Rend. Lincei Mat. Appl. 24 (2013), 527–548
DOI 10.4171/RLM/665

Functional Analysis — *On the regularity theory of bi-Sobolev mappings*, by C. Capone, M. R. Formica, R. Giova and R. Schiattarella, communicated on 21 June 2013.

Abstract. — Let $\Omega, \Omega' \subset \mathbb{R}^2$ be bounded domains and let $f : \Omega \to \Omega'$ be a bi-Sobolev mapping. We provide regularity properties for the inverse map f^{-1} under suitable assumptions on q-distortion function of f.

Key words: Bi-Sobolev mappings, q-distortion function, Lusin $\mathcal{N}$ condition.

2010 Mathematics Subject Classification: 46E30, 46E99.

1. Introduction

Let $\Omega, \Omega' \subset \mathbb{R}^2$ be bounded domains and let $f : \Omega \to \Omega' = f(\Omega)$ be a homeomorphism. In the last few years, the relations between regularity properties of f and those of its inverse have attracted a great interest (see [1], [2], [24], [18], [31], [32], [23]).

In the class of planar homeomorphisms an important role is played by bi-Sobolev mappings, originally proposed in [27]. We recall that a homeomorphism f is called a bi-Sobolev mapping if $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ and its inverse $f^{-1} \in W^{1,1}(\Omega', \mathbb{R}^2)$.

Our interest in this class of mappings is motivated by their close connection with homeomorphisms of finite distortion. More precisely, if $n = 2$, we recall that bi-Sobolev mappings are exactly those that have finite distortion ([27]). On the other hand, the inverse of a homeomorphism $W^{1,1}$ belongs to BV only (see [26], [7], [9]).

The theory of mappings of finite distortion has received considerable attention in recent years thanks to its connection with elliptic partial differential equations (see e.g. [1], [6], [3], [16]).

We say that a homeomorphism $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ has finite distortion if its Jacobian is strictly positive on a set where $|Df|$ does not vanish, i.e.,

$$J_f(x) = 0 \quad \Rightarrow \quad |Df(x)| = 0 \quad \text{a.e. } x \in \Omega,$$

where $|Df|$ denotes the operator norm of the gradient and J_f is the Jacobian determinant of f.

For $q \in (1, +\infty)$, we define the q-distortion function of f as

$$K_{q,f}(x) = \begin{cases} \dfrac{|Df(x)|^q}{J_f(x)}, & \text{if } J_f(x) > 0; \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

In the case $q = 2$, $K_{q,f}$ coincides with the classical distortion function K_f.

If we further assume that $K_{q,f}$ is bounded, we recover the class of q-quasiconformal mappings recently introduced by Hencl and Kleprlík ([22]).

In [2] the authors studied the integrability condition on K_f which guarantees better regularity for the inverse of f. In particular they showed that, if $f \in W^{1,2}$ is a homeomorphism of finite distortion such that $K_f \in L^1$, then the inverse map f^{-1} belongs to $W^{1,2}$. Later, in [24] was established the same result under the relaxed assumption $f \in W^{1,1}$.

In this direction, some interpolation-type results are given in the setting of Orlicz-Zygmund classes (see [18], [31]) and in the setting of grand Lebesgue spaces (see [27]).

Here, we are interested in establishing some regularity properties of the inverse of a homeomorphism under suitable integrability conditions on the q-distortion function. We state some results in this direction.

THEOREM 1.1. *Let $1 < q \le 2$ and let f be a bi-Sobolev mapping. Then*

$$K_{q,f} \in L^{\frac{1}{q-1}}(\Omega)$$

if, and only if,

$$|Df^{-1}| \in L^{\frac{q}{q-1}}(\Omega').$$

Moreover,

$$\int_{\Omega'} |Df^{-1}|^{\frac{q}{q-1}}\, dy = \int_{\Omega} (K_{q,f})^{\frac{1}{q-1}}\, dx. \tag{2}$$

For $q = 2$ Theorem 1.1 reduces to the result due to [24] already mentioned.

THEOREM 1.2. *Let $1 < q \le 2$, $1 \le p < \infty$ and let f be a bi-Sobolev map, such that*

$$K_{q,f} \in L^{\frac{p}{q-1}}(\Omega).$$

Then

$$J_{f^{-1}} \in L \log^p L(\Omega'), \tag{3}$$

$$|Df^{-1}| \in L^{\frac{q}{q-1}} \log^{p-1} L(\Omega'). \tag{4}$$

Moreover, we examine the case in which $K_{q,f}$ belongs to classes of functions not too far from $L^{\frac{1}{q-1}}$; we refer to Zygmund class and grand Lebesgue space (see Section 2 and Section 3 for definitions and results).

In [29] the author proved that if $K_{q,f} \in L^{\frac{1}{q-1}}(\Omega)$ then f^{-1} satisfies Lusin $\mathcal{N}$ condition, i.e. f^{-1} maps every set of measure zero to a set of measure zero. It is worth pointing out that this property is proved by assuming that $K_{q,f}$ is zero on the set where J_f vanishes. Nevertheless, one can easily see that it is also true in the case of definition (1).

We prove that this condition can be relaxed into the following one

$$\sup_{0<\varepsilon<1} \Big(\varepsilon \int_\Omega K_{q,f}^{\frac{1}{q-1}-\varepsilon}(x)\, dx \Big)^{\frac{1}{\frac{1}{q-1}-\varepsilon}} < \infty, \tag{5}$$

when $f \in W^{1,p}$, $1 < p \leq q < 2$, is a homeomorphism of finite distortion (see Corollary 3.6). When (5) occurs, we write $K_{q,f} \in L^{\frac{1}{q-1})}(\Omega)$.

Finally, in Section 4, an integrability property for the q-distortion of the composition map $g \circ f$ of two bi-Sobolev mappings is given.

The following result is a particular case of the more general Theorem 4.3.

Theorem 1.3. *Let $1 < q \leq 2$, $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ be bi-Sobolev maps. If*

$$K_{q,f} \in L^2(\Omega) \quad \text{and} \quad (K_{q,g})^{\frac{1}{q-1}} \in \mathrm{EXP}(\Omega')$$

then the composition $h = g \circ f \in W^{1,1}(\Omega; \mathbb{R}^2)$ is a bi-Sobolev map and

$$K_{q,h} \in L^1(\Omega).$$

2. Preliminaries

2.1. *Some function spaces*

Let us recall the definitions of some function spaces, which will be useful in the sequel.

Let P be an increasing function from $P(0) = 0$ to $\lim_{t\to\infty} P(t) = \infty$ and continuously differentiable on $(0, \infty)$. We denote by $L^P(\Omega)$ the Orlicz space generated by the function $P(t)$. It consists of all functions h for which there exists a constant $\lambda = \lambda(h) > 0$ such that $P\big(\frac{|h|}{\lambda}\big) \in L^1(\Omega)$.

In particular, the Orlicz-Zygmund space $L^p \log^\alpha L$, $1 \leq p \leq \infty$, $\alpha \in \mathbb{R}$ is the Orlicz space generated by the function $P(t) = t^p \log^\alpha(e + t)$.

For $\alpha > 0$ the dual Orlicz space to $L \log^{\frac{1}{\alpha}} L(\Omega)$ is the space $\mathrm{EXP}_\alpha(\Omega)$, generated by the function $P(t) = \exp(t^\alpha) - 1$, i.e. it consists of all measurable functions f for which there exists $\lambda > 0$ such that

$$\int_\Omega \exp\Big\{ \Big(\frac{|f(x)|}{\lambda} \Big)^\alpha \Big\} < \infty. \tag{6}$$

It is well known that $L^\infty(\Omega)$ is not dense into $\mathrm{EXP}_\alpha(\Omega)$. We denote by $\exp_\alpha(\Omega)$ the closure of $L^\infty(\Omega)$ in $\mathrm{EXP}_\alpha(\Omega)$ and it consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that (6) is fulfilled for any $\lambda > 0$ ([5]).

