1/q + 1/q' = 1$ in (6.61)). As a consequence, all operators in (6.56)–(6.61) are invertible for any $p \in (1, \infty)$, $q \in (0, \infty]$, and $s \in (0, 1)$ (with the same conventions as above on $q$, $p'$, $q'$) if $\nu \in \text{VMO}(\partial\Omega)$ hence, in particular, if $\Omega$ is a $\mathscr{C}^1$ domain.*

PROOF. Fix $\varepsilon > 0$ as in the proof of Theorem 5.1 and let $I_\varepsilon$ and $I_\varepsilon'$ be as in (5.6) and (5.7), respectively. From (5.2)–(5.3) and the compatibility of inverses stated just below (5.3) we obtain that

$$(6.62) \qquad \begin{aligned} \left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1} &: \dot{L}_{1,0}^{p_0}(\partial\Omega) \to \dot{L}_{1,0}^{p_0}(\partial\Omega) \text{ boundedly } \forall p_0 \in I_\varepsilon', \\ \left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1} &: \dot{L}_{1,1}^{p_1}(\partial\Omega) \to \dot{L}_{1,1}^{p_1}(\partial\Omega) \text{ boundedly } \forall p_1 \in I_\varepsilon. \end{aligned}$$

Based on this and interpolation (cf. [15]) we eventually deduce that

$$(6.63) \qquad \left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1} : \dot{B}_{1,s}^{p,q}(\partial\Omega) \to \dot{B}_{1,s}^{p,q}(\partial\Omega) \text{ is bounded}$$

for every $q \in (0, \infty]$ and $p, s$ as in (6.58).

Since $\frac{1}{2}I + \dot{K}_\theta : \dot{B}_{1,s}^{p,q}(\partial\Omega) \to \dot{B}_{1,s}^{p,q}(\partial\Omega)$ is also bounded, thanks to (4.16), we finally arrive at the conclusion that the operator in (6.56) is an isomorphism whenever $q \in (0, \infty]$ and $p, s$ are as in (6.58). In fact, all the other claims pertaining to (6.57)–(6.61) may be handled analogously. Finally, under the additional assumption that (6.38) holds, we reason similarly, starting with the invertibility results proved in Theorem 6.8. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The invertibility results established in Theorem 6.9 are the key ingredients in the proofs of the well-posedness theorems discussed in the remaining portion of this section. We begin by treating the inhomogeneous Dirichlet problem for the bi-Laplacian with boundary data from Whitney-Besov spaces.

THEOREM 6.10. *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Then there exists $\varepsilon > 0$ such*

*that the inhomogeneous Dirichlet problem*

(6.64)
$$\begin{cases} u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega), \\ \Delta^2 u = w \in B^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ (\mathrm{Tr}\, u, \mathrm{Tr}(\nabla u)) = \dot{f} \in \dot{B}^{p,q}_{1,s}(\partial\Omega), \end{cases}$$

*is well-posed whenever $0 < q \le \infty$ while $p \in (1, \infty)$ and $s \in (0,1)$ satisfy*

(6.65)
$$\frac{n-3-\varepsilon}{2} < \frac{n-1}{p} - s < \frac{n-1+\varepsilon}{2} \quad \text{when } n \ge 4,$$

$$0 < \frac{1}{p} - \left(\frac{1-\varepsilon}{2}\right)s < \frac{1+\varepsilon}{2} \quad \text{when } n \in \{2, 3\}.$$

*Moreover, if $w = 0$ then the unique solution $u$ of (6.64) admits the following integral representation*

(6.66)
$$u(X) = \dot{\mathcal{D}}_\theta\left[\left(\frac{1}{2}I + \dot{K}_\theta\right)^{-1}\dot{f}\right](X), \quad \forall X \in \Omega.$$

*Furthermore, given any $p \in (1, \infty)$, $q \in (0, \infty]$, $s \in (0,1)$ there exists $\varepsilon > 0$, depending only on $p$, $q$, $s$, the Lipschitz character of $\Omega$, $n$, and $\theta$, with the property that if the outward unit normal $v$ to $\Omega$ satisfies (6.38) then the problem (6.64) is well-posed. As a consequence, the problem (6.64) is well-posed for any $p \in (1, \infty)$, $q \in (0, \infty]$, and $s \in (0,1)$ if $v \in \mathrm{VMO}(\partial\Omega)$ hence, in particular, if $\Omega$ is a $\mathscr{C}^1$ domain.*

