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Functional Analysis — *Spectral analysis and long-time behaviour of a Fokker-Planck equation with a non-local perturbation*, by Dominik Stürzer and Anton Arnold, communicated on 8 November 2013.

Abstract. — In this article we consider a Fokker-Planck equation on $\mathbb{R}^d$ with a non-local, mass preserving perturbation. We first give a spectral analysis of the unperturbed Fokker-Planck operator in an exponentially weighted L^2-space. In this space the perturbed Fokker-Planck operator is an isospectral deformation of the Fokker-Planck operator, i.e. the spectrum of the Fokker-Planck operator is not changed by the perturbation. In particular, there still exists a unique (normalized) stationary solution of the perturbed evolution equation. Moreover, the perturbed Fokker-Planck operator generates a strongly continuous semigroup of bounded operators. Any solution of the perturbed equation converges towards the stationary state with exponential rate -1, the same rate as for the unperturbed Fokker-Planck equation. Moreover, for any $k \in \mathbb{N}$ there exists an invariant subspace with codimension k (if $d = 1$) in which the exponential decay rate of the semigroup equals $-k$.

Key words: Fokker-Planck, non-local perturbation, spectral analysis, exponential stability.

Mathematics Subject Classification: 35B20, 35P99, 35Q84, 47D06.

1. Introduction

This work deals with the analysis of the following class of perturbed Fokker-Planck equations:

$$\partial_t f = \nabla \cdot (\nabla f + \mathbf{x} f) + \Theta f =: Lf + \Theta f \tag{1.1 a}$$

$$f|_{t=0} = \varphi(\mathbf{x}), \tag{1.1 b}$$

where $t \geq 0$, $\mathbf{x} \in \mathbb{R}^d$ with $d \in \mathbb{N}$, and $f = f(t, \mathbf{x})$. Here, $\partial_t f$ denotes the time derivative. The linear, non-local operator Θ is given by a convolution $\Theta f = \vartheta * f$ with respect to $\mathbf{x}$, where its kernel ϑ is assumed to be time-independent and with zero mean, i.e. $\int_{\mathbb{R}^d} \vartheta(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$. Also, it is assumed to satisfy certain regularity conditions, which will be specified in the Sections 3 and 4.

The above equation is mainly motivated by the quantum-kinetic Wigner-Fokker-Planck equation, describing so-called open quantum systems, see [3, 4]. It is of the form

Presented by Giuseppe Toscani.

$$\partial_t u = \nabla_{\mathbf{x},\mathbf{v}} \cdot (\nabla_{\mathbf{x},\mathbf{v}} u + (\nabla_{\mathbf{x},\mathbf{v}} A + \mathbf{F})u) + \Xi[V]u$$
$$u|_{t=0} = u_0,$$

where $u = u(t, \mathbf{x}, \mathbf{v})$ is the phase-space quasi-density, with $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$ denoting position and momentum. The given coefficient function $\nabla_{\mathbf{x},\mathbf{v}} A + \mathbf{F}$ is affine in $(\mathbf{x}, \mathbf{v})$ and models the confinement and friction of the system. $\Xi[V]$ is a non-local operator (convolution in $\mathbf{v}$) determined by an external potential $V(\mathbf{x})$. One question of interest in this problem is to show the existence of a unique normalized stationary state, and to prove uniform exponential convergence of the solution to the stationary state. In the case of a quadratic confinement potential with a small perturbation these questions have been answered positively in [3], see also [2] for an operator-theoretic approach. However, from the physical point of view, the restriction to nearly quadratic potentials seems quite artificial. This raises the question if the results can be extended to a more general family of (confining) potentials. In order to gain insight into what can be expected and what mechanisms are responsible for the actual behaviour, we shall consider here (1.1) as a similar, yet simplified model, which still preserves the essential structure. The non-local operator $\Xi[V]$, which is a convolution in $\mathbf{v}$, is replaced by a convolution with kernel ϑ. This represents a first step towards the full analysis.

Other examples of non-local perturbations in Fokker-Planck equations appear e.g. in the linearized vorticity formulation of the 2D Navier-Stokes equations (cf. (12)–(14) in [12]) or in electronic transport models (cf. the linearization of equations (1), (6), (7) in [20]).

For the unperturbed equation (1.1), i.e. the case $\vartheta = 0$, the natural functional setting is the space $L^2(\mu^{-1})$, with the weight function $\mu(\mathbf{x}) = \exp(-|\mathbf{x}|^2/2)$. Here, $\mu/(2\pi)^{d/2}$ is the unique steady state with normalized mass, i.e. $\int_{\mathbb{R}^d} \mu/(2\pi)^{d/2}\, d\mathbf{x} = 1$, and all solutions to initial conditions with mass one decay towards this state with exponential rate of at least -1, see e.g. [5]. However, if Θ is added, the situation often becomes more complicated. One reason is that many non-local (convolution) operators are unbounded in the space $L^2(\mu^{-1})$. This can be illustrated for the simple example with the convolution kernel $\vartheta = \delta_{-\alpha} - \delta_\alpha$, $\alpha \in \mathbb{R}$, in one dimension. It corresponds to the operator $(\Theta f)(x) = f(x+\alpha) - f(x-\alpha)$, $x \in \mathbb{R}$, which is unbounded in $L^2(\mu^{-1})$. In this case one can show (with an eigenfunction expansion) that every (non-trivial) stationary state of (1.1) is *not* even an element of $L^2(\mu^{-1})$. Thus, this space is not suitable for our intended large-time analysis, since it is "too small". This motivates to consider (1.1) in some larger space $L^2(\omega)$, with a weight ω growing slower than μ^{-1}. Due to the previous discussion we shall choose ω such that a large class of non-local operators becomes bounded. But the new space should not be "too large" either, since we would risk to loose many convenient properties (like the spectral gap) of the unperturbed Fokker-Planck operator. In $L^2(\mathbb{R}^d)$, e.g., the spectrum of L is the left half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \le d/2\}$, cf. [21]. It will turn out that $\omega(\mathbf{x}) := \cosh \beta|\mathbf{x}|$, $\beta > 0$, is a convenient choice. Moreover, there is a useful characterization of the functions of $L^2(\omega)$ in terms of their Fourier transform, see Lemma 2.2.

Here we focus on the Fokker-Planck operator in exponentially weighted spaces. For L^2-spaces with polynomial weights, the spectrum of L was studied in [11]. Furthermore, our results complement the analysis of Metafune [21], where a larger class of Ornstein-Uhlenbeck operators is investigated in unweighted L^p-spaces with $p \geq 1$.

This paper is organized as follows. Since the analysis in the d-dimensional case is very similar to the one-dimensional case, we first discuss (in Sections 2 and 3) the one-dimensional problem in great detail, to keep the notation and arguments more concise. In Section 4, we generalize the proofs to higher dimensions.

In Section 2 we investigate the one-dimensional Fokker-Planck operator in $L^2(\omega)$ (denoted by $\mathcal{L}$), and show that its spectrum is $-\mathbb{N}_0$, and consists entirely of eigenvalues. All eigenspaces are one-dimensional, in particular the stationary state is unique up to normalization. Moreover, the operator $\mathcal{L}$ generates a C_0-semigroup of uniformly bounded operators on $L^2(\omega)$, and any solution of (1.1) for $\Theta = 0$ converges towards the (appropriately scaled) stationary solution with exponential rate of at least -1. More generally, for any $k \in \mathbb{N}_0$ there exists an $\mathcal{L}$-invariant subspace of $L^2(\omega)$ with codimension k in which the associated semigroup has an exponential decay rate of $-k$. Section 3 is dedicated to the perturbed Fokker-Planck operator $\mathcal{L} + \Theta$ in one dimension. Using the compactness of the resolvent of $\mathcal{L}$ and ladder operators we show that $\mathcal{L} + \Theta$ is an isospectral deformation of the unperturbed operator $\mathcal{L}$, i.e. $\sigma(\mathcal{L} + \Theta) = \sigma(\mathcal{L}) = -\mathbb{N}_0$. The spectrum still consists only of eigenvalues with one-dimensional eigenspaces, which ensures the existence of a unique normalized steady state of (1.1) in $L^2(\omega)$. On a formal level this isospectral property of $\mathcal{L} + \Theta$ can be understood as follows: In the eigenbasis of $\mathcal{L}$, Θ corresponds to a strictly lower triangular (infinite) matrix. Finally we show that the semigroup generated by $\mathcal{L} + \Theta$ still has the same decay properties as the one generated by $\mathcal{L}$. In particular the solutions of (1.1) with normalized mass decay to the stationary state with exponential rate of at least -1. In Section 5 we present simulation results, which illustrate the decay rates obtained before.