Let $p > 1$, *the grand Lebesgue space*, denoted by $L^{p)}(\Omega)$, is a space slightly larger than $L^p(\Omega)$. It was introduced in [28] and consists of all functions $f \in \bigcap_{0<\varepsilon\leq p-1} L^{p-\varepsilon}$ such that

$$\|f\|_{p)} = \sup_{0<\varepsilon<1} \left(\varepsilon \int_\Omega |f(x)|^{p-\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}} < \infty. \tag{7}$$

It is known that $L^\infty(\Omega)$ is not dense in $L^{p)}(\Omega)$ and in [5] the following formula for the distance to L^∞ in this space was established

$$dist_{L^{p)}}(f, L^\infty) = \limsup_{\varepsilon\to 0}\left(\varepsilon \int_\Omega |f(x)|^{p-\varepsilon}\, dx\right)^{\frac{1}{p-\varepsilon}}.$$

For a further generalization see also [14].

We shall denote by $L^{p)}_b(\Omega)$ the closure of $L^\infty(\Omega)$ into $L^{p)}(\Omega)$ which represents the subclass of $L^{p)}(\Omega)$ of all functions f such that

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega |f(x)|^{p-\varepsilon}\, dx = 0. \tag{8}$$

In [19], Greco showed the following inclusions

$$L^n(\Omega) \subset \frac{L^n}{\log L}(\Omega) \subset L^{n)}_b(\Omega) \subset L^{n)}(\Omega) \subset \bigcap_{p<n} L^p(\Omega).$$

Recently a generalization of grand Lebesgue space has been given in [4].

In the sequel we shall deal with the *grand Sobolev space*, denoted by $W^{1,p)}(\Omega)$ which consists of all measurable functions $f \in \bigcap_{0<\varepsilon\leq p-1} W^{1,p-\varepsilon}(\Omega)$ such that $|\nabla f| \in L^{p)}(\Omega)$. For some applications see [10].

2.2. *Differentiability and area formula*

Let $f : \Omega \to \Omega'$ be a homeomorphism. We denote by $|Df|$ the operator norm of the differential matrix, that is

$$|Df| = \sup\{|(Df)(\xi)| : \xi \in \mathbb{R}^n, |\xi| = 1\}.$$

The adjugate $\operatorname{adj} Df$ is the transpose of the cofactor matrix. We have the formula

$$Df(\operatorname{adj} Df) = (\operatorname{adj} Df)Df = \mathbf{I}J_f,$$

where $\mathbf{I}$ denotes the identity matrix. Thus, if Df is nonsingular,

$$\frac{1}{J_f} \operatorname{adj} Df = (Df)^{-1}. \tag{9}$$

The well known Hadamard's inequality implies

$$|J_f| \leq |\mathrm{adj}\, Df|^{\frac{n}{n-1}} \leq |Df|^n.$$

We say that $f : \Omega \to \mathbb{R}^n$ verifies the *Lusin $\mathcal{N}$* condition if

$$|E| = 0 \quad \Rightarrow \quad |f(E)| = 0$$

for any measurable set $E \subset \Omega$.

It is well known that if the homeomorphism f satisfies the natural assumption $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, then f verifies the Lusin $\mathcal{N}$ condition. This is due to Reshetnyak ([33]) and it is a sharp result in the scale of $W^{1,p}(\Omega; \mathbb{R}^2)$-homeomorphisms. In [30] an example of a homeomorphism $f \in \bigcap_{1 \leq p < n} W^{1,p}(\Omega, \mathbb{R}^n)$ such that $|Df| \in L^{n)}(\Omega)$ and f does not satisfy Lusin $\mathcal{N}$ condition is provided. Moreover, the authors show that the sharp regularity assumption to rule out the failure of the condition $\mathcal{N}$ can be $|Df| \in L_b^{n)}(\Omega)$ under the condition that the Jacobian determinant is non negative a.e..

We underline that if $|Df| \in L_b^{n)}(\Omega)$ then the weak Jacobian of the mapping f coincides with the pointwise Jacobian by a result of L. Greco (see [19]).

We decompose the domain Ω of a given mapping f as follows

$$\Omega = \mathcal{R}_f \cup \mathcal{Z}_f \cup \mathcal{E}_f$$

where

$$\mathcal{R}_f = \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) \neq 0\}, \tag{10}$$

$$\mathcal{Z}_f = \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) = 0\}, \tag{11}$$

$$\mathcal{E}_f = \{x \in \Omega : f \text{ is not differentiable at } x\}. \tag{12}$$

Differentiability is understood in the classical sense. If f is a homeomorphism, these are Borel sets. Moreover, $f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}$ and for all $x \in \mathcal{R}_f$,

$$Df^{-1}(f(x)) = (Df(x))^{-1}, \quad J_{f^{-1}}(f(x)) = \frac{1}{J_f(x)}. \tag{13}$$

Let us recall that if $f \in W^{1,1}(\Omega, \mathbb{R}^2)$ is a homeomorphism, then f is differentiable a.e. in Ω and either $J_f \geq 0$ or $J_f \leq 0$ a.e. (see [34, Lemma 4.11]). Without loss of generality, we will assume $J_f \geq 0$.

For a study of the regularity and differentiability points of bi-Sobolev maps see [8] and for bi-ACL maps see [11].

The *Lusin $\mathcal{N}$* condition is strongly connected with the validity of the area formula which is crucial in the next developments.

Let $B \subset \Omega$ be a Borel measurable set and assume that $f : \Omega \to \Omega'$ is a homeomorphism such that f is differentiable at every point of B. From ([15, Theorem 3.2.3]) and from the fact that the set of differentiability can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz ([15, Theorem 3.1.8]), we can deduce the following *weak area formula*

$$\int_B \eta(f(x))|J_f(x)|\,dx \le \int_{f(B)} \eta(y)\,dy \tag{14}$$

for any $\eta : \mathbb{R}^2 \to [0, +\infty)$ Borel measurable function.

The equality

$$\int_B \eta(f(x))|J_f(x)|\,dx = \int_{f(B)} \eta(y)\,dy \tag{15}$$

occurs if f satisfies *Lusin $\mathscr{N}$* condition.

It is well known that the *Lusin $\mathscr{N}$* condition holds on the set $\mathscr{R}_f \cup \mathscr{Z}_f$ where f is differentiable. Hence, by the area formula (15) with $B = \mathscr{R}_f \cup \mathscr{Z}_f$, we get the following version of the Sard's Lemma

$$|f(\mathscr{Z}_f)| = 0. \tag{16}$$

Therefore, if f^{-1} satisfies the *Lusin $\mathscr{N}$* condition, then $J_f(x) > 0$ for a.e. $x \in \Omega$ (see [8]).

2.3. Some elementary inequalities

We recall the following elementary inequalities.

Lemma 2.1. *Let $a \ge 0$. Then*

$$\log(\mathrm{e} + a^\alpha) \le \alpha \log(\mathrm{e} + a), \quad \alpha \ge 1, \tag{17}$$

$$\log(\mathrm{e} + a^\alpha) \ge \alpha \log(\mathrm{e} + a), \quad 0 < \alpha < 1. \tag{18}$$

Lemma 2.2. *Fix $\lambda > 0$ and $\alpha > 0$. Then for all $a \ge 0$, $b \ge 0$*

$$a^\alpha b \le C[\exp(\lambda a) + b \log^\alpha(\mathrm{e} + b)], \tag{19}$$

where

$$C = \left(\frac{\mathrm{e} + \alpha}{\lambda \mathrm{e}}\right)^\alpha.$$

As a consequence of Lemma 2.2, we obtain the following:

Proposition 2.3. *Let $1 \le p < \infty$ and f be a bi-Sobolev mapping. The following conditions are equivalent:*

i) $J_{f^{-1}} \log^p(\mathrm{e} + J_{f^{-1}}) \in L^1(\Omega')$
ii) $J_{f^{-1}} \log^p(\mathrm{e} + |Df^{-1}|) \in L^1(\Omega')$.