*Finally, similar results are valid for the inhomogeneous Dirichlet problem on Triebel-Lizorkin spaces, i.e., for*

(6.67)
$$\begin{cases} u \in F^{p,q}_{s+\frac{1}{p}+1}(\Omega), \\ \Delta^2 u = w \in F^{p,q}_{s+\frac{1}{p}-3}(\Omega), \\ (\mathrm{Tr}\, u, \mathrm{Tr}(\nabla u)) = \dot{f} \in \dot{B}^{p,p}_{1,s}(\partial\Omega). \end{cases}$$

PROOF. All well-posedness claims may be proved by relying mapping properties for multi-layer potential operators on Besov and Triebel-Lizorkin scales (cf. [15]), and on the invertibility results from Theorem 6.9. □

There are three corollaries to the above theorem which we wish to single out. To state the first, recall the weighted Sobolev spaces from (2.37). Mapping properties for generic double multi-layers acting from Besov spaces and taking values in these weighted Sobolev spaces have been proved in [15]. Based on this and Theorem 6.10 we may then conclude the following.

COROLLARY 6.11. *Suppose that $\Omega \subset \mathbb{R}^n$, with $n \ge 2$, is a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon > 0$ with the property that*

*whenever $0 < q \leq \infty$ and $p \in (1, \infty)$, $s \in (0, 1)$ satisfy (6.65), one has*

$$(6.68) \qquad \|u\|_{B^{p,q}_{s+\frac{1}{p}+1}(\Omega)} \approx \|\mathrm{Tr}\, u\|_{\dot{B}^{p,q}_{1,s}(\partial\Omega)} + \|\mathrm{Tr}(\nabla u)\|_{\dot{B}^{p,q}_{1,s}(\partial\Omega)},$$

*uniformly for biharmonic functions $u$ belonging to $B^{p,q}_{s+1/p+1}(\Omega)$, and*

$$(6.69) \qquad \|u\|_{W^{2,p}_{1-s-\frac{1}{p}}(\Omega)} \approx \|u\|_{F^{p,q}_{s+\frac{1}{p}+1}(\Omega)} \approx \|\mathrm{Tr}\, u\|_{\dot{B}^{p,p}_{1,s}(\partial\Omega)} + \|\mathrm{Tr}(\nabla u)\|_{\dot{B}^{p,p}_{1,s}(\partial\Omega)},$$

*uniformly for biharmonic functions $u$ belonging to $F^{p,q}_{s+\frac{1}{p}+1}(\Omega)$.*

Here is the second corollary alluded to above.

COROLLARY 6.12. *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary. Then there exists $\varepsilon > 0$ with the following significance. Whenever $p \in (1, \infty)$ satisfies*

$$(6.70) \qquad \begin{aligned} \frac{2n}{n+1+\varepsilon} &< p < \frac{2n}{n-1-\varepsilon} \quad \text{if } n \geq 3, \\[2mm] \frac{3}{2+\varepsilon} &< p < \frac{3}{1-\varepsilon} \quad \text{if } n = 2, \end{aligned}$$

*one can find a finite constant $C = C(\Omega, p) > 0$ with the property that if $p' \in (1, \infty)$ is such that $1/p + 1/p' = 1$ then for every function $v \in \mathring{W}^{2,p}(\Omega)$,*

$$(6.71) \qquad \|v\|_{W^{2,p}(\Omega)} \leq C \sup\left\{ \int_\Omega \Delta v \Delta u\, dX : u \in \mathscr{C}^\infty_c(\Omega) \text{ with } \|u\|_{W^{2,p'}(\Omega)} \leq 1 \right\}.$$

*Moreover, if the outward unit normal $v$ to $\Omega$ belongs to $\mathrm{VMO}(\partial\Omega)$ (hence, in particular, if $\Omega$ is a domain of class $\mathscr{C}^1$), it follows that (6.71) holds for any $p \in (1, \infty)$.*