2. The Fokker-Planck operator in weighted L^2-spaces

Here and in Section 3 we shall consider the one-dimensional Fokker-Planck equation, i.e. $d = 1$. For the Fourier transform we use the convention

$$\mathcal{F}_{x \to \xi} f \equiv \hat{f}(\xi) := \int_{\mathbb{R}} f(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x.$$

With this scaling we may identify $\hat{f}(0)$ with the *mass* of f.

For an analytic function f on a simply connected domain Ω we denote the line integral of f along a path from a to b inside of Ω by

$$\int_{a \to b} f(\zeta) \, \mathrm{d}\zeta.$$

In order to properly define complex powers, we specify a branch of the logarithm. For $\xi \in \mathbb{C}\backslash\{0\}$ we set $\ln \xi := \log|\xi| + \mathrm{i} \arg \xi$, with $\arg \xi \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\log(\cdot)$ is the natural logarithm on $\mathbb{R}^+$. For $\zeta \in \mathbb{C}$ we may then define $\xi^\zeta := \exp(\zeta \ln(\xi))$.

On a domain $\Omega \subseteq \mathbb{R}$ we call a real-valued function $w \in L^1_{\mathrm{loc}}(\Omega)$ a *weight function* if it is bounded from below by a positive constant a.e. on every compact subset of Ω. We denote the corresponding weighted L^p-space by $L^p(\Omega; w) \equiv L^p(\Omega; w(x)\,\mathrm{d}x)$, where $1 \le p \le \infty$. The space $L^2(\Omega; w)$ is equipped with the inner product

$$\langle f, g \rangle_{\Omega, w} = \int_\Omega f \bar{g} w \,\mathrm{d}x,$$

and the norm $\| \cdot \|_{\Omega, w}$.

Also, we introduce weighted Sobolev spaces. For two weight functions w_0 and w_1 and $1 \le p \le \infty$, the space $W^{1,p}(\Omega; w_0, w_1)$ consists of all functions $f \in L^p(\Omega; w_0)$, whose distributional derivative satisfies $f' \in L^p(\Omega; w_1)$. We equip the space $W^{1,2}(\Omega; w_0, w_1)$ with the norm

$$\|f\|_{\Omega, w_0, w_1} := (\|f\|^2_{\Omega, w_0} + \|f'\|^2_{\Omega, w_1})^{\frac{1}{2}},$$

see [18]. If $\Omega = \mathbb{R}$ we shall omit the symbol Ω in these notations.

Furthermore, we present some definitions and properties concerning unbounded operators and their spectrum. Let X, $\mathcal{X}$ be Hilbert spaces. If X is continuously and densely embedded in $\mathcal{X}$, we write $X \hookrightarrow \mathcal{X}$, and $X \hookrightarrow\hookrightarrow \mathcal{X}$ indicates that the embedding is compact. $\mathscr{C}(X)$ denotes the set of all closed operators A in X with dense domain $D(A)$. The set of all bounded operators $A : X \to \mathcal{X}$ is $\mathscr{B}(X, \mathcal{X})$; if $X = \mathcal{X}$ we just write $\mathscr{B}(X)$. A closed, linear subspace $Y \subset X$ is said to be *invariant* under $A \in \mathscr{C}(X)$ (or A-*invariant*) iff $D(A) \cap Y$ is dense in Y and $\operatorname{ran} A|_Y \subset Y$, see e.g. [1]. For an operator $A \in \mathscr{C}(X)$ its range is $\operatorname{ran} A$, its null space is $\ker A$, and its algebraic null space is $M(A) := \bigcup_{k \ge 0} \ker A^k$. For any $\zeta \in \mathbb{C}$ lying in the resolvent set $\rho(A)$, we denote the resolvent by $R_A(\zeta) := (\zeta - A)^{-1}$. The complement of $\rho(A)$ is the spectrum $\sigma(A)$, and $\sigma_p(A)$ is the point spectrum. For an isolated subset $\sigma' \subset \sigma(A)$ the corresponding *spectral projection* $\mathrm{P}_{A,\sigma'}$ is defined via the line integral

$$\mathrm{P}_{A,\sigma'} := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma R_A(\zeta)\,\mathrm{d}\zeta, \tag{2.1}$$

where Γ is a closed Jordan curve with counter-clockwise orientation, strictly separating σ' from $\sigma(A)\backslash\sigma'$, with σ' in the inside of Γ and $\sigma(A)\backslash\sigma'$ on the outside. The following results can be found in [17, Section III.6.4] and [29, Section V.9]: The spectral projection is a bounded projection operator, decomposing X into two A-invariant subspaces, namely $\operatorname{ran} \mathrm{P}_{A,\sigma'}$ and $\ker \mathrm{P}_{A,\sigma'}$. This property is referred to as the *reduction of A by* $\mathrm{P}_{A,\sigma'}$. A remarkable property of this decomposition is the fact that $\sigma(A|_{\operatorname{ran} \mathrm{P}_{A,\sigma'}}) = \sigma'$ and $\sigma(A|_{\ker \mathrm{P}_{A,\sigma'}}) = \sigma(A)\backslash\sigma'$. Most of the time we will be concerned with the situation where $\sigma' = \{\lambda\}$, i.e. an isolated point of the spectrum. For further results see the Appendix A.

A final remark concerns constants occurring in estimates: Throughout this article, C denotes some positive constant, not necessarily always the same. Dependence on certain parameters will be indicated in brackets, e.g. $C(t)$ for dependence on t.

We begin our analysis by investigating the unperturbed one-dimensional Fokker-Planck operator $Lf := f'' + xf' + f$ in various weighted spaces. The natural space to consider L in is $E := L^2(1/\mu)$ with $\mu(x) := \exp(-x^2/2)$. We use the notation $\|\cdot\|_E$ for the norm and $\langle \cdot, \cdot \rangle_E$ for the inner product. Writing the operator in the form

$$Lf = \left(\left(\frac{f}{\mu}\right)'\mu\right)'$$

shows that $L|_{C_0^\infty}$ is symmetric and dissipative in E. Then, the proper definition of L is obtained by the closure of $L|_{C_0^\infty}$, and this procedure yields its domain $D(L) \subset E$. In the subsequent theorem we summarize some important properties of L in E, see [21, 5, 16]. Since L in E is isometrically equivalent to the (dimensionless) quantum harmonic oscillator Hamiltonian $H = -\Delta - 1/2 + x^2/4$ in $L^2(\mathbb{R})$, we transfer many results of H (see [23] and [26, Theorem XIII.67]) to L. For the properties of the spectral projections, see also [17, Section V.3.5].

THEOREM 2.1. *The Fokker-Planck operator L in E has the following properties:*

(i) *L with $D(L) = \{f \in E : f'' + xf' + f \in E\}$ is self-adjoint and has a compact resolvent.*
(ii) *The spectrum is $\sigma(L) = -\mathbb{N}_0$, and it consists only of eigenvalues.*
(iii) *For each eigenvalue $-k \in \sigma(L)$ the corresponding eigenspace is one-dimensional, spanned by $\mu_k := \frac{1}{\sqrt{2\pi}} H_k \mu$, where*

$$H_k(x) = \mu(x)^{-1} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mu(x)$$

is the k-th Hermite polynomial.
(iv) *The eigenvectors $(\mu_k)_{k \in \mathbb{N}_0}$ form an orthogonal basis of E.*
(v) *There holds the spectral representation*

$$L = \sum_{k \in \mathbb{N}_0} -k\Pi_{L,k}, \quad \text{where } \Pi_{L,k} := \frac{\sqrt{2\pi}}{k!} \mu_k \langle \cdot, \mu_k \rangle_E$$

is the spectral projection onto the k-th eigenspace.
(vi) *The operator L generates a C_0-semigroup of contractions on E_k for all $k \in \mathbb{N}_0$, where $E_k := \ker(\Pi_{L,0} + \cdots + \Pi_{L,k-1})$, $k \geq 1$, and $E_0 := E$ are L-invariant subspaces of E. The semigroup satisfies the estimate*

$$\|\mathrm{e}^{tL}|_{E_k}\|_{\mathscr{B}(E_k)} \leq \mathrm{e}^{-kt}, \quad \forall k \in \mathbb{N}_0.$$

Hence, the Fokker-Planck equation $\partial_t f = Lf$ has a unique stationary solution with normalized mass, given by μ_0. Its orthogonal complement E_1 consists of all elements of E with zero mass. And according to Result (vi) for $k = 1$, any solution of $\partial_t f = Lf$ with unit mass converges towards μ_0 with exponential rate of at least -1 in the E-norm.