PROOF. i) $\Rightarrow$ ii) follows by (19) choosing $\alpha = p$, $a = \log(e + |Df^{-1}|)$ and $b = J_{f^{-1}}$. ii) $\Rightarrow$ i) is a consequence of Hadamard's inequality and (17). □

3. Regularity of the inverse map

In this Section we deduce regularity of f^{-1} as a consequence of integrability assumptions on the distortion $K_{q,f}$ in Sobolev spaces, in Zygmund-type spaces and finally in grand Sobolev spaces.

3.1. Regularity of the inverse map in Sobolev-Zygmund spaces

It is well known ([24]) that if f is a bi-Sobolev map such that $K_f \in L^1(\Omega)$, then $|Df^{-1}| \in L^2(\Omega')$.

Now we prove an analogous result concerning the integrability of the q-distortion function.

THEOREM 3.1. *Let $q > 1$ and let f be a bi-Sobolev mapping such that*

$$K_{q,f} \in L^{\frac{1}{q-1}}(\Omega).$$

Then

$$|Df^{-1}| \in L^{\frac{q}{q-1}}(\Omega').$$

PROOF. Applying basic linear algebra and weak area formula (14), we get

$$\begin{aligned}
(20)\qquad & \int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}}\,dy \\
&= \int_{\mathcal{R}_{f^{-1}}} |Df^{-1}(y)|^{\frac{q}{q-1}}\,dy \\
&= \int_{\mathcal{R}_{f^{-1}}} |Df(f^{-1}(y))|^{\frac{q}{q-1}}(J_{f^{-1}}(y))^{\frac{q}{q-1}}\,dy \\
&= \int_{\mathcal{R}_{f^{-1}}} |Df(f^{-1}(y))|^{\frac{q}{q-1}}(J_{f^{-1}}(y))^{\frac{1}{q-1}}J_{f^{-1}}(y)\,dy \\
&= \int_{\mathcal{R}_{f^{-1}}} \frac{|Df(f^{-1}(y))|^{\frac{q}{q-1}}}{(J_f(f^{-1}(y)))^{\frac{1}{q-1}}}J_{f^{-1}}(y)\,dy \\
&\le \int_{\mathcal{R}_f} \frac{|Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{1}{q-1}}}\,dx \\
&\le \int_{\Omega} (K_{q,f}(x))^{\frac{1}{q-1}}\,dx,
\end{aligned}$$

which is finite thanks to the assumption. □

In previous Theorem we deduced regularity of the inverse mapping under some regularity assumption on the distortion. Now we proceed with the proof of Theorem 1.1 which also goes in the opposite direction.

Proof of Theorem 1.1. From Theorem 3.1, we know that if $K_{q,f}(x) \in L^{\frac{1}{q-1}}(\Omega)$ then $|Df^{-1}| \in L^{\frac{q}{q-1}}(\Omega')$.

Conversely, if $|Df^{-1}| \in L^{\frac{q}{q-1}}(\Omega')$ then f^{-1} verifies the *Lusin $\mathcal{N}$* condition. Therefore we deduce that f has positive Jacobian a.e. in Ω and (15) holds. Using the same arguments of (20), we get

$$\int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}}\,dy = \int_{\mathcal{R}_f} (K_{q,f}(x))^{\frac{1}{q-1}}\,dx = \int_{\Omega} (K_{q,f}(x))^{\frac{1}{q-1}}\,dx. \qquad \square \tag{21}$$

When setting in the Orlicz-Zygmund classes Theorem 1.2 reveals higher integrability property of the Jacobian determinant of the inverse mapping. For $q = 2$ it reduces to Lemma 4.2 in [21].

Proof of Theorem 1.2. By Theorem 3.1 $|Df^{-1}| \in L^{\frac{q}{q-1}}(\Omega')$.

Since $\dfrac{q}{q-1} \geq 2$, f^{-1} satisfies the *Lusin $\mathcal{N}$* condition and hence $J_f > 0$ a.e..

Arguing as in the proof of [21, Lemma 4.2], we show that, for all $p \geq 1$, (3) and (4) are equivalent to each other.

Firstly, (4) implies (3) with no conditions on the distortion. Indeed the assumption on q yields $|Df^{-1}| \in L^2 \log^{p-1} L(\Omega')$. By higher integrability of the Jacobian determinant $J_{f^{-1}} \in L \log^p L(\Omega')$ ([20]).

Now, we prove that (3) implies (4).

By Proposition 2.3, $J_{f^{-1}} \log^p(\mathrm{e} + J_{f^{-1}}) \in L^1(\Omega')$ iff $J_{f^{-1}} \log^p(\mathrm{e} + |D_{f^{-1}}|) \in L^1(\Omega')$.

Let us assume $p > 1$.

There is an interesting iterative argument which proves that

$$|Df^{-1}|^{\frac{q}{q-1}} \log^{\alpha-1}(\mathrm{e} + |Df^{-1}|) \in L^1(\Omega') \tag{22}$$

for any α such that

$$1 \leq \alpha < p. \tag{23}$$

We describe quickly this argument now, and shall give more details later. Let $\gamma = 1 - 1/p$. Assume that

$$\log^{\beta}\left(\mathrm{e} + \frac{1}{J_f}\right) \in L^1(\Omega), \tag{24}$$

for some $\beta \geq 0$. Then, arguing as in the proof of [25, Lemma 6.2], essentially using the area formula, we can show that

$$|Df^{-1}|^{\frac{q}{q-1}} \log^{\gamma\beta}(\mathrm{e} + |Df^{-1}|) \in L^1(\Omega'). \tag{25}$$

By higher integrability of Jacobian determinant, (25) implies

$$J_{f^{-1}} \log^{\gamma\beta+1}(\mathrm{e} + J_{f^{-1}}) \in L^1(\Omega'). \tag{26}$$

By area formula again, as in the proof of [24, Theorem 6.1], we have

$$\log^{\gamma\beta+1}\Big(e+\frac{1}{J_f}\Big) \in L^1(\Omega). \tag{27}$$

If $\beta < \gamma\beta + 1$, then (27) is stronger than condition (24) we have started with, and we can iterate the above argument. Clearly, (24) holds with $\beta = 0$, hence we find in turn that

$$J_{f^{-1}} \log(\mathrm{e} + J_{f^{-1}}),\ J_{f^{-1}} \log^{\gamma+1}(\mathrm{e} + J_{f^{-1}}),\ J_{f^{-1}} \log^{\gamma(\gamma+1)+1}(\mathrm{e} + J_{f^{-1}}), \dots$$

are locally integrable. As

$$1 + \gamma + \gamma^2 + \cdots = \frac{1}{1-\gamma} = p,$$

obviously with a finite number of steps we get (22), for every fixed α satisfying (23).

To prove (3) and (4) we need to make the above argument more precise. Let $\mu \in C_0^\infty(\Omega')$, $\mu \geq 0$. We start with the following estimate

$$\begin{aligned}\int_{\Omega'} \mu^2(y) J_{f^{-1}}(y) \log^\alpha(\mathrm{e} + \mu(y)|Df^{-1}(y)|)\, dy \\ \leq C \int_{\Omega'} F(y)\, dy \\ + C \int_{\Omega'} \mu^2(y) |Df^{-1}(y)|^2 \log^{\alpha-1}(\mathrm{e} + \mu(y)|Df^{-1}(y)|)\, dy,\end{aligned} \tag{28}$$

with

$$\begin{aligned} F = |f^{-1}(y) \otimes \nabla\mu(y)|(|f^{-1}(y) \otimes \nabla\mu(y)| + \mu(y)|Df^{-1}(y)|) \\ \times \log^p(\mathrm{e} + |f^{-1}(y) \otimes \nabla\mu(y)| + \mu(y)|Df^{-1}(y)|). \end{aligned}$$

Estimate (28) follows from Corollary 3.2 and Example 2.8 in [17]. Notice that the constant $C = C(p) > 0$ in (28) can be chosen independent on α satisfying (23) and $F \in L^1(\Omega')$.