PROOF. Let $\varepsilon > 0$ be as in Theorem 6.10 and assume that the exponent $p$ is as in (6.70) and that $1/p + 1/p' = 1$. Finally, pick an arbitrary function $v \in \mathring{W}^{2,p}(\Omega)$. Note that by (2.39) and (2.40),

$$(6.72) \qquad (\mathring{W}^{2,p}(\Omega))^* = W^{-2,p'}(\Omega) = F^{p',2}_{-2}(\Omega),$$

Hence, with $\langle \cdot, \cdot \rangle$ standing for a natural duality pairing, there exists a finite constant $C = C(\Omega, p) > 0$ with the property that

$$(6.73) \qquad \|v\|_{W^{2,p}(\Omega)} \leq C \sup\{\langle v, h \rangle : h \in F^{p',2}_{-2}(\Omega) \text{ with } \|h\|_{F^{p',2}_{-2}(\Omega)} \leq 1\}.$$

Fix some $h \in F^{p',2}_{-2}(\Omega)$ with $\|h\|_{F^{p',2}_{-2}(\Omega)} \leq 1$. The incisive observation is that, together, $p' \in (1, \infty)$ and $s := 1/p \in (0, 1)$ satisfy the conditions in (6.65) and, as such, Theorem 6.10 (augmented with (2.40)–(2.41)) guarantees the existence of

some function $u \in \mathring{W}^{2,p'}(\Omega)$ with the property that

$$(6.74) \qquad \Delta^2 u = h \quad \text{and} \quad \|u\|_{W^{2,p'}(\Omega)} \le C(\Omega, p).$$

Consequently,

$$(6.75) \qquad \langle v, h \rangle = \langle v, \Delta^2 u \rangle = \langle \Delta v, \Delta u \rangle = \int_\Omega \Delta v \Delta u \, dX.$$

At this stage, (6.71) follows from (6.73), (6.74), (6.75), and (2.38). □

It is instructive to formulate the well-posedness results from Theorem 6.10 in a fashion which emphasizes the smoothing properties of the Green operator for the inhomogeneous Dirichlet boundary value problem for the bi-Laplacian. Recall that this Green operator, call it $\mathbf{G}$, is formally defined as

$$(6.76) \qquad \mathbf{G}w := u \text{ where } u \text{ solves } \quad \Delta^2 u = w \text{ in } \Omega, \quad u = \partial_v u = 0 \text{ on } \partial\Omega.$$

Variational considerations based on the Lax-Milgram lemma and trace results ultimately yield that

$$(6.77) \qquad \mathbf{G} : W^{-2,2}(\Omega) \to \mathring{W}^{2,2}(\Omega) \quad \text{isomorphically,}$$

and we wish to explore the extent to which the Green operator continues to be smoothing of order 4 when considered on more general scales of Besov and Triebel-Lizorkin spaces. In this regard, we have the following result.

COROLLARY 6.13. *Assume that $\Omega \subset \mathbb{R}^n$, with $n \ge 2$, is a bounded Lipschitz domain with connected boundary. Then there exists some $\varepsilon = \varepsilon(\Omega) > 0$ such that the Green operators*

$$(6.78) \qquad \mathbf{G} : B^{p,q}_{s+\frac{1}{p}-3}(\Omega) \to \{u \in B^{p,q}_{s+\frac{1}{p}+1}(\Omega) : (\mathrm{Tr}\, u, \mathrm{Tr}(\nabla u)) = \dot{0}\},$$

*and*

$$(6.79) \qquad \mathbf{G} : F^{p,q}_{s+\frac{1}{p}-3}(\Omega) \to \{u \in F^{p,q}_{s+\frac{1}{p}+1}(\Omega) : (\mathrm{Tr}\, u, \mathrm{Tr}(\nabla u)) = \dot{0}\},$$

*are isomorphisms whenever $p \in (1, \infty)$ and $s \in (0, 1)$ satisfy*

$$(6.80) \qquad \begin{aligned} \frac{n-3-\varepsilon}{2} &< \frac{n-1}{p} - s < \frac{n-1+\varepsilon}{2} \quad \text{if } n \ge 4, \\[2mm] 0 &< \frac{1}{p} - \left(\frac{1-\varepsilon}{2}\right)s < \frac{1+\varepsilon}{2} \quad \text{if } n \in \{2,3\}, \end{aligned}$$