In order to analyze the perturbed equation (1.1), we quickly find that E is not appropriate. For example, for the simple (unbounded) perturbation $\Theta f(x) := f(x+\alpha) - f(x-\alpha)$, $\alpha \in \mathbb{R}$, we can explicitly compute the stationary solution f_0 of (1.1) and expand it with respect to the orthogonal basis $(\mu_k)_{k\in\mathbb{N}}$ of E. The obtained Fourier coefficients form a divergent sequence, and so $f_0 \notin E$. Therefore we consider some larger space $L^2(\omega)$ instead of E, with a weight function ω growing more slowly than μ^{-1}. Thereby we choose ω such that Θ becomes a bounded operator in $L^2(\omega)$ for a large family of convolution kernels. E.g., one can easily verify that $\Theta f(x) = f(x+\alpha) - f(x-\alpha)$ is bounded in $L^2(\exp(\beta|x|^\gamma))$ iff $\gamma \in [0,1]$ (for $\beta > 0$). At the same time, ω should grow fast enough such that L still has a spectral gap in $L^2(\omega)$, i.e. there exists some $a < 0$ such that $\{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > a\} \cap \sigma(L) = \{0\}$. These requirements suggest that exponentially growing weights would be good candidates, growing as fast as permissible while still admitting a large class of non-local operators. So, for the rest of this paper, we choose the weight function $\omega(x) = \cosh \beta x$ for some fixed $\beta > 0$, and use the corresponding space $\mathcal{E} := L^2(\cosh \beta x)$. As we will see in the following, the space $\mathcal{E}$ is very convenient also for technical purposes, since it can easily be characterized using the Fourier transform.

Lemma 2.2. *For $f \in \mathcal{E}$ we have the following properties:*

(i) *There holds $f \in \mathcal{E}$ iff its Fourier transform $\hat{f}$ possesses an analytic continuation (still denoted by $\hat{f}$) to the open strip $\Omega_{\beta/2} := \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta/2\}$, which satisfies*

$$\sup_{\substack{|b|<\beta/2 \\ b\in\mathbb{R}}} \|\hat{f}(\cdot + \mathrm{i}b)\|_{L^2(\mathbb{R})} < \infty. \tag{2.2}$$

(ii) *For $\xi \in \mathbb{R}$ and $|b| < \beta/2$, $\hat{f}$ is explicitly given by $\hat{f}(\xi + \mathrm{i}b) = \mathcal{F}_{x\to\xi}(\mathrm{e}^{bx} f(x))$.*

(iii) *The following function lies in $L^2(\mathbb{R})$:*

$$\xi \mapsto \hat{f}\left(\xi \pm \mathrm{i}\frac{\beta}{2}\right) := \mathcal{F}_{x\to\xi}(\mathrm{e}^{\pm\frac{\beta}{2}x} f(x)), \quad \textit{for a.e. } \xi \in \mathbb{R}. \tag{2.3}$$

Moreover, $b \mapsto \hat{f}(\cdot + \mathrm{i}b)$ lies in $C([-\beta/2, \beta/2]; L^2(\mathbb{R}))$. In particular (2.3) is a natural continuation of $\hat{f}$ from $\Omega_{\beta/2}$ to the closure $\overline{\Omega_{\beta/2}}$.

The proof is deferred to the Appendix C. In the following, $\hat{f}$ always denotes the extension of the Fourier transform of $f \in \mathcal{E}$ according to Lemma 2.2

(ii)–(iii). Using this convention, we introduce an alternative norm on the space $\mathcal{E}$:

$$\| f \|_{\omega}^2 := \|\hat{f}(\cdot + \mathrm{i}\beta/2)\|_{L^2(\mathbb{R})}^2 + \|\hat{f}(\cdot - \mathrm{i}\beta/2)\|_{L^2(\mathbb{R})}^2, \tag{2.4}$$

which is equal to $4\pi \|f\|_{\omega}^2$.

Furthermore, we notice that there holds a Poincaré-type inequality in $\mathcal{E}$:

Lemma 2.3 (Poincaré inequality). *The inequality*

$$\|f\|_{\omega} \leq C_{\beta} \|f'\|_{\omega} \tag{2.5}$$

holds for all $f \in W^{1,2}(\omega, \omega)$*, where* $C_{\beta} > 0$ *is a constant only depending on* β*.*

Proof. Use $|\widehat{f'}(\xi)| = |\xi \hat{f}(\xi)|$, and $|\xi| \geq \beta/2$ on $|\mathrm{Im}\,\xi| = \beta/2$. Then apply the norm $\|\cdot\|_{\omega}$. □

Our next step is to properly define the Fokker-Planck operator in $\mathcal{E}$. To this end we first define the distributional Fokker-Planck operator $\mathfrak{L}f := f'' + xf' + f$ for $f \in \mathscr{S}'$.

Lemma 2.4. *Let* $\zeta \in \mathbb{C}$ *with* $\mathrm{Re}\,\zeta \geq 1 + \beta^2/2$*, and consider the resolvent equation* $(\zeta - \mathfrak{L})f = g$ *for* $f, g \in \mathcal{E}$*. Then there exists a constant* $C > 0$ *independent of* f*,* g*, such that*

$$\|f\|_{\varpi} + \|f'\|_{\omega} \leq C\|g\|_{\omega}, \tag{2.6}$$

where $\varpi(x) = (1 + |x|)\omega(x)$*.*

Proof. Let us fix $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta \geq 1 + \beta^2/2$. Now we consider the resolvent equation $(\zeta - \mathfrak{L})f = g$ for $f, g \in \mathcal{E} \subset \mathscr{S}'$. Applying $\langle \cdot, f \rangle_{\omega}$ to both sides yields:

$$\begin{aligned} \int_{\mathbb{R}} \bar{f} g \omega \,\mathrm{d}x &= \int_{\mathbb{R}} \zeta |f|^2 \omega - (f' + xf)' \bar{f} \omega \,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \zeta\omega) + f' \bar{f} \omega' + f \bar{f}' x\omega \,\mathrm{d}x. \end{aligned}$$

Next we take the real part:

$$\begin{aligned} \mathrm{Re} \int_{\mathbb{R}} \bar{f} g \omega \,\mathrm{d}x &= \int_{\mathbb{R}} |f'|^2 \omega + |f|^2 (x\omega' + \mathrm{Re}(\zeta)\omega) + \frac{1}{2} |f^2|' (\omega' + x\omega) \,\mathrm{d}x \\ &= \int_{\mathbb{R}} |f'|^2 \omega + \frac{1}{2} |f|^2 \tilde{\omega} \,\mathrm{d}x, \end{aligned} \tag{2.7}$$

with $\tilde{\omega} := -\omega'' + x\omega' + (2\,\mathrm{Re}\,\zeta - 1)\omega$. For our choice $\omega(x) = \cosh \beta x$ we obtain $\tilde{\omega}(x) = (2\,\mathrm{Re}\,\zeta - 1 - \beta^2)\omega(x) + x\beta \sinh \beta x$. For $\mathrm{Re}\,\zeta \geq 1 + \beta^2/2$, $\tilde{\omega}$ is strictly positive. Thus, $\tilde{\omega}$ is a weight function, and it has the asymptotic behaviour

$\tilde{\omega}(x) \sim \beta|x|\omega(x)$ as $x \to \pm\infty$. Applying the Cauchy-Schwarz inequality to the left hand side of (2.7) yields

$$\frac{1}{2}\|f\|_{\tilde{\omega}}^2 + \|f'\|_{\omega}^2 \leq \|f\|_{\omega}\|g\|_{\omega}.$$

For the left hand side we use $\omega(x) \leq \tilde{\omega}(x)$ and the Poincaré inequality (2.5) to obtain

$$\frac{1}{2}\|f\|_{\tilde{\omega}} + \frac{1}{C_\beta}\|f'\|_{\omega} \leq \|g\|_{\omega}.$$

The result follows, since the weight functions $\tilde{\omega}$ and ϖ define equivalent norms. □

COROLLARY 2.5. *The operator* $(L - 1 - \beta^2/2)|_{C_0^\infty(\mathbb{R})}$ *is dissipative in* $\mathcal{E}$.