Moreover, (28) is equivalent to

$$\begin{aligned}\int_{\Omega'} \mu^2(y) J_{f^{-1}}(y) \log^\alpha(\mathrm{e} + \mu(y)|Df^{-1}(y)|)\, dy \\ \leq C \int_{\Omega'} F(y)\, dy \\ + C \int_{\Omega' \cap \{|Df^{-1}| \leq 1\}} \mu^2(y) |Df^{-1}(y)|^2 \log^{\alpha-1}(\mathrm{e} + \mu(y)|Df^{-1}(y)|)\, dy \\ + C \int_{\Omega' \cap \{|Df^{-1}| > 1\}} \mu^2(y) |Df^{-1}(y)|^2 \log^{\alpha-1}(\mathrm{e} + \mu(y)|Df^{-1}(y)|)\, dy\end{aligned} \tag{29}$$

$$\leq C\int_{\Omega'} F(y)\,dy + C(\alpha,\Omega)$$

$$+\,C\int_{\Omega'} \mu^2(y)|Df^{-1}(y)|^{\frac{q}{q-1}}\log^{\alpha-1}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy.$$

Now we consider last term in (29). Since f^{-1} has finite distortion, by Young's inequality with conjugate exponents α and $\frac{\alpha}{\alpha-1}$ and for $\varepsilon \in \,]0,1[$, we can write

$$\int_{\Omega'} \mu^2(y)|Df^{-1}(y)|^{\frac{q}{q-1}}\log^{\alpha-1}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy \tag{30}$$

$$= \int_{\Omega'} \mu^2(y)\frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{(\varepsilon J_{f^{-1}}(y))^{\frac{\alpha-1}{\alpha}}}(\varepsilon J_{f^{-1}}(y))^{\frac{\alpha-1}{\alpha}}$$

$$\times \log^{\alpha-1}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy$$

$$\leq \varepsilon^{1-p}\int_{\Omega'} \mu^2(y)\Big(\frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{J_{f^{-1}}(y)}\Big)^{\alpha} J_{f^{-1}}(y)\,dy$$

$$+\,\varepsilon\int_{\Omega'} \mu^2(y)J_{f^{-1}}(y)\log^{\alpha}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy.$$

Inserting (30) into (29) and choosing ε such that $C\varepsilon = 1/2$, we get

$$\frac{1}{2}\int_{\Omega'} \mu^2(y)J_{f^{-1}}(y)\log^{\alpha}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy \tag{31}$$

$$\leq C\int_{\Omega'} F(y)\,dy + C(\alpha,\Omega)$$

$$+\,C\int_{\Omega'} \mu^2(y)\Big(\frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{J_{f^{-1}}(y)}\Big)^{\alpha} J_{f^{-1}}(y)\,dy.$$

Notice that we can absorb in the left hand side a term appearing in the right hand side, since by our iterative argument we already know that it is converging. Now we pass to the limit in (31) as $\alpha \to p$, using monotone convergence theorem, and obtain

$$\int_{\Omega'} \mu^2(y)J_{f^{-1}}(y)\log^{p}(\mathrm{e}+\mu(y)|Df^{-1}(y)|)\,dy \tag{32}$$

$$\leq C\int_{\Omega'} F(y)\,dy + C(\alpha,\Omega)$$

$$+\,C\int_{\Omega'} \mu^2(y)\Big(\frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{J_{f^{-1}}(y)}\Big)^{p} J_{f^{-1}}(y)\,dy.$$

We conclude showing that the last integral in (32) is finite, under the assumption $K_{q,f} \in L^{\frac{p}{q-1}}(\Omega)$. As

$$\frac{|Df^{-1}(f(x))|^{\frac{q}{q-1}}}{J_{f^{-1}}(f(x))} = J_f(x)|(Df(x))^{-1}|^{\frac{q}{q-1}} = \frac{|Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{q}{q-1}-1}} = (K_{q,f}(x))^{\frac{1}{q-1}},$$

using the weak area formula (15), we find

$$\int_{\Omega'} \mu^2(y)\Big(\frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{J_{f^{-1}}(y)}\Big)^p J_{f^{-1}}(y)\,dy \le \int_{\Omega} \mu^2(f(x))(K_{q,f}(x))^{\frac{p}{q-1}}\,dx$$

which conclude the proof. □

Now we examine how the regularity of the q-distortion function of a homeomorphism f reflects on the regularity of the inverse f^{-1} in the scale of Orlicz spaces. We deal with the case of spaces smaller and larger than $L^{\frac{1}{q-1}}$.

THEOREM 3.2. *Let $1 < q \le 2$ and let f be a bi-Sobolev mapping such that*

$$K_{q,f} \in L^{\frac{1}{q-1}} \log^{\alpha} L(\Omega)$$

for some $\alpha \ge 0$. Then

$$|Df^{-1}| \in L^{\frac{q}{q-1}} \log^{\alpha} \log L(\Omega').$$

PROOF. From Theorem 3.1 we know that $f^{-1} \in W^{1,\frac{q}{q-1}}(\Omega';\mathbb{R}^2)$ has finite distortion, satisfies *Lusin $\mathcal{N}$* condition and then $J_f > 0$ a.e. in Ω.

$$\begin{aligned}(33)\quad &\int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}} \log^{\alpha}(e + \log(e + |Df^{-1}(y)|))\,dy \\ &= \int_{\mathcal{R}_{f^{-1}}} |Df^{-1}(y)|^{\frac{q}{q-1}} \log^{\alpha}(e + \log(e + |Df^{-1}(y)|))\,dy \\ &= \int_{\mathcal{R}_{f^{-1}}} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{J_{f^{-1}}(y)} J_{f^{-1}}(y) \log^{\alpha}(e + \log(e + |Df^{-1}(y)|))\,dy \\ &\le \int_{\mathcal{R}_{f^{-1}}} \frac{|\operatorname{adj} Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{q}{q-1}}} J_f(x) \log^{\alpha}\Big(e + \log\Big(e + \frac{|\operatorname{adj} Df(x)|}{J_f(x)}\Big)\Big)\,dx \\ &= \int_{\mathcal{R}_f} \frac{|Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{1}{q-1}}} \log^{\alpha}\Big(e + \log\Big(e + \frac{|Df(x)|}{J_f(x)}\Big)\Big)\,dx\end{aligned}$$

$$= \int_{\mathscr{R}_f \cap \{|Df| \geq 1\}} \frac{|Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{1}{q-1}}} \log^\alpha\Big(e + \log\Big(e + \frac{|Df(x)|}{J_f(x)}\Big)\Big)\, dx$$

$$+ \int_{\mathscr{R}_f \cap \{|Df| < 1\}} \frac{|Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{1}{q-1}}} \log^\alpha\Big(e + \log\Big(e + \frac{|Df(x)|}{J_f(x)}\Big)\Big)\, dx$$

$$= I + II$$

Let us estimate I:

$$I \leq \int_{\mathscr{R}_f \cap \{|Df| \geq 1\}} (K_{q,f}(x))^{\frac{1}{q-1}} \log^\alpha\Big(e + \log\Big(e + \frac{|Df(x)|^q}{J_f(x)}\Big)\Big)\, dx \tag{34}$$

$$\leq \int_\Omega (K_{q,f}(x))^{\frac{1}{q-1}} \log^\alpha(e + \log(e + K_{q,f}(x)))\, dx < \infty$$

thanks to the assumption.