*and $0 < q \le \infty$ for the Besov scale, and $\min\{p, 1\} \le q < \infty$ for the Triebel-Lizorkin scale.*

*In particular,*

$$(6.81) \qquad \mathbf{G} : W^{-2,p}(\Omega) \to \mathring{W}^{2,p}(\Omega) \quad \text{isomorphically,}$$

*provided*

$$(6.82) \quad \begin{aligned} \frac{2n}{n+1} - \varepsilon < p < \frac{2n}{n-1} + \varepsilon \quad \textit{if } n \geq 3, \\ \frac{3}{2} - \varepsilon < p < 3 + \varepsilon \quad \textit{if } n = 2. \end{aligned}$$

*Furthermore, given any $p \in (1, \infty)$, $q \in (0, \infty)$, and $s \in (0, 1)$ there exists $\varepsilon > 0$, depending only on $p$, $q$, $s$, and the Lipschitz character of $\Omega$ with the property that if the outward unit normal $v$ to $\Omega$ satisfies (6.38) then the operators (6.78)–(6.79) are isomorphisms (also assuming the inequality $\min\{p, 1\} \leq q < \infty$ in the case of (6.79)). As a consequence, the operators (6.78)–(6.79) are isomorphisms for any $p \in (1, \infty)$, $q \in (0, \infty)$, and $s \in (0, 1)$ (also assuming that $\min\{p, 1\} \leq q < \infty$ in the case of (6.79)) if $v \in \mathrm{VMO}(\partial\Omega)$ hence, in particular, if $\Omega$ is a $\mathscr{C}^1$ domain.*

PROOF. The fact that the operator in (6.78) is an isomorphism follows from the well-posedness of (6.64) and the definition of the Green operator in (6.76). The argument for (6.79) is similar, relying on the well-posedness of (6.67). Having proved this, (6.81) follows by specializing (6.79) to the case when $s + 1/p = 1$ and $q = 2$ and keeping in mind (2.40)–(2.41). The remaining claims in the statement of the corollary are established similarly, making use of appropriate well-posedness results from Theorem 6.16.                                                     □

Regarding the optimality of Theorem 6.10, we have the following result.

PROPOSITION 6.14. *In the class of Lipschitz domains in $\mathbb{R}^n$, the range of indices $p$, $s$ in (6.65) for which the inhomogeneous Dirichlet problems (6.64), (6.67) are well-posed is sharp when $n \in \{4, 5\}$.*

PROOF. We begin by recording a consequence of [21, Theorem 2.6, p. 623]: if $n \in \{2, 3, 4, 5\}$ then for each $\theta \in (0, \pi)$ there exist a bounded Lipschitz domain $\Omega_\theta$ in $\mathbb{R}^n$, with connected boundary, such that $0 \in \partial\Omega_\theta$ and

$$(6.83) \quad \Omega_\theta \cap B(0, 1) = \{X = (x_1, \dots, x_n) \in B(0, 1) :$$
$$x_n < (\cot\theta)\sqrt{x_1^2 + \cdots + x_{n-1}^2}\},$$

along with a non-zero function $u : \Omega_\theta \to \mathbb{R}$ satisfying

$$(6.84) \qquad u \in C^\infty \text{ in } \overline{\Omega_\theta} \text{ away from the origin,}$$

$$(6.85) \qquad u(X) \equiv |X|^{\lambda(\theta)} \varphi(X/|X|) \text{ for } X \text{ near } 0,$$

$$(6.86) \qquad \varphi \in C^\infty(S^{n-1}) \text{ and } \lambda(\theta) \searrow \frac{5-n}{2} \text{ as } \theta \searrow 0,$$

$$(6.87) \qquad \Delta^2 u \in C^\infty(\overline{\Omega_\theta}), \quad u = \partial_v u = 0 \text{ on } \partial\Omega_\theta.$$