PROOF. We use the result (2.7) for $\zeta = 1 + \beta^2/2$. We then estimate the right hand side for $f \in C_0^\infty(\mathbb{R})$:

$$\mathrm{Re}\int_{\mathbb{R}} \bar{f}(L - \zeta)f \,\mathrm{d}x \leq -\Big(C_\beta + \frac{1}{2}\Big)\|f\|_{\omega}^2 \leq 0,$$

where we used the Poincaré inequality and $\tilde{\omega} \geq \omega$. □

The above results can be used to establish the proper definition of the Fokker-Planck operator in $\mathcal{E}$:

LEMMA 2.6. *The operator* $L|_{C_0^\infty(\mathbb{R})}$ *is closable in* $\mathcal{E}$. *Its closure* $\mathcal{L} := \mathrm{cl}_{\mathcal{E}}\, L|_{C_0^\infty(\mathbb{R})}$ *has the domain of definition* $D(\mathcal{L}) = \{f \in \mathcal{E} : \mathfrak{L}f \in \mathcal{E}\}$. *For* $f \in D(\mathcal{L})$ *we have* $\mathcal{L}f = \mathfrak{L}f$.

The proof is deferred to the Appendix C. It also yields the following result:

COROLLARY 2.7. *The resolvent set* $\rho(\mathcal{L})$ *is non-empty. It contains the half-plane* $\{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \geq 1 + \beta^2/2\}$.

As it turns out, the resolvent estimate (2.6) is strong enough to prove compactness of the resolvent. To this end we shall use the following simplified version of [22, Theorem 2.4]:

LEMMA 2.8. *Let* w, w_0, w_1 *be weight functions, and* $(\Omega_n)_{n\in\mathbb{N}}$ *a monotonically increasing sequence of subsets of* $\mathbb{R}$ *that converges to* $\mathbb{R}$. *Assume that for all* $n \in \mathbb{N}$ *there holds the compact embedding* $W^{1,2}(\Omega_n; w_0, w_1) \hookrightarrow\hookrightarrow L^2(\Omega_n; w)$. *Then*

$$W^{1,2}(w_0, w_1) \hookrightarrow\hookrightarrow L^2(w) \quad \Leftrightarrow \quad \lim_{n\to\infty} \sup_{\|f\|_{w_0,w_1}\leq 1} \|f\|_{\mathbb{R}\setminus\Omega_n; w} = 0.$$

From this we deduce immediately the following lemma:

LEMMA 2.9. *Let w, w_0, w_1 be weight functions. If* $\lim_{|x|\to\infty} w(x)/w_0(x) = 0$, *then the compact embedding holds:*

$$W^{1,2}(w_0, w_1) \hookrightarrow\hookrightarrow L^2(w).$$

This compact embedding allows to prove that $R_{\mathcal{L}}(\zeta)$ is compact:

THEOREM 2.10. *For any $\zeta \in \rho(\mathcal{L})$ the resolvent operator $R_{\mathcal{L}}(\zeta) : \mathcal{E} \to \mathcal{E}$ is compact. In particular $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L})$, i.e. the spectrum of $\mathcal{L}$ consists entirely of eigenvalues.*

PROOF. To begin with, we fix some $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq 1 + \beta^2/2$. According to Lemma 2.4 we have the estimate (2.6), which we can reformulate: There exists a constant $C > 0$ such that

$$\|R_{\mathcal{L}}(\zeta)g\|_{\varpi,\omega} \leq C\|g\|_{\omega}, \quad \forall g \in \mathcal{E}.$$

Hence $R_{\mathcal{L}}(\zeta) \in \mathscr{B}(\mathcal{E}, W^{1,2}(\varpi, \omega))$. Now there holds the asymptotic behaviour $\omega(x)/\varpi(x) \sim 1/|x| \to 0$ as $x \to \pm\infty$. Therefore we may apply Lemma 2.9 for $w = w_1 = \omega$ and $w_0 = \varpi$, which yields the compact embedding $W^{1,2}(\varpi, \omega) \hookrightarrow\hookrightarrow \mathcal{E}$. Thus, the resolvent $R_{\mathcal{L}}(\zeta) \in \mathscr{B}(\mathcal{E})$ is compact for $\operatorname{Re}\zeta \geq 1 + \beta^2/2$. But this already implies the compactness of $R_{\mathcal{L}}(\zeta)$ for all $\zeta \in \rho(\mathcal{L})$, cf. [17, Theorem III.6.29]. The same reference confirms that $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L})$. □

With these preparations we can now characterize the spectrum of $\mathcal{L}$:

PROPOSITION 2.11. *We have $\sigma(\mathcal{L}) = -\mathbb{N}_0$. Each eigenspace is one-dimensional, and for $k \in \mathbb{N}_0$ we have* $\ker(k + \mathcal{L}) = \operatorname{span}\{\mu_k\}$.

PROOF. We consider the Fourier transform of the eigenvalue equation $(\zeta - \mathfrak{L})f = 0$ for $f \in \mathcal{E}$. The general solution of the Fourier-transformed equation on the real line reads:

$$\hat{f}(\xi) = C_{\pm}\mu(\xi)\xi^{-\zeta}, \quad \xi \in \mathbb{R}^{\pm}. \tag{2.8}$$

For details see the computation in the beginning of the Appendix B for $g = \vartheta = 0$. Since $f \in \mathcal{E}$, $\hat{f}$ has to be analytic in $\Omega_{\beta/2}$, see Lemma 2.2. With the specification of the complex logarithm in Section 2 we may extend both parts of $\hat{f}$ from (2.8) analytically to the complex half-planes $\{\operatorname{Re}\xi > 0\}$ and $\{\operatorname{Re}\xi < 0\}$ respectively. However, if $\zeta \in \mathbb{C}\backslash\mathbb{Z}$, the two extensions do not meet continuously at the imaginary axis, thus $\hat{f}$ is not analytic in $\Omega_{\beta/2}$ (except for the trivial case $C_{\pm} = 0$). If $\zeta \in \mathbb{Z}$, we obtain continuity of $\hat{f}$ at the imaginary axis (without $\xi = 0$) iff $C_- = C_+$. But for $\zeta \in \mathbb{N}$, $\hat{f}$ still has a pole at $\xi = 0$, thus it is not analytic. In the remaining case $\zeta \in -\mathbb{N}_0$ the function $\hat{f}$ from (2.8) has an analytic extension to $\mathbb{C}$, when we choose $C_- = C_+$. So $f \in \mathcal{E}$ solves the eigenvalue equation for ζ iff $\zeta \in -\mathbb{N}_0$. And according to (2.8) the eigenspaces are still spanned by the μ_k, $k \in \mathbb{N}_0$, since $\hat{\mu}_k(\xi) = (\mathrm{i}\xi)^k \mu(\xi)$. □

The main difference to L in E is that the eigenfunctions do not form an orthogonal basis any more. However, we are still able to transfer the concept of the L-invariant subspaces $E_k \subset E$ to $\mathcal{E}$.