In order to estimate II, we recall that since $K_{q,f} \in L^{\frac{1}{q-1}} \log^\alpha L(\Omega)$, we may apply Theorem 6.1 in [24] to find that $\log\big(\mathrm{e} + \frac{1}{J_f}\big) \in L^1(\Omega)$. In order to estimate II, using inequalities (19) with $a = \log\big(\mathrm{e} + \log\big(\mathrm{e} + \frac{1}{J_f}\big)\big)$, $b = K_{q,f}^{\frac{1}{q-1}}$ and (17), we get

$$II \leq \int_{\mathscr{R}_f \cap \{|Df| < 1\}} (K_{q,f}(x))^{\frac{1}{q-1}} \log^\alpha\Big(e + \log\Big(e + \frac{1}{J_f(x)}\Big)\Big)\, dx \tag{35}$$

$$\leq C \int_{\mathscr{R}_f \cap \{|Df| < 1\}} (K_{q,f}(x))^{\frac{1}{q-1}} \log^\alpha(e + (K_{q,f}(x))^{\frac{1}{q-1}})\, dx$$

$$+ C \int_{\mathscr{R}_f \cap \{|Df| < 1\}} \exp\Big(\log\Big(e + \log\Big(e + \frac{1}{J_f(x)}\Big)\Big)\Big)\, dx$$

$$\leq c(\alpha, q) \int_{\mathscr{R}_f \cap \{|Df| < 1\}} (K_{q,f}(x))^{\frac{1}{q-1}} \log^\alpha(e + K_{q,f}(x))\, dx$$

$$+ c(\alpha) \int_{\mathscr{R}_f \cap \{|Df| < 1\}} \Big(e + \log\Big(e + \frac{1}{J_f(x)}\Big)\Big)\, dx. \qquad \square$$

Next result is obtained when $K_{q,f}$ distortion enjoys a degree of integrability weaker than $L^{\frac{1}{q-1}}$.

Theorem 3.3. *Let $1 < q \leq 2$, $p > 1$ and let $f \in W^{1,p}(\Omega, \mathbb{R}^2)$ be a homeomorphism of finite distortion such that, for some $\alpha \geq 0$,*

$$K_{q,f} \in \frac{L^{\frac{1}{q-1}}}{\log^\alpha L}(\Omega).$$

Then

$$|Df^{-1}| \in \frac{L^{\frac{q}{q-1}}}{\log^{\alpha} L}(\Omega').$$

PROOF. Since f is a homeomorphism of finite distortion, then also f^{-1} has finite distortion ([24]). By (13), (14) and (18) we get

$$\begin{aligned}
&\int_{\Omega'} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{\log(e+|Df^{-1}(y)|)}\,dy \\
&= \int_{R_{f^{-1}}\cup Z_{f^{-1}}} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{\log(e+|Df^{-1}(y)|)}\,dy = \int_{R_{f^{-1}}} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{\log(e+|Df^{-1}(y)|)}\,dy \\
&= \int_{R_{f^{-1}}} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}}}{\log(e+|Df^{-1}(y)|)} \cdot \frac{J_{f^{-1}}(y)}{J_{f^{-1}}(y)}\,dy \\
&\le \int_{R_f} \frac{|\operatorname{adj} Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{q}{q-1}} \log\left(e+\frac{|\operatorname{adj} Df(x)|}{J_f(x)}\right)} J_f(x)\,dx \\
&= \int_{R_f} \frac{|\operatorname{adj} Df(x)|^{\frac{q}{q-1}}}{(J_f(x))^{\frac{1}{q-1}} \log\left(e+\frac{|\operatorname{adj} Df(x)|}{J_f(x)}\right)}\,dx = \int_{R_f} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log\left(e+\frac{|\operatorname{adj} Df(x)|}{J_f(x)}\right)}\,dx \\
&= \int_{R_f} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log\left(e+\frac{|Df(x)|^q}{J_f(x)}\cdot\frac{|\operatorname{adj} Df(x)|}{|Df(x)|^q}\right)}\,dx \le \int_{\Omega} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log\left(e+\frac{K_{q,f}(x)}{|Df(x)|^{q-1}}\right)}\,dx \\
&= \int_{\{K_{q,f}^{\frac{1}{q-1}} \le |Df|^p\}} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log\left(e+\frac{K_{q,f}(x)}{|Df(x)|^{q-1}}\right)}\,dx + \int_{\{K_{q,f}^{\frac{1}{q-1}} > |Df|^p\}} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log\left(e+\frac{K_{q,f}(x)}{|Df(x)|^{q-1}}\right)}\,dx \\
&\le \int_{\Omega} |Df(x)|^p\,dx + c\int_{\Omega} \frac{(K_{q,f}(x))^{\frac{1}{q-1}}}{\log(e+K_{q,f}(x))}\,dx < +\infty
\end{aligned}$$

and theorem is completely proved. □

For $q = 2$ Theorems 3.2 and 3.3 have been established in [31].

3.2. Regularity of the inverse map and of the K_f distortion in grand Sobolev spaces

Assuming $K_{q,f}$ in grand Lebesgue spaces we are able to prove the following regularity results for the inverse map f^{-1}. In case $q = 2$ Theorem 3.4 and Corollary 3.5 have been proved in [27] and [18] respectively.

THEOREM 3.4. *Let $1 < p \le q \le 2$. If $f \in W^{1,p}(\Omega;\mathbb{R}^2)$ is a homeomorphism of finite distortion such that*

$$K_{q,f} \in L^{\frac{1}{q-1})}(\Omega),$$

then

$$|Df^{-1}| \in L^{\frac{q}{q-1})}(\Omega').$$

PROOF. By standard linear algebra, we get

$$\begin{aligned}\int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}\, dy &= \int_{\mathscr{R}_{f^{-1}}} \frac{|Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}}{J_{f^{-1}}(y)} J_{f^{-1}}(y)\, dy \\ &\le \int_{\Omega} \frac{|(Df(x))^{-1}|^{\frac{q}{q-1}-\varepsilon}}{J_{f^{-1}}(f(x))}\, dx \\ &= \int_{\Omega} |\mathrm{adj}\, Df(x)|^{\frac{q}{q-1}-\varepsilon} (J_{f^{-1}}(f(x)))^{\frac{q}{q-1}-\varepsilon-1}\, dx \\ &= \int_{\Omega} \frac{|Df(x)|^{q(\frac{1}{q-1}-\varepsilon)}}{(J_f(x))^{\frac{1}{q-1}-\varepsilon}} |Df(x)|^{\varepsilon(q-1)}\, dx \\ &= \int_{\Omega} (K_{q,f}(x))^{\frac{1}{q-1}-\varepsilon} |Df(x)|^{\varepsilon(q-1)}\, dx.\end{aligned}$$

Multiplying both sides by ε and using Hölder's inequality of exponents $\dfrac{p}{(q-1)\varepsilon}$ and $\dfrac{p}{p-(q-1)\varepsilon}$, we get

$$\begin{aligned}&\left(\varepsilon \int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}\, dy\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}} \\ &\quad\le \left(\varepsilon \int_{\Omega} (K_{q,f}(x))^{\frac{1}{q-1}-\varepsilon} |Df(x)|^{\varepsilon(q-1)}\, dx\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}} \\ &\quad\le \|Df\|_p^{\frac{(q-1)\varepsilon}{q/(q-1)-\varepsilon}} \\ &\qquad\times \left(\varepsilon^{\frac{p}{p-(q-1)\varepsilon}} \int_{\Omega} (K_{q,f}(x))^{(\frac{1}{q-1}-\varepsilon)\frac{p}{p-(q-1)\varepsilon}}\, dx\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}\frac{p-(q-1)\varepsilon}{p}}.\end{aligned}$$

Let us consider $\delta > 0$ such that

$$\left(\frac{1}{q-1}-\varepsilon\right)\frac{p}{p-(q-1)\varepsilon} = \frac{1}{q-1} - \delta \tag{36}$$

or equivalently

$$\varepsilon = \frac{\delta p}{\delta(q-1)+p-1} \le \frac{\delta p}{p-1}. \tag{37}$$

Therefore we have

$$\varepsilon^{\frac{p}{p-(q-1)\varepsilon}} \le \delta\left(\frac{p}{p-1}\right)^{\frac{p}{p-(q-1)\varepsilon}} \quad \text{for every } 0<\delta<1$$

and

$$\frac{1}{\frac{q}{q-1}-\varepsilon}\,\frac{p-(q-1)\varepsilon}{p} = \frac{(p-1)(q-1)}{\delta(q-1)(q-p)+q(p-1)} < \frac{q-1}{q}.$$