Note that, in concert with Lemma 2.4 in [15], conditions (6.84)–(6.87) ensure that the function $u \in W^{2,2}(\Omega)$. Hence, if we set $f := \Delta^2 u \in C^\infty(\overline{\Omega_\theta})$, then $\mathbf{G}f = u$. On the other hand, (6.84)–(6.87) and Lemma 2.4 in [15] give that for any $p, q \in (0, \infty)$ and $s > n(1/p - 1)_+$

$$(6.88) \quad u \in F^{p,q}_{1+s+1/p}(\Omega) \Leftrightarrow 1 + s + \frac{1}{p} < \frac{n}{p} + \lambda(\theta) \Leftrightarrow 1 - \lambda(\theta) < \frac{n-1}{p} - s,$$

and note that, by (6.86),

$$(6.89) \qquad\qquad 1 - \lambda(\theta) \nearrow \frac{n-3}{2} \quad \text{as } \theta \searrow 0.$$

This proves that, when $n \in \{4, 5\}$, the lower bound for $\frac{n-1}{p} - s$ in (6.65) is sharp as far as the well-posedness of (6.67) is concerned. In fact, the same argument also shows that the aforementioned lower bound is optimal in relation to the operator $\mathbf{G}$ being boundedly invertible in the context of (6.79) if $n \in \{4, 5\}$. In the later setting, by relying on the self-adjointness of the Green operator $\mathbf{G}$, it follows by duality that the upper bound for $\frac{n-1}{p} - s$ in (6.80) is also sharp when $n \in \{4, 5\}$. Ultimately, this result implies that the range in (6.65) is sharp as far as the well-posedness of (6.67) is concerned. Finally, the argument on the scale of Besov spaces is similar. $\qquad\qquad\square$

We conclude with a well-posedness result for the inhomogeneous Neumann problem for the bi-Laplacian with boundary data from duals of Whitney-Besov spaces. This requires some preparations. For starters, let us agree to associate to any functional $w \in (\mathcal{X}^{p,q}_s(\Omega))^*$ (where either $\mathcal{X} = B$, or $\mathcal{X} = F$, $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, and $s \in \mathbb{R}$, $1 \le p, q \le \infty$) the distribution $w\lfloor_\Omega \in \mathcal{D}'(\Omega)$ defined by

$$(6.90) \qquad\qquad \langle w\lfloor_\Omega, \varphi \rangle := \langle w, \varphi \rangle, \quad \forall \varphi \in \mathscr{C}^\infty_c(\Omega),$$

where the brackets in the left-hand side correspond to distributional pairing, and the brackets in the right-hand side stand for the natural (Banach space) duality pairing. The reader is alerted to the fact that, while linear and continuous, generally speaking

$$(6.91) \quad \text{the assignment } (A^{p,q}_s(\Omega))^* \ni w \mapsto w\lfloor_\Omega \in \mathcal{D}'(\Omega) \text{ is not injective.}$$

The definition below is modeled upon Green's formula for the bi-Laplacian in the case of sufficiently regular functions.

**DEFINITION 6.15.** *Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and recall the writing of the bi-Laplacian from (3.4) corresponding to the choice of the tensor coefficient $A_\theta$ as in (3.1)–(3.2) for some fixed $\theta \in \mathbb{R}$. Finally, suppose that $1 < p, q < \infty$, $0 < s < 1$ and let $p', q' \in (1, \infty)$ be such that $1/p + 1/p' = 1/q + 1/q' = 1$. In*

*this context, define the conormal derivative operator $\partial_\nu^{A_\theta}$ as the mapping acting on the space*

$$(6.92) \quad \{(u, w) \in \mathcal{X}_{2-s+1/p}^{p,q}(\Omega) \oplus (\mathcal{X}_{1+s+1/p'}^{p',q'}(\Omega))^* : \Delta^2 u = w \lfloor_\Omega \text{ in } \mathcal{D}'(\Omega)\}$$