PROPOSITION 2.12. *For every $k \in \mathbb{N}$ we have the following facts:*

(i) *The subspace $\mathcal{E}_k := \mathrm{cl}_{\mathcal{E}} E_k$ is $\mathcal{L}$-invariant, and $\sigma(\mathcal{L}|_{\mathcal{E}_k}) = \{-k, -k-1, \ldots\}$*
(ii) *The spectral projection $\Pi_{\mathcal{L},k}$ of $\mathcal{L}$ associated to the eigenvalue $-k$ satisfies*

$$\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \mathrm{span}\{\mu_{k-1}, \ldots, \mu_0\}, \quad \mathrm{ran}\, \Pi_{\mathcal{L},k} = \mathrm{span}\{\mu_k\}.$$

Moreover, $\ker \Pi_{\mathcal{L},0} = \mathcal{E}_1$ *and* $\mathrm{ran}\, \Pi_{\mathcal{L},0} = \mathrm{span}\{\mu_0\}$.
(iii) *There holds $\mathcal{E} = \mathcal{E}_k \oplus \mathrm{span}\{\mu_{k-1}, \ldots, \mu_0\}$.*

PROOF. Since $\sigma(L) = \sigma(\mathcal{L})$, and $R_L(\zeta) \subset R_{\mathcal{L}}(\zeta)$ for all $\zeta \in \mathbb{C} \backslash (-\mathbb{N}_0)$, we conclude from (2.1) that for any $\sigma' \subset \sigma(\mathcal{L})$ there holds $\Pi_{L,\sigma'} \subset \Pi_{\mathcal{L},\sigma'}$, and they are bounded projections in E and $\mathcal{E}$, respectively. For $\sigma' := \{0, \ldots, -k+1\}$, $k \in \mathbb{N}$, we apply Lemma C.1 from the appendix: $\mathrm{ran}\, \Pi_{\mathcal{L},\sigma'} = \mathrm{cl}_{\mathcal{E}}\, \mathrm{ran}\, \Pi_{L,\sigma'} = \mathrm{cl}_{\mathcal{E}}\, \mathrm{span}\{\mu_0, \ldots, \mu_{k-1}\} = \mathrm{span}\{\mu_0, \ldots, \mu_{k-1}\}$ and $\ker \Pi_{\mathcal{L},\sigma'} = \mathrm{cl}_{\mathcal{E}} \ker \Pi_{L,\sigma'} = \mathrm{cl}_{\mathcal{E}} E_k =: \mathcal{E}_k$. This shows (i). Since the projection $\Pi_{\mathcal{L},\sigma'}$ is bounded, the range and kernel indeed represent a decomposition of $\mathcal{E}$, thus we also obtain Result (iii).
For (ii) we use the same arguments as before, with $\sigma' = \{-k\}$ instead. □

Next we characterize the subspaces $\mathcal{E}_k$.

PROPOSITION 2.13. *For $k \in -\mathbb{N}$ the subspace $\mathcal{E}_k$ is explicitly given by*

$$\mathcal{E}_k = \left\{ f \in \mathcal{E} : \int_{\mathbb{R}} f(x) x^j \,\mathrm{d}x = 0, 0 \le j \le k-1 \right\}. \tag{2.9}$$

Furthermore, there holds

$$\mathcal{E}_k = \{ f \in \mathcal{E} : \hat{f}^{(j)}(0) = 0, 0 \le j \le k-1 \}, \tag{2.10}$$

where $\hat{f}^{(j)}$ denotes the j-th derivative of the Fourier transform of f.

PROOF. The functionals $\psi_j : f \mapsto \int_{\mathbb{R}} f(x) x^j \,\mathrm{d}x$, $j \in \mathbb{N}$, are continuous in $\mathcal{E}$. We define $\tilde{\psi}_j := \psi_j|_E$. Let $f \in E_k = \{\mu_0, \ldots, \mu_{k-1}\}^{\perp_E}$. The orthogonality condition then reads

$$0 = \langle f, \mu_j \rangle_E = \int_{\mathbb{R}} f(x) \mu_j(x) \mu(x)^{-1} \,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) H_j(x) \,\mathrm{d}x, \quad \forall 0 \le j \le k-1,$$

which is equivalent to $\tilde{\psi}_0(f) = \cdots = \tilde{\psi}_{k-1}(f) = 0$. Applying Lemma C.2 from the appendix with $\mathcal{X} = \mathcal{E}$ and $X = E$ yields $\mathrm{cl}_{\mathcal{E}}\, E_k = \{f \in \mathcal{E} : \psi_j(f) = 0, 0 \le j \le k-1\}$, which is equal to $\mathcal{E}_k$ by definition. This proves (2.9).

The second equality (2.10) immediately follows from

$$\int_{\mathbb{R}} f(x)x^j \,\mathrm{d}x = \mathcal{F}_{x\to\xi}[f(x)x^j](0) = \mathrm{i}^j \hat{f}^{(j)}(0), \quad \forall j \in \mathbb{N}_0. \qquad \square$$

Remark 2.14. The representation (2.9) of the $\mathcal{E}_k$ also holds in polynomially weighted spaces, which is shown in [11, Appendix A].

The final result of this section deals with the analysis of the semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\ge0}$ generated by $\mathcal{L}$ in $\mathcal{E}$. We already know that L generates a C_0-semigroup $(\mathrm{e}^{tL})_{t\ge0}$ of bounded operators in E, and from [11, Appendix A] we get its representation (for $f \in E$):

$$\mathcal{F}_{x\to\xi}[\mathrm{e}^{tL}f] = \exp\Big(-\frac{\xi^2}{2}(1-\mathrm{e}^{-2t})\Big)\hat{f}(\xi\mathrm{e}^{-t}), \quad t \ge 0. \tag{2.11}$$

This formula can be extended to $f \in \mathcal{E}$, yielding a family $(S(t))_{t\ge0}$ of operators in $\mathcal{E}$.

Lemma 2.15. *The family of operators $(S(t))_{t\ge0}$ given by (2.11) is a family of bounded operators in $\mathcal{E}$.*

Proof. In order to show that the operators $S(t)$ are bounded, we use the norm $|||\cdot|||_\omega$. So we estimate $\|\mathcal{F}[S(t)f](\xi + \mathrm{i}\beta/2)\|$, the estimate for the other term in $|||\cdot|||_\omega$ is analogous:

$$\begin{aligned} &\|\mathcal{F}[S(t)f](\cdot + \mathrm{i}\beta/2)\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \exp\bigg(\Big[-\xi^2 + \frac{\beta^2}{4}\Big](1-\mathrm{e}^{-2t})\bigg)\bigg|\hat{f}\Big(\Big[\xi + \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big)\bigg|^2 \mathrm{d}\xi \\ &\le \exp\Big(\frac{\beta^2}{4}\Big)\int_{\mathbb{R}}\bigg|\hat{f}\Big(\Big[\xi + \mathrm{i}\frac{\beta}{2}\Big]\mathrm{e}^{-t}\Big)\bigg|^2 \mathrm{d}\xi \\ &= \exp\Big(\frac{\beta^2}{4} + t\Big)\int_{\mathbb{R}}\bigg|\hat{f}\Big(\xi + \mathrm{i}\mathrm{e}^{-t}\frac{\beta}{2}\Big)\bigg|^2 \mathrm{d}\xi \\ &\le \exp\Big(\frac{\beta^2}{4} + t\Big)|||f|||^2_{\cosh(\mathrm{e}^{-t}\beta x)} \le \exp\Big(\frac{\beta^2}{4} + t\Big)|||f|||^2_\omega \end{aligned} \tag{2.12}$$

So $(S(t))_{t\ge0}$ is a family of bounded operators in $\mathcal{E}$, and there exists a constant $M > 0$ with

$$\|S(t)\|_{\mathscr{B}(\mathcal{E})} \le M\mathrm{e}^{t/2}, \quad t \ge 0. \qquad \square$$

LEMMA 2.16. *The operator $\mathcal{L}$ is the infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$ in $\mathcal{E}$.*

PROOF. According to [24, Theorem 1.4.5], Corollary 2.5 implies that $\mathcal{L} - 1 - \beta^2/2 = \mathrm{cl}_{\mathcal{E}}(L|_{C_0^\infty} - 1 - \beta^2/2)$ is dissipative in $\mathcal{E}$. From Proposition 2.11 we also know that any $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta > 0$ lies in $\rho(\mathcal{L})$. So we can apply the Lumer-Phillips Theorem [24, Theorem 1.4.3] and find that $\mathcal{L}$ generates a C_0-semigroup $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$ of bounded operators. Since $\mathrm{e}^{t\mathcal{L}}$ and $S(t)$ are both bounded in $\mathcal{E}$ and coincide on the dense subspace $D(L) \subset \mathcal{E}$, we get $\mathrm{e}^{t\mathcal{L}} = S(t)$ in $\mathcal{E}$ for all $t \geq 0$. □

As a consequence we write $\mathrm{e}^{t\mathcal{L}} := S(t)$ for the semigroup generated by $\mathcal{L}$, and the representation (2.11) holds for all $f \in \mathcal{E}$.