Then we can write

$$\begin{aligned}
&\left(\varepsilon\int_{\Omega'}|Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}\,dy\right)^{\frac{1}{\frac{1}{q-1}-\varepsilon}}\\
&\le \|Df\|_p^{\frac{(q-1)\varepsilon}{q/(q-1)-\varepsilon}}\left(\frac{p}{p-1}\right)^{\frac{1}{\frac{1}{q-1}-\varepsilon}}\\
&\quad\times\left(\delta\int_{\Omega}(K_{q,f}(x))^{\frac{1}{q-1}-\delta}\,dx\right)^{\frac{(p-1)(q-1)}{\delta(q-1)(q-p)+q(p-1)}}\\
&\le \|Df\|_p^{\frac{(q-1)\varepsilon}{q/(q-1)-\varepsilon}}\sup_{0<\varepsilon<1}\left(\frac{p}{p-1}\right)^{\frac{1}{\frac{1}{q-1}-\varepsilon}}\\
&\quad\times\left(\delta\int_{\Omega}(K_{q,f}(x))^{\frac{1}{q-1}-\delta}\,dx\right)^{\frac{(p-1)(q-1)}{\delta(q-1)(q-p)+q(p-1)}}\\
&\le \max\{1,\|Df\|_p^{(q-1)^2}\}\left(\frac{p}{p-1}\right)^{q-1}\\
&\quad\cdot\sup_{0<\delta<1}\left[\left(\delta\int_{\Omega}(K_{q,f}(x))^{\frac{1}{q-1}-\delta}\,dx\right)^{\frac{1}{\frac{1}{q-1}-\delta}}\right]^{\frac{\left(\frac{1}{q-1}-\delta\right)(p-1)(q-1)}{\delta(q-1)(q-p)+q(p-1)}}\\
&\le \max\{1,\|Df\|_p^{(q-1)^2}\}c(p,q)\max\{1,\|K_{q,f}\|_{\frac{1}{q-1})}^{1/q}\}.
\end{aligned}$$

Taking the supremum for $0<\varepsilon<1$, we conclude the proof. □

COROLLARY 3.5. *Let $1<p\le q\le 2$. Assume that $f\in W^{1,p}(\Omega;\mathbb{R}^2)$ is a homeomorphism of finite distortion such that*

$$K_{q,f}\in L_b^{\frac{1}{q-1})}(\Omega).$$

Then

$$|Df^{-1}|\in L_b^{\frac{q}{q-1})}(\Omega')$$

and hence f^{-1} satisfies Lusin $\mathcal{N}$ condition.

PROOF. Arguing as in the proof of Theorem 3.4, we get

$$\left(\varepsilon \int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}\, dy\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}}$$

$$\leq \|Df\|_p^{\frac{(q-1)\varepsilon}{q/(q-1)-\varepsilon}} \left(\frac{p}{p-1}\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}}$$

$$\cdot \left[\left(\delta \int_{\Omega} (K_{q,f}(x))^{\frac{1}{q-1}-\delta}\, dx\right)^{\frac{1}{\frac{1}{q-1}-\delta}}\right]^{\frac{\left(\frac{1}{q-1}-\delta\right)(p-1)(q-1)}{\delta(q-1)(q-p)+q(p-1)}}.$$

From (37) letting $\varepsilon \to 0^+$, it follows that $\delta \to 0^+$ and therefore, thanks to the assumption on the q-distortion function $K_{q,f}$, we get

$$\lim_{\varepsilon\to 0}\left(\varepsilon \int_{\Omega'} |Df^{-1}(y)|^{\frac{q}{q-1}-\varepsilon}\, dy\right)^{\frac{1}{\frac{q}{q-1}-\varepsilon}} = 0,$$

that is $|Df^{-1}| \in L_b^{\frac{q}{q-1})}(\Omega')$. It follows that f^{-1} satisfies *Lusin* $\mathcal{N}$ condition ([30]). □

The connection between the regularity properties of q-distortion of f and the classical one is given by the following results.

COROLLARY 3.6. *Let* $1 < p \leq q < 2$ *and let* $f \in W^{1,p}(\Omega;\mathbb{R}^2)$ *be a homeomorphism of finite distortion. If*

$$K_{q,f} \in L^{\frac{1}{q-1})}(\Omega),$$

then

$$K_f \in L^1(\Omega)$$

and f^{-1} *satisfies Lusin* $\mathcal{N}$ *condition.*

PROOF. By Theorem 3.4 we have $|Df^{-1}| \in L^{\frac{q}{q-1})}(\Omega')$. Since $\frac{q}{q-1} > 2$, then $L^{\frac{q}{q-1})}(\Omega') \subset L^2(\Omega')$. From this inclusion we get f^{-1} satisfies Lusin $\mathcal{N}$ condition ([33]) and $K_f \in L^1(\Omega)$ ([24]). □

In this subsection up to now we have deduced regularity of $|Df^{-1}|$ and of K_f under some $W^{1,p}$-homeomorphic assumption on the mapping f. Next results show that the regularity of K_f decreases if we only assume $f \in W^{1,1}$.

THEOREM 3.7. *Let* $1 < q < 2$ *and let* f *be a bi-Sobolev mapping such that*

$$K_{q,f} \in L^{\frac{1}{q-1})}(\Omega).$$

Then

$$\sup_{0<\delta<\delta_0}\left(\delta \int_{\Omega} K_f^{1-\delta}(x)\, dx\right)^{\frac{1}{1-\delta}} < \infty,$$

where $0 < \delta_0 < 1$.

PROOF.

$$\begin{aligned}\int_\Omega (K_f(x))^{1-\delta}\,dx &\le |\Omega| + \int_{\mathscr{R}_f} (K_f(x))^{1-\delta}\,dx\\ &= |\Omega| + \int_{\mathscr{R}_f} \Big(\frac{|Df(x)|^q}{J_f(x)}\Big)^{1-\delta} |Df(x)|^{(1-\delta)(2-q)}\,dx\\ &= |\Omega| + \int_{\mathscr{R}_f} (K_{q,f}(x))^{1-\delta}|Df(x)|^{(1-\delta)(2-q)}\,dx\\ &\le |\Omega| + \int_\Omega (K_{q,f}(x))^{1-\delta}|Df(x)|^{(1-\delta)(2-q)}\,dx.\end{aligned}$$

Using Hölder inequality of exponents $\dfrac{1}{(1-\delta)(2-q)}$ and $\dfrac{1}{1-(1-\delta)(2-q)}$, we get

$$\begin{aligned}&\int_\Omega (K_f(x))^{1-\delta}\,dx\\ &\quad\le |\Omega| + \Big(\int_\Omega (K_{q,f}(x))^{\frac{1-\delta}{1-(1-\delta)(2-q)}}\,dx\Big)^{1-(1-\delta)(2-q)} \|Df\|_1^{(1-\delta)(2-q)}.\end{aligned}$$

Let us choose $\varepsilon > 0$ such that

$$\frac{1-\delta}{1-(1-\delta)(2-q)} = \frac{1}{q-1} - \varepsilon$$

or equivalently

$$\delta = \frac{\varepsilon(q-1)^2}{1-\varepsilon(q-1)(2-q)} < \frac{\varepsilon}{1-(q-1)(2-q)} \quad \text{for every } 0<\varepsilon<1.$$