*(where the convention introduced in (6.90) has been used), and taking values in the space $(\dot{B}_{1,s}^{p',q'}(\partial\Omega))^*$ if $\mathcal{X} = B$, and $(\dot{B}_{1,s}^{p',p'}(\partial\Omega))^*$ if $\mathcal{X} = F$, by setting, for each Whitney array $\dot{f}$ in these spaces*

$$(6.93) \qquad \langle \partial_\nu^{A_\theta}(u, w), \dot{f} \rangle := - \sum_{|\alpha|=|\beta|=2} \langle A_{\alpha\beta}(\theta)\partial^\beta u, \partial^\alpha F \rangle + \langle w, F \rangle,$$

*where $F \in \mathcal{X}_{1+s+1/p'}^{p',q'}(\Omega)$ is such that $(\operatorname{Tr} F, \operatorname{Tr}(\nabla F)) = \dot{f}$. In (6.93), the first bracket denotes the duality pairing between elements of the space $\mathcal{X}_{-s+1/p}^{p,q}(\Omega)$ and elements in its dual, $\mathcal{X}_{s-1+1/p'}^{p',q'}(\Omega)$, while the second bracket denotes the duality pairing between elements of the space $\mathcal{X}_{1+s+1/p'}^{p',q'}(\Omega)$ and its dual, $(\mathcal{X}_{1+s+1/p'}^{p',q'}(\Omega))^*$.*

It is important to point out that definition (6.93) is independent of the choice of the extension $F$ of $\dot{f}$ (also, such an extension always exists). Here we also wish to note that, in general, definition (6.93) of $\partial_\nu^{A_\theta}(u, w)$ is not an ordinary generalization of the conormal derivative $\partial_\nu^{A_\theta} u$ considered in a pointwise sense in (3.24) when $u$ is regular enough, since this is not the case here. In fact, it is more appropriate to regard the former as a "renormalization" of the latter, in a fashion that depends strongly on the choice of an extension of the distribution $\Delta^2 u \in \mathcal{D}'(\Omega)$ to a functional $w \in (\mathcal{X}^{p',q'}(\Omega))^*$. This phenomenon, which may be traced back to (6.91), also accounts for the more elaborate notation $\partial_\nu^{A_\theta}(u, w)$, presently used in order to stress the dependence of this object on $w$.

In anticipation to stating the aforementioned well-posedness result, we also need to discuss some notation. In order to be specific, fix $p, q \in (1, \infty)$, $s \in (0, 1)$ and suppose that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Also, consider a functional $\Lambda \in (\dot{B}_{1,1-s}^{p',q'}(\partial\Omega))^*$ satisfying the compatibility condition

$$(6.94) \qquad\qquad \langle \Lambda, \dot{P} \rangle = 0 \quad \text{for each } \dot{P} \in \dot{\mathcal{P}}(\partial\Omega).$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Then define

$$(6.95) \qquad\qquad \tilde{\Lambda} \in (\dot{B}_{1,1-s}^{p',q'}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^*,$$

by setting

$$(6.96) \qquad\qquad \langle \tilde{\Lambda}, [\dot{f}] \rangle := \langle \Lambda, \dot{f} \rangle, \quad \forall \dot{f} \in \dot{B}_{1,1-s}^{p',q'}(\partial\Omega),$$

where $[\dot{f}]$ denotes the equivalence class of the Whitney array $\dot{f} \in \dot{B}_{1,1-s}^{p',q'}(\partial\Omega)$ in the quotient space $\dot{B}_{1,1-s}^{p',q'}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$. Thanks to (6.94) this definition is unambiguous. Going further, let

$$(6.97) \qquad\qquad \pi : \dot{B}_{1,1-s}^{p',q'}(\partial\Omega) \to \dot{B}_{1,1-s}^{p',q'}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$$

denote the canonical projection, taking an arbitrary Whitney array $\dot{f} \in \dot{B}_{1,1-s}^{p';q'}(\partial\Omega)$ into $[\dot{f}] \in \dot{B}_{1,1-s}^{p';q'}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega)$. Its adjoint then becomes

$$(6.98) \qquad \pi^* : (\dot{B}_{1,1-s}^{p';q'}(\partial\Omega)/\dot{\mathcal{P}}(\partial\Omega))^* \to (\dot{B}_{1,1-s}^{p';q'}(\partial\Omega))^*.$$

We are now ready to state and prove the following theorem.