PROPOSITION 2.17. *For every $k \in \mathbb{N}_0$ we have:*

(i) *The space $\mathcal{E}_k$ is invariant under the family $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$.*
(ii) *There exists some $C_k > 0$ such that*

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{E}_k}\|_{\mathscr{B}(\mathcal{E}_k)} \leq C_k \mathrm{e}^{-kt}, \quad t \geq 0.$$

PROOF. The closed subspaces $\mathcal{E}_k$ are $\mathcal{L}$-invariant, so they are also invariant under $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$.

In order to show (ii), we use the first line of (2.12) and make the additional assumption $t \geq 1$:

$$\begin{aligned}(2.13)\qquad &\|\mathcal{F}[\mathrm{e}^{t\mathcal{L}}f](\cdot + \mathrm{i}\beta/2)\|^2_{L^2(\mathbb{R}_\xi)} \\ &\leq \mathrm{e}^{\frac{\beta^2}{4}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{\xi^2}{2}} \left| \left[\xi + \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right|^{2k} \left| \frac{\hat{f}\left(\left[\xi + \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)}{\left(\left[\xi + \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)^k}\right|^2 \mathrm{d}\xi\end{aligned}$$

Here we used the inequality $\frac{1}{2} < 1 - \mathrm{e}^{-2t} < 1$ for $t \geq 1$. In the following we use the Poincaré inequality (2.5):

$$\begin{aligned}\left\| \frac{\hat{f}\left(\left[\xi + \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)}{\left(\left[\xi + \mathrm{i}\frac{\beta}{2}\right]\mathrm{e}^{-t}\right)^k}\right\|_{L^\infty(\mathbb{R}_\xi)} &= \left\| \mathcal{F}_{x\to\xi}\left(\exp\left(\frac{\beta}{2}\mathrm{e}^{-t}x\right) \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right)\right\|_{L^\infty(\mathbb{R}_\xi)} \\ &\leq \left\| \exp\left(\frac{\beta}{2}\mathrm{e}^{-t}x\right) \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_{L^1(\mathbb{R}_x)} \\ &\leq \tilde{C}(t) \left\| \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_\omega \\ &\leq C(t) \left\| (\mathrm{i}\partial_x)^k \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\hat{f}(\xi)}{\xi^k}\right]\right\|_\omega = C(t)\|f\|_\omega.\end{aligned}$$

Thereby, the constant $\tilde{C}(t)$ is given by

$$\tilde{C}(t) = \int_{\mathbb{R}} \frac{\exp(\beta e^{-t} x)}{\cosh \beta x} dx,$$

which is uniformly bounded for $t \geq 1$. Inserting this result in (2.13) yields for $t \geq 1$

$$\begin{aligned} \|\mathcal{F}[e^{t\mathcal{L}} f](\cdot + i\beta/2)\|^2_{L^2(\mathbb{R}_\xi)} &\leq C e^{\frac{\beta^2}{4}} e^{-2kt} \|f\|^2_\omega \int_{\mathbb{R}} e^{-\frac{\xi^2}{2}} \left| \xi + i\frac{\beta}{2} \right|^{2k} d\xi \\ &= C e^{-2kt} \|f\|^2_\omega. \end{aligned}$$

Thus there exists a constant $C > 0$ such that $|||e^{t\mathcal{L}} f|||_\omega \leq C e^{-kt} |||f|||_\omega$ for all $t \geq 1$. From Lemma 2.15 we also know that the semigroup is uniformly bounded for $t \in [0, 1]$, so altogether we get the desired decay estimate for the semigroup in $\mathcal{E}_k$. □

Before we turn to the perturbed Fokker-Planck equation, we summarize our results so far:

THEOREM 2.18. *Let $\omega(x) := \cosh \beta x$ for some $\beta > 0$. Then the Fokker-Planck operator $L|_{C_0^\infty(\mathbb{R})}$ is closable in $\mathcal{E} = L^2(\omega)$, and its closure $\mathcal{L} = \mathrm{cl}_{\mathcal{E}} L|_{C_0^\infty(\mathbb{R})}$ has the following properties:*

- (i) *The spectrum satisfies $\sigma(\mathcal{L}) = -\mathbb{N}_0$, and $\ker(\mathcal{L} + k) = \mathrm{span}\{\mu_k\}$ for any $k \in \mathbb{N}_0$. The eigenfunctions satisfy the relation $\mu_k = \mu_0^{(k)}$, the k-th derivative of μ_0.*
- (ii) *The resolvent $R_{\mathcal{L}}(\zeta)$ is compact in $\mathcal{E}$ for all $\zeta \notin -\mathbb{N}_0$.*
- (iii) *For any $k \in \mathbb{N}_0$ the closed subspace $\mathcal{E}_k := \mathrm{cl}_{\mathcal{E}}\, \mathrm{span}\{\mu_k, \mu_{k+1}, \dots\}$ is an $\mathcal{L}$-invariant subspace of $\mathcal{E}$, and $\mathrm{span}\{\mu_0, \dots, \mu_{k-1}\}$ is a complement. In particular $\mathcal{E}_0 = \mathcal{E}$.*
- (iv) *The spectral projection $\Pi_{\mathcal{L},k}$ to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\mathrm{ran}\, \Pi_{\mathcal{L},k} = \mathrm{span}\{\mu_k\}$ and $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \mathrm{span}\{\mu_{k-1}, \dots, \mu_0\}$ for $k \in \mathbb{N}_0$.*
- (v) *For any $k \in \mathbb{N}_0$ the operator $\mathcal{L}$ generates a C_0-semigroup on $\mathcal{E}_k$, and there exists a constant $C_k \geq 1$ such that we have the estimate*

$$\left\| e^{t\mathcal{L}}|_{\mathcal{E}_k} \right\|_{\mathscr{B}(\mathcal{E}_k)} \leq C_k e^{-kt}, \quad \forall t \geq 0.$$

REMARK 2.19. More generally, the results of Theorem 2.18 hold for all weight functions $\omega(x) = \exp(\beta |x|^\gamma)$ with either $\gamma \in (0, 2)$ and $\beta > 0$ or $\gamma = 2$ and $\beta \in \left(0, \frac{1}{2}\right]$. This can be shown by using the results from [13], where an operator decomposition method is used to transfer spectral properties of operators from a Banach space to a larger Banach space. For a detailed discussion of the application of [13], see [27].

REMARK 2.20. The sequence of eigenfunctions $(\mu_k)_{k\in\mathbb{N}_0}$ is an orthogonal basis of E. In the larger space $\mathcal{E}$, the linear hull $\operatorname{span}\{\mu_k : k \in \mathbb{N}_0\}$ is still dense, due to the continuous embedding $E \hookrightarrow \mathcal{E}$.

Also, each $f \in \mathcal{E}$ can (formally) uniquely be decomposed according to the sequence of spectral projections $(\Pi_{\mathcal{L},k})_{k\in\mathbb{N}_0}$, see the proof of Proposition 3.9. But the obtained series may diverge in $\mathcal{E}$. As an example we consider $f(x) := \exp(-|x|) \in L^2(\cosh x)$. Since f is symmetric, we have $\Pi_{\mathcal{L},k} f = 0$ if k is odd. For $k = 2n$, $n \in \mathbb{N}_0$, one can show the asymptotic behaviour for $n \to \infty$:

$$\|\Pi_{\mathcal{L},2n} f\|_\omega = \mathcal{O}\Big(\frac{\sqrt{(2n)!}}{n^{1/4}}\Big),$$

where we use the explicit representation for the Hermite polynomials H_{2n} from (5.5.4) in [28], and the asymptotic expansions for H_{2n} given in [28, Theorem 8.22.9]. Therefore, the formal series $\sum_{n\in\mathbb{N}_0} \Pi_{\mathcal{L},2n} f$ is divergent in $\mathcal{E}$. So the sequence $(\mu_k)_{k\in\mathbb{N}_0}$ is neither a Schauder basis nor a representation system of $\mathcal{E}$. However, the sequence $(\mu_k/\|\mu_k\|_E)_{k\in\mathbb{N}_0}$ is still a Bessel system, see [7, 6] for the definitions.