Therefore we get

$$\begin{aligned}&\Big(\delta \int_\Omega (K_f(x))^{1-\delta}\,dx\Big)^{\frac{1}{1-\delta}}\\ &\quad\le C(\delta|\Omega|)^{\frac{1}{1-\delta}}\\ &\qquad + C\Big(\delta\int_\Omega (K_{q,f}(x))^{\frac{1-\delta}{1-(1-\delta)(2-q)}}\,dx\Big)^{\frac{1-(1-\delta)(2-q)}{1-\delta}} \|Df\|_1^{2-q}\\ &\quad\le C(\delta|\Omega|)^{\frac{1}{1-\delta}}\\ &\qquad + C\Big(\frac{\varepsilon}{1-(q-1)(2-q)}\int_\Omega (K_{q,f}(x))^{\frac{1}{q-1}-\varepsilon}\,dx\Big)^{\frac{1}{\frac{1}{q-1}-\varepsilon}} \|Df\|_1^{2-q}\end{aligned}$$

$$\leq C(\delta|\Omega|)^{\frac{1}{1-\delta}} + C\Big(\frac{1}{1-(q-1)(2-q)}\Big)^{\frac{q-1}{2-q}}$$

$$\cdot \sup_{0<\varepsilon<1}\Big(\varepsilon\int_\Omega (K_{q,f}(x))^{\frac{1}{q-1}-\varepsilon}\,dx\Big)^{\frac{1}{\frac{1}{q-1}-\varepsilon}}\|Df\|_1^{2-q}$$

$$\leq C(\delta|\Omega|)^{\frac{1}{1-\delta}} + c(q)\|K_{q,f}\|_{\frac{1}{q-1})}\|Df\|_1^{2-q}.$$

The claim is proved by taking the supremum for $0<\delta<\delta_0$. □

Corollary 3.8. *Let* $1<q<2$ *and let* f *be a bi-Sobolev mapping such that*

$$K_{q,f}\in L_b^{\frac{1}{q-1})}(\Omega),$$

then

$$\lim_{\delta\to 0}\delta\int_\Omega K_f^{1-\delta}(x)=0.$$

Proof. Arguing as in the proof of Theorem 3.7 we get

$$\Big(\delta\int_\Omega (K_f(x))^{1-\delta}\,dx\Big)^{\frac{1}{1-\delta}}$$

$$\leq \Big(\frac{1}{1-(q-1)(2-q)}\Big)^{\frac{q-1}{2-q}}$$

$$\times\Big(\varepsilon\int_\Omega (K_{q,f}(x))^{\frac{1}{q-1}-\varepsilon}\,dx\Big)^{\frac{1}{\frac{1}{q-1}-\varepsilon}}\|Df\|_1^{2-q}.$$

The proof is over by observing that $\varepsilon\to 0$ implies $\delta\to 0$. □

4. Integrability of the q-distortion of the composition map

In [24] and in [21] the authors study under which conditions the composition of two bi-Sobolev mappings is a bi-Sobolev mapping. Moreover, they give some integrability properties of the distortion of the composition map. Similarly the integrability property of the composition of a scalar function with a bi-Sobolev mapping has been considered e.g. in [24], [12], [13].

In this Section we present some results on this subject concerning q-distortion function.

Proposition 4.1. *Let* $1<q\leq 2$ *and* $\alpha\geq 0$. *Assume that* $f:\Omega\to\Omega'$ *and* $g:\Omega'\to\Omega''$ *are bi-Sobolev mappings. If*

$$|Df^{-1}|\in L^{\frac{q}{q-1}}\log^\alpha L(\Omega')\quad \textit{and}\quad |Dg|\in L^q\log^{-\alpha(q-1)}L(\Omega),$$

then the composition $h = g \circ f : \Omega \to \Omega''$ is a bi-Sobolev mapping and

$$K_{q,h}(x) \leq K_{q,g}(f(x))K_{q,f}(x) \quad \textit{for a.e. } x \in \mathcal{R}_h. \tag{38}$$

PROOF. By Theorem 3.1 in [21], we know that $h \in W^{1,1}(\Omega; \mathbb{R}^2)$. In particular h is differentiable a.e. in Ω and we have

$$Dh(x) = Dg(f(x))Df(x) \quad J_h(x) = J_g(f(x))J_f(x) \quad \text{a.e. in } \Omega. \tag{39}$$

From (39) and since g has finite distortion we deduce that h has finite distortion. Moreover, as f^{-1} verifies *Lusin* $\mathcal{N}$ condition, we get

$$x \in \mathcal{R}_h \quad \Leftrightarrow \quad f(x) \in \mathcal{R}_g.$$

The above arguments also give

$$K_{q,h}(x) = \frac{|Dh(x)|^q}{J_h(x)} \leq \frac{|Dg(f(x))|^q}{J_g(f(x))} \cdot \frac{|Df(x)|^q}{J_f(x)} = K_{q,g}(f(x))K_{q,f}(x)$$

on a.e. $x \in \mathcal{R}_h$. □

Under the assumption of exponential integrability of $K_{q,f}$ we get the following regularity result for the map f.

LEMMA 4.2. *Let $1 < q \leq 2$, $\alpha > 0$ and f be a bi-Sobolev mapping such that*

$$K_{q,f}^{\alpha} \in \operatorname{Exp}(\Omega).$$

Then

$$|Df| \in L^q \log^{-\frac{1}{\alpha}} L(\Omega).$$

PROOF. By Hadamard's inequality, (17) and (19) with $a = K_{q,f}^{\alpha}(x)$ and $b = J_f(x)$, we get

$$\begin{aligned} \int_{\Omega} \frac{|Df(x)|^q}{\log^{\frac{1}{\alpha}}(e + |Df(x)|)} dx \qquad \qquad (40)\\ = \int_{\mathcal{R}_f} \frac{|Df(x)|^q}{\log^{\frac{1}{\alpha}}(e + |Df(x)|)} dx \\ \leq C \int_{\mathcal{R}_f} \frac{(K_{q,f}^{\alpha}(x))^{\frac{1}{\alpha}} J_f(x)}{\log^{\frac{1}{\alpha}}(e + J_f(x))} dx \\ \leq c_1(\alpha) \int_{\Omega} \frac{\exp(\lambda K_{q,f}^{\alpha}(x)) + J_f(x) \log^{\frac{1}{\alpha}}(e + J_f(x))}{\log^{\frac{1}{\alpha}}(e + J_f(x))} dx \\ \leq c_1(\alpha) \left[\int_{\Omega} \exp(\lambda K_{q,f}^{\alpha}(x))\, dx + \int_{\Omega} J_f(x)\, dx \right] < +\infty. \end{aligned}$$

□

THEOREM 4.3. *Let* $p \geq 1$, $1 < q \leq 2$, $r = \dfrac{\alpha p(q-1)}{\alpha(q-1)+1}$ *and* $\alpha > 0$.

Let $f : \Omega \to \Omega'$ *and* $g : \Omega' \to \Omega''$ *be bi-Sobolev maps. If*

$$K_{q,f} \in L^{\frac{p}{q-1}}(\Omega) \quad \text{and} \quad (K_{q,g})^{\alpha} \in \mathrm{EXP}(\Omega')$$

and, for $\alpha \leq 1$ *and* $\alpha(p-1)(q-1) < 1$ *we also assume that*

$$|Dg| \in L^q \log^{(1-p)(q-1)} L(\Omega'),$$

then the composition $h = g \circ f \in W^{1,1}(\Omega; \mathbb{R}^2)$ *is a bi-Sobolev map and*

$$(K_{q,h})^{\frac{r}{q-1}} \in L^1(\Omega).$$

PROOF. By Theorem 1.2 we have

$$J_{f^{-1}}(y) \log^p(e + J_{f^{-1}}(y)) \in L^1(\Omega')$$

and

$$|Df^{-1}(y)|^{\frac{q}{q-1}} \log^{p-1}(e + |Df^{-1}(y)|) \in L^1(\Omega').$$

By Proposition 4.1, with $\alpha = p - 1$, we have that $h = g \circ f \in W^{1,1}(\Omega; \mathbb{R}^2)$ has finite distortion. By Hölder inequality, in order to prove theorem, we only have to prove that

$$K_{q,g}(f(x)) \in L^{\frac{rp}{(q-1)(p-r)}}(\Omega).$$

To this aim, since $|Df^{-1}| \in L^{\frac{q}{q-1}} \log^{p-1} L(\Omega')$, then f^{-1} verifies the *Lusin* $\mathcal{N}$ condition. Hence, area formula (14) gives

$$\int_{\Omega} K_{q,g}(f(x))^{\frac{rp}{(q-1)(p-r)}}\, dx = \int_{\Omega'} K_{q,g}(y)^{\frac{rp}{(q-1)(p-r)}} J_{f^{-1}}(y)\, dy.$$

The assumption $K^{\alpha}_{q,g}(x) \in \mathrm{Exp}(\Omega')$ yields that there exists $\lambda > 0$ such that $\int_{\Omega'} \exp(\lambda K^{\alpha}_{q,g}(y))\, dy < +\infty$.