**THEOREM 6.16.** *Assume that $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, is a bounded Lipschitz domain with connected boundary, and fix $\theta \in \mathbb{R}$ with $\theta > -\frac{1}{n}$. Then there exists $\varepsilon > 0$ such that the inhomogeneous Neumann problem for the biharmonic operator*

$$(6.99) \qquad \begin{cases} u \in B_{1+s+1/p}^{p,q}(\Omega), \\ \Delta^2 u = w\lfloor_\Omega, \quad w \in (B_{2-s+1/p'}^{p';q'}(\Omega))^*, \\ \partial_\nu^{A_\theta}(u,w) = \Lambda \in (\dot{B}_{1,1-s}^{p';q'}(\partial\Omega))^*, \end{cases}$$

*where the boundary datum satisfies the necessary compatibility condition*

$$(6.100) \qquad \langle \Lambda, \dot{P} \rangle = \langle w, P \rangle \quad \text{for each } P \in \mathcal{P}(\Omega),$$

*is well-posed, with uniqueness understood modulo polynomials of degree $\leq 1$, whenever $s \in (0,1)$ and $p, p', q, q' \in (1, \infty)$, satisfy $1/p + 1/p' = 1/q + 1/q' = 1$, and*

$$(6.101) \qquad \begin{aligned} \frac{-n+1-\varepsilon}{2} &< -\frac{n-1}{p} + s < \frac{-n+3+\varepsilon}{2} \quad \text{when } n \geq 4, \\ -\frac{1+\varepsilon}{2} &< -\frac{1}{p} + \left(\frac{1-\varepsilon}{2}\right)s < 0 \quad \text{when } n \in \{2,3\}. \end{aligned}$$

*Moreover, if $w = 0$ then a solution $u$ of (6.99) is given by the following integral formula*

$$(6.102) \qquad u(X) = \dot{\mathcal{S}}\left[\pi^*\left(-\frac{1}{2}I + \dot{K}_\theta^*\right)^{-1}\tilde{\Lambda}\right](X), \quad \forall X \in \Omega,$$

*where $\pi^*$ and $\tilde{\Lambda}$ are as in (6.97)–(6.98) and (6.95)–(6.96), respectively.*

*Furthermore, given any $p \in (1, \infty)$, $q \in (1, \infty)$, and $s \in (0,1)$ there exists $\varepsilon > 0$, depending only on $p, q, s$, the Lipschitz character of $\Omega$, $n$, and $\theta$, with the property that if the outward unit normal $\nu$ to $\Omega$ satisfies (6.38) then the problem (6.99) is well-posed. As a consequence the problem (6.99) is well-posed for any $p \in (1, \infty)$, $q \in (1, \infty)$, and $s \in (0,1)$ if $\nu \in \text{VMO}(\partial\Omega)$ hence, in particular, if $\Omega$ is a $\mathscr{C}^1$ domain.*

*Finally, similar results hold for the inhomogeneous Neumann problem formulated in Triebel-Lizorkin spaces, i.e., for*

$$(6.103) \quad \begin{cases} u \in F^{p,q}_{1+s+1/p}(\Omega), \\ \Delta^2 u = w\lfloor_\Omega, \quad w \in (F^{p',q'}_{2-s+1/p'}(\Omega))^*, \\ \partial^{A_\theta}_\nu(u,w) = \Lambda \in (\dot{B}^{p',p'}_{1,1-s}(\partial\Omega))^*, \end{cases}$$

*where the boundary datum satisfies the necessary compatibility condition* (6.100).

PROOF. The well-posedness claims formulated in the statement of the theorem may be justified making use of boundedness properties of the multi-layer operators involved as well as the invertibility results from Theorem 6.9. □

## REFERENCES

[1] S. AGMON, *Remarks on self-adjoint and semi-bounded elliptic boundary value problems*, Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), Jerusalem Academic Press, Jerusalem, 1960, pp. 1–13.

[2] C. BOURGEOIS - S. NICAISE, *Prewavelet approximations for a system of boundary integral equations for plates with free edges on polygons*, Math. Meth. Appl. Sci., 21 (1998), 1233–1267.

[3] M. BOURLARD - S. NICAISE, *Abstract Green formula and applications to boundary integral equations*, Num. Funct. Anal. Opt., 18 (1997), 667–689.