3. ANALYSIS OF THE PERTURBED OPERATOR

So far we have discussed the one-dimensional Fokker-Planck operator $\mathcal{L}$ in $\mathcal{E} = L^2(\omega)$, with $\omega(x) = \cosh \beta x$. In this section we investigate the properties of the perturbed (one-dimensional) operator $\mathcal{L} + \Theta$ in $\mathcal{E}$, and we shall summarize the results in Theorem 3.19. We begin by specifying the assumptions we make on the perturbation Θ.

(C) Conditions on Θ: We assume that $\Theta f = \vartheta * f$, for $f \in \mathcal{E}$, where ϑ is a tempered distribution that fulfills the following properties in $\Omega_{\beta/2}$ for some $\beta > 0$:

(i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^\infty(\Omega_{\beta/2})$.
(ii) It holds $\hat{\vartheta}(0) = 0$, i.e. ϑ has zero mean.
(iii) The mapping $\xi \mapsto \operatorname{Re} \displaystyle\int_0^1 \hat{\vartheta}(\xi s)/s \,\mathrm{d}s$ is essentially bounded in $\Omega_{\beta/2}$.

REMARK 3.1. If the conditions **(C)(i)–(ii)** hold for ϑ, then the mapping $\xi \mapsto \displaystyle\int_0^1 \hat{\vartheta}(\xi s)/s \,\mathrm{d}s$ is analytic in $\Omega_{\beta/2}$. This becomes clear when writing $\hat{\vartheta}(\xi s)/s = \xi\hat{\vartheta}(\xi s)/(\xi s)$, which is analytic for all $s \in (0, 1]$ and can be continuously extended to $\hat{\vartheta}'(0)\xi$ for $s = 0$. The analyticity of $\xi \mapsto \displaystyle\int_0^1 \hat{\vartheta}(\xi s)/s \,\mathrm{d}s$ on $\Omega_{\beta/2}$ then follows from [9, Theorem 4.9.1].

LEMMA 3.2. *There holds $\Theta f \in \mathcal{E}$ for all $f \in \mathcal{E}$ iff the condition* **(C)(i)** *holds.*

PROOF. Clearly, $\widehat{\Theta f} = \hat{\vartheta}\hat{f}$ is analytic in $\Omega_{\beta/2}$ for $f \in \mathcal{E}$. According to Lemma 2.2 there holds $\Theta f \in \mathcal{E}$ iff

$$\sup_{|b|<\beta/2} \|(\hat{\vartheta}\hat{f})(\cdot + \mathrm{i}b)\|_{L^2(\mathbb{R})} < \infty, \tag{3.1}$$

where we use $\widehat{\Theta f} = \hat{\vartheta}\hat{f}$. Now we apply Hölder's inequality and find that (3.1) holds for all $f \in \mathcal{E}$ iff ϑ satisfies **(C)(i)**. □

As a consequence of the above lemma and (3.1), the product $\hat{\vartheta}\hat{f}$ itself is the Fourier transform of an element of $\mathcal{E}$. So we may define $(\hat{\vartheta}\hat{f})(\cdot \pm \mathrm{i}\beta/2) \in L^2(\mathbb{R})$ for $f \in \mathcal{E}$ according to (2.3) whenever ϑ satisfies **(C)(i)**. With this we obtain according to Lemma 2.2 (iii):

$$b \mapsto (\hat{\vartheta}\hat{f})(\cdot + \mathrm{i}b) \in C([-\beta/2, \beta/2]; L^2(\mathbb{R})). \tag{3.2}$$

COROLLARY 3.3. *The convolution Θ is bounded in $\mathcal{E}$ if the condition* **(C)(i)** *holds.*

PROOF. We apply the norm (2.4) to Θf. The Fourier transform turns the convolution into a multiplication, so we get according to (3.2) and **(C)(i)**

$$\begin{aligned}
|||\Theta f|||_\omega^2 &= \int_{\mathbb{R}} |\hat{\vartheta}\hat{f}(\xi - \mathrm{i}\beta/2)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}} |\hat{\vartheta}\hat{f}(\xi + \mathrm{i}\beta/2)|^2 \,\mathrm{d}\xi \\
&= \lim_{b\nearrow\beta/2} \left[\int_{\mathbb{R}} |\hat{\vartheta}\hat{f}(\xi - \mathrm{i}b)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}} |\hat{\vartheta}\hat{f}(\xi + \mathrm{i}b)|^2 \,\mathrm{d}\xi \right] \\
&\le \|\hat{\vartheta}\|^2_{L^\infty(\Omega_{\beta/2})} \lim_{b\nearrow\beta/2} \left[\int_{\mathbb{R}} |\hat{f}(\xi - \mathrm{i}b)|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}} |\hat{f}(\xi + \mathrm{i}b)|^2 \,\mathrm{d}\xi \right] \\
&= \|\hat{\vartheta}\|^2_{L^\infty(\Omega_{\beta/2})} |||f|||_\omega^2.
\end{aligned}$$

□

LEMMA 3.4. *Under the assumption* **(C)** *there holds $\Theta : \mathcal{E}_k \to \mathcal{E}_{k+1} \subset \mathcal{E}_k$ for every $k \in \mathbb{N}$.*

PROOF. According to Proposition 2.13, $f \in \mathcal{E}_k$ iff $\xi = 0$ is a zero of $\hat{f}(\xi)$ of order greater or equal to k. Because of the assumption $\hat{\vartheta}(0) = 0$ the Fourier transform $\widehat{\Theta f} = \hat{\vartheta}\hat{f}$ has a zero at least of order $k + 1$ for $f \in \mathcal{E}_k$, so $\Theta f \in \mathcal{E}_{k+1}$. □

COROLLARY 3.5. *Let* **(C)** *hold, and $k \in \mathbb{N}_0$. Then the space $\mathcal{E}_k$ is an $(\mathcal{L} + \Theta)$-invariant subspace of $\mathcal{E}$.*

Since the conditions **(C)** are not very handy for direct applications, the following lemma gives some criteria that are simpler to verify and sufficient for **(C)**.

LEMMA 3.6. *Let $\beta > 0$ and $\omega(x) = \cosh \beta x$, and assume that $\vartheta \in \mathscr{S}'$ fulfills*

(i) $\hat{\vartheta}(0) = 0$,
(ii) $\vartheta = \vartheta_W + \vartheta_D$ *with* $\vartheta_W \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$ *and* $\vartheta_D \in D := \{\sum_{j=1}^n a_j \delta_{x_j} : a_j \in \mathbb{C}, x_j \in \mathbb{R}, n \in \mathbb{N}\}$, *where* δ_{x_j} *denotes the delta distribution located at* x_j.

Then $\Theta f = \vartheta * f$ *satisfies* **(C)** *for this* $\beta > 0$.

PROOF. In general $\hat{\vartheta}_W(0)$ and $\hat{\vartheta}_D(0)$ are not zero, so it is convenient to define $\vartheta_W^* := \vartheta_W + M\mu$ and $\vartheta_D^* := \vartheta_D - M\mu$, where $M := \hat{\vartheta}_D(0)/\sqrt{2\pi}$. Then $\hat{\vartheta}_W^*$ and ϑ_D^* have zero mass, and we still have $\hat{\vartheta}_W^* \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$. Since $\mathcal{F}_{x\to\xi}\delta_{x_j} = e^{-i\xi x_j}$ and $\hat{\mu}(\xi) = \sqrt{2\pi}\mu(\xi)$, it is immediate that ϑ_D^* satisfies **(C)(i)**. In order to see **(C)(iii)** for ϑ_D^*, we note that the integral occurring in this condition can be rewritten as the line integral from 0 to ξ:

$$\int_{0\to\xi} \frac{\hat{\vartheta}_D^*(z)}{z} dz$$

which is path-independent in $\mathbb{C}$ (and thus in $\Omega_{\beta/2}$), since $\hat{\vartheta}_D^*$ is an entire function and has a zero at 0. Therefore the integral itself is analytic, and thus uniformly bounded on every compact subset of $\mathbb{C}$. Because of this, it is sufficient to show uniform boundedness of this integral as $|\xi| \to \infty$ in $\Omega_{\beta/2}$. We outline this for the map $\xi \mapsto e^{-ix_j\xi}$ for any fixed $x_j \in \mathbb{R}$ and $\operatorname{Re}\xi > 1$, the case $\operatorname{Re}\xi < -1$ is analogous. Thereby we choose the following integration path (note that we may start from $z = 1$, since the integral from 0 to 1 is a constant)

$$\left|\int_{1\to\xi} \frac{e^{-ix_jz}}{z} dz\right| \leq \left|\int_1^{\operatorname{Re}(\xi)} \frac{e^{-ix_jz}}{z} dz\right| + \left|\int_{\operatorname{Re}(\xi)\to\operatorname{Re}(\xi)+i\operatorname{Im}(\xi)} \frac{e^{-ix_jz}}{z} dz\right|$$
$$\leq \left|\int_{x_j}^{\operatorname{Re}(\xi)x_j} \frac{e^{-iz}}{z} dz\right| + \frac{\beta}{2} e^{|x_j|\beta/2}.$$