Moreover $\dfrac{rp}{(q-1)(p-r)} \cdot \dfrac{1}{\alpha} = p$, then, by the inequality (19), with $a = K^{\alpha}_{q,g}$, $b = J_{f^{-1}}$ and $\alpha = p$, we obtain

$$\begin{aligned} K_{q,g}(y)^{\frac{rp}{(q-1)(p-r)}} \cdot J_{f^{-1}}(y) &= (K_{q,g}(y)^{\frac{rp}{(q-1)(p-r)}\cdot\frac{1}{\alpha}})^{\alpha} \cdot J_{f^{-1}}(y) \\ &= (K^{\alpha}_{q,g}(y))^p \cdot J_{f^{-1}}(y) \\ &\leq c[\exp(\lambda K^{\alpha}_{q,g}(y)) + J_{f^{-1}}(y) \log^p(e + J_{f^{-1}}(y))]. \end{aligned} \tag{41}$$

The proof is over by integrating on Ω'. □

REFERENCES

[1] K. ASTALA - T. IWANIEC - G. J. MARTIN, *Elliptic Partial Differential equations and Quasiconformal Mappings in the plane*, Princeton University Press, 2009.

[2] K. Astala - T. Iwaniec - G. J. Martin - J. Onninen, *Extremal mappings of finite distortion*, Proc. London Math. Soc. 91 (2005), no. 3, 655–702.

[3] C. Capone, *Quasiharmonic fields and Beltrami operators*, Comment. Math. Univ. Carolinae 43, 2 (2002), 363–377.

[4] C. Capone - M. R. Formica - R. Giova, *Grand Lebesgue spaces with respect to measurable functions*, Nonlinear Analysis 85 (2013), 125–131.

[5] M. Carozza - C. Sbordone, *The distance to L^∞ in some function spaces and applicatons*, Diff. Integr. Equations, 10 (1997), 599–607.

[6] G. David, *Solutions de l'e'quation de Beltrami avec* $\|\mu\|_\infty = 1$ (Solutions of the Beltrami equation with $\|\mu\|_\infty = 1$) Ann. Acad. Sci. Fenn. Math. Ser. A I Math. 13 (1), (1988), 25–70.

[7] P. Di Gironimo - L. D'Onofrio - C. Sbordone - R. Schiattarella, *Anisotropic Sobolev homeomorphisms*, Ann. Acad. Sci. Fenn. Math., 36, (2011), 593–602.

[8] L. D'Onofrio - C. Sbordone - R. Schiattarella, *The Grand Sobolev homeomorphisms and their measurability properties*, Advanced Nonlinear Studies, 12 (2012), no. 4, 767–782.

[9] L. D'Onofrio - R. Schiattarella, *On the total variations for the inverse of a BV-homeomorphism*, Adv. Calc. Var. 6, (2013), 321–338.

[10] L. D'Onofrio - R. Schiattarella, *On the continuity of Jacobian of orientation preserving mappings in the grand Sobolev space*, Differential Integral Equations, 26, (2013), no. 9/10, 1139–1148.

[11] F. Farroni - R. Giova - C. Sbordone, *Regularity points of ACL-homeomorphisms in the plane*, Contemporary Mathematics, 594 (2013), 167–178.

[12] F. Farroni - R. Giova, *Quasiconformal mappings and exponentially integrable functions*, Studia Math. 203 (2011), no. 2, 195–203.

[13] F. Farroni - R. Giova, *Quasiconformal mappings and sharp estimates for the distance to L^∞ in some function spaces*, J. Math. Anal. Appl. 395 (2012), no. 2, 694–704.

[14] F. Farroni - R. Giova, *The distance to L^∞ in the grand Orlicz spaces*, J. Funct. Spaces Appl., (2013), doi:10.1155/2013/658527.

[15] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag, New York (1969) (Second edition 1996).

[16] M. R. Formica, *On the Γ-convergence of Laplace-Beltrami operators in the plane*, Ann. Acad. Sci. Fenn. Math. 25, (2000), no. 2, 423–438.

[17] F. Giannetti - L. Greco - A. Passarelli di Napoli, *Regularity of mappings of finite distortion*, Funct. Approx. Comment. Math. 40, (2009), no. 1, 91–103.

[18] F. Giannetti - A. Passarelli di Napoli, *Bi-Sobolev mappings with differential matrices in Orlicz Zygmund classes*, J. Math. Anal. Appl. 369 (2010), no. 1, 346–356.

[19] L. Greco, *A remark on the equality det $Df = $ Det Df*, Differential Integral Equations 6 (1993), 1089–1100.

[20] L. Greco - T. Iwaniec - G. Moscariello, *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J., 44, (1995), 305–339.

[21] L. Greco - C. Sbordone - R. Schiattarella, *Composition of bi-Sobolev homeomorphisms*, Proc. Roy. Soc. Edinburgh Sect. A 142 (2012), no. 1, 61–80.

[22] S. Hencl - L. Kleprlík, *Composition of q-Quasiconformal mappings and functions in Orlicz-Sobolev spaces*, to appear in Illinois Journal of Mathematics.

[23] S. Hencl - P. Koskela, *Lectures on Mappings of finite distortion*, Monograph, (2012), preprint.

[24] S. Hencl - P. Koskela, *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Ration. Mech. Anal., 180, (2006), 75–95.

[25] S. Hencl - P. Koskela, *Mappings of finite distortion: composition operator*, Ann. Acd. Sci. Fenn. Math., 33, (2008), 65–80.

[26] S. Hencl - P. Koskela - J. Onninen, *Homeomorphisms of Bounded Variation*, Arch. Ration. Mech. Anal. 186, 2007, 351–360.

[27] S. Hencl - G. Moscariello - A. Passarelli di Napoli - C. Sbordone, *Bi-Sobolev mappings and elliptic equations in the plane*, J. Math. Anal. Appl., 355, (2009), 22–32.

[28] T. Iwaniec - C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch., Rat., Mech., Anal., 119, (1992), 129–143.

[29] L. Kleprlík, *Mappings of finite signed distortion: Sobolev spaces and composition of mappings*, J. Math. Anal. Appl. 386 (2012), no. 2, 870–881.

[30] J. Kauhanen - P. Koskela - J. Maly, *Mappings of finite distortion: Condition N*, Michigan Math. J., 49 (2001), 169–180.

[31] G. Moscariello - A. Passarelli di Napoli, *The regularity of the inverses of Sobolev homeomorphisms with finite distortion*, J. Geom. Anal., DOI:10.1007/s12220-012-9345-x.

[32] A. Passarelli di Napoli, *Bisobolev mappings and homeomorphisms with finite distortion*, Rend. Lincei Mat. Appl. 23 (2012), 437–454.

[33] Yu. G. Reshetnyak, *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Math. Zh., 7, (1966), 886–919.

[34] S. Rickman, *Quasiregular mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 26. Springer-Verlag, Berlin, 1993.

Received 10 May 2013,
and in revised form 17 June 2013.

Claudia Capone
Istituto per le Applicazioni del Calcolo "M. Picone"
Consiglio Nazionale delle Ricerche
Via Pietro Castellino 111, 80131, Napoli, Italy
capone@na.iac.cnr.it

Maria Rosaria Formica, Raffaella Giova
Dipartimento di Statistica e Matematica per la Ricerca Economica
Università degli Studi di Napoli "Parthenope"
Via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy
mara.formica@uniparthenope.it
raffaella.giova@uniparthenope.it

Roberta Schiattarella
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II"
Complesso Monte S. Angelo
Via Cintia, 80126, Napoli, Italy
roberta.schiattarella@unina.it