[4] R. DAUTRAY - J.-L. LIONS, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 2: Functional and Variational Methods*, Springer-Verlag, Berlin, Heidelberg, 2000.

[5] G. FICHERA, *On some general integration methods employed in connection with linear differential equations*, J. Math. Physics, 29 (1950), 59–68.

[6] G. FICHERA, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Mathematics, Vol. 8, Springer-Verlag, Berlin-New York, 1965.

[7] G. FICHERA - P. E. RICCI, *The single layer potential approach in the theory of boundary value problems for elliptic equations*, pp. 39–50 in "Function Theoretic Methods for Partial Differential Equations" (Proc. Internat. Sympos., Darmstadt, 1976), Lecture Notes in Math., Vol. 561, Springer, Berlin, 1976.

[8] M. FRAZIER - B. JAWERTH, *A discrete transform and decompositions of distribution spaces*, J. Funct. Anal., Vol. 93 No. 1 (1990), 34–170.

[9] M. FRAZIER - B. JAWERTH - G. WEISS, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, Vol. 79, AMS, Providence, RI, 1991.

[10] F. GAZZOLA - H.-C. GRUNAU - G. SWEERS, *Polyharmonic Boundary Value Problems*, Vol. 1991, Lecture Notes in Mathematics, Springer-Verlag, 2010.

[11] J. GIROIRE - J.-C. NÉDÉLEC, *A new system of boundary integral equations for plates with free edges*, Mathematical Methods in the Applied Sciences, 18 (1995), no. 10, 755–772.

[12] S. HOFMANN - M. MITREA - M. TAYLOR, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*, International Math. Research Notices, 2010 (2010), 2567–2865.

[13] J. KILTY - Z. SHEN, *A bilinear estimate for biharmonic functions in Lipschitz domains*, to appear in Math. Ann., (2012).

[14] D. MITREA - I. MITREA - M. MITREA - S. MONNIAUX, *A Treatise on the Theory of Elliptic Boundary Value Problems, Singular Integral Operators, and Smoothness Spaces in Rough Domains*, book manuscript, 2012.

[15] I. MITREA - M. MITREA, *Multi-Layer Potentials and Boundary Problems for Higher-Order Elliptic Systems in Lipschitz Domains*, Lecture Notes in Mathematics, Vol. 2063, Springer-Verlag, 2012.

[16] M. MITREA - M. WRIGHT, *Boundary Value Problems for the Stokes System in Arbitrary Lipschitz Domains*, Astérisque, Societé Mathématique de France, Vol. 344, 2012.

[17] A. NADAI, *Theory of Flow and Fracture of Solids*, Vol. II, McGraw-Hill, New York, 1963.

[18] C. NAZARET, *A system of boundary integral equations for polygonal plates with free edges*, Math. Methods Appl. Sci., 21 (1998), no. 2, 165–185.

[19] J. NEČAS, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et Cie, Éditeurs, Paris, Academia, Éditeurs, Prague, 1967.

[20] S. NICAISE, *Polygonal interface problems for the biharmonic operator*, Mathematical Methods in the Applied Sciences, 17 (1994), 21–39.

[21] J. PIPHER - G. C. VERCHOTA, *Maximum principles for the polyharmonic equation on Lipschitz domains*, Potential Anal., 4 (1995), no. 6, 615–636.

[22] T. RUNST - W. SICKEL, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators*, de Gruyter, Berlin, New York, 1996.

[23] G. SCHMIDT, *Boundary integral operators for plate bending in domains with corners*, Zeitschrift für Analysis und ihre Anwendungen (Journal for Analysis and its Applications), 20 (2001), no. 1, 131–154.

[24] Z. SHEN, *The $L^p$ Boundary Value Problems on Lipschitz Domains*, Adv. Math., 216 (2007), 212–254.

[25] H. TRIEBEL, *Theory of function spaces. II*, Monographs in Mathematics, 84. Birkhäuser Verlag, Basel, 1992.

[26] G. VERCHOTA, *The biharmonic Neumann problem in Lipschitz domains*, Acta Math., 194 (2005), 217–279.

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