The first integral is known to remain uniformly bounded as $\operatorname{Re}(\xi) \to +\infty$. For estimating the second integral we used $\xi \in \Omega_{\beta/2}$ and $\operatorname{Re}\xi \geq 1$. Since $\hat{\mu} = \sqrt{2\pi}\mu$ decays sufficiently fast in $\Omega_{\beta/2}$, it is clear that the integral of $\hat{\mu}(z)/z$ from 1 to ξ also remains uniformly bounded as $\xi \to +\infty$. Altogether, we conclude that $\hat{\vartheta}_D^*$ satisfies **(C)(iii)**.

Now we verify the same properties for ϑ_W^*. Since $\vartheta_W^* \in L^1(\omega^{\frac{1}{2}})$, we may extend $\hat{\vartheta}_W^*$ to an analytic function in $\Omega_{\beta/2}$, and there holds (2.3), cf. [8, Proposition XVI.1.3]. The Fourier transform is a continuous map from $L^1(\mathbb{R})$ to $B_0(\mathbb{R})$, i.e. the continuous functions decaying at infinity, equipped with the uniform norm. Therefore, $\vartheta_W^* \in L^1(\omega^{\frac{1}{2}})$ implies

$$\|\hat{\vartheta}_W^*\|_{L^\infty(\Omega_{\beta/2})} = \sup_{|b|<\frac{\beta}{2}} \sup_{\xi\in\mathbb{R}} |\hat{\vartheta}_W^*(\xi + ib)| \leq \sup_{|b|<\frac{\beta}{2}} \|\vartheta_W^*(x)e^{bx}\|_{L^1(\mathbb{R})}$$
$$\leq \|\vartheta_W^*(x)e^{\frac{\beta}{2}|x|}\|_{L^1(\mathbb{R})} < \infty.$$

So **(C)(i)** is satisfied. For **(C)(iii)** it is sufficient to show that for some $c > 0$ and all $\xi \in \Omega_{\beta/2}$ with $|\xi| \geq 1$ there holds $|\hat{\vartheta}^*_W(\xi)| \leq c/|\xi|$, which is fulfilled if $\mathcal{F}({\vartheta^*_W}') \in L^\infty(\Omega_{\beta/2})$. Analogously to the previous part of the proof we obtain that this is satisfied if ${\vartheta^*_W}' \in L^1(\omega^{\frac{1}{2}})$. We conclude that ϑ^*_W fulfills **(C)(i)** and **(C)(iii)** if $\vartheta^*_W \in W^{1,1}(\omega^{\frac{1}{2}}, \omega^{\frac{1}{2}})$.

Finally, ϑ satisfies the condition **(C)(ii)** due to the assumption (i). □

For the rest of the article, we shall always assume that Θ satisfies the condition **(C)** for some fixed $\beta > 0$, and we choose the weight function $\omega(x) = \cosh \beta x$ with this particular β. The first result about the perturbed Fokker-Planck operator is the following lemma:

Lemma 3.7. *The operator $\mathcal{L} + \Theta$ has compact resolvent in $\mathcal{E}$.*

Proof. A bounded perturbation of an infinitesimal generator with compact resolvent has compact resolvent again, see [10, Proposition III.1.12]. Then the result follows by combining the results of Theorems 2.10 and 2.18 for $\mathcal{L}$, and Corollary 3.3 for Θ. □

As a consequence, the spectrum of $\mathcal{L} + \Theta$ in $\mathcal{E}$ is non-empty and consists only of eigenvalues. In order to characterize the entire spectrum, we introduce the following ladder operators[1], namely the *annihilation operator*

$$\alpha^- : \mathcal{E}_1 \to \mathcal{E} : f \mapsto \int_{-\infty}^{x} f(y)\,\mathrm{d}y,$$

and its formal inverse $\alpha^+ : f \mapsto f'$, the *creation operator*.

Lemma 3.8. *The annihilation operator α^- has the following properties:*

(i) *For any $k \in \mathbb{N}$ there holds $\alpha^- \in \mathscr{B}(\mathcal{E}_k, \mathcal{E}_{k-1})$.*
(ii) *In $\mathcal{E}_1$ the operators Θ and α^- commute.*
(iii) *Let $f \in \mathcal{E}_1$, $\zeta \in \mathbb{C}$ such that $(\mathcal{L} + \Theta)f = \zeta f$. Then*

$$(\mathcal{L} + \Theta)(\alpha^- f) = (\zeta + 1)(\alpha^- f).$$

Proof. First we show (i). The property $\alpha^- : \mathcal{E}_k \to \mathcal{E}_{k-1}$ can be verified by using the explicit representation (2.9) of the $\mathcal{E}_k$, and integration by parts (first for $f \in C_0^\infty(\mathbb{R})$). The boundedness of α^- follows immediately from the Poincaré inequality (2.5). Property (ii) holds true since Θ is a convolution. For Result (iii) one applies α^- to the equation $(\mathcal{L} + \Theta)f = \zeta f$, and uses the identity $\alpha^-(\mathcal{L}f) = \mathcal{L}(\alpha^- f) - \alpha^- f$ and the Property (ii). □

[1] One of the best-known applications of ladder operators occurs in the spectral analysis of the quantum harmonic oscillator, see e.g. [15].

By using the annihilation operator, we are able to prove:

PROPOSITION 3.9. *We have the following spectral properties of $\mathcal{L}+\Theta$ in $\mathcal{E}$:*

(i) $\sigma(\mathcal{L}+\Theta) = -\mathbb{N}_0$.
(ii) *For each $k \in \mathbb{N}_0$, the eigenspace* $\ker(\mathcal{L}+\Theta+k)$ *is one-dimensional.*
(iii) *The eigenfunction f_k to the eigenvalue $-k \in \mathbb{N}_0$ is explicitly given by (up to a normalization constant)*

$$f_k = (\alpha^+)^k f_0 = f_0^{(k)}, \quad \textit{and} \quad \hat{f}_0(\xi) = \exp\Big(-\frac{\xi^2}{2} + \int_0^1 \frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\Big), \quad \xi \in \Omega_{\beta/2}. \tag{3.3}$$

In particular, f_0 is the unique stationary solution with unit mass of the perturbed Fokker-Planck equation (1.1) *in one dimension.*

PROOF. In order to show (i) we first prove that $\bigcap_{k\in\mathbb{N}} \mathcal{E}_k = \{0\}$. According to (2.10) there holds

$$\bigcap_{k\in\mathbb{N}} \mathcal{E}_k = \{f \in \mathcal{E} : \hat{f}^{(k)}(0) = 0, k \in \mathbb{N}_0\}.$$

But for $f \in \mathcal{E}$, $\hat{f}$ is analytic, and the only analytic function with a zero of infinite order is the zero function, which proves the statement.

Thus, for any eigenfunction f, there exists a unique $k \in \mathbb{N}_0$ such that $f \in \mathcal{E}_k \backslash \mathcal{E}_{k+1}$, which is the minimal $k \in \mathbb{N}_0$ with the property $\Pi_{\mathcal{L},k} f \neq 0$. Applying this projection to the eigenvalue equation yields

$$\Pi_{\mathcal{L},k}(\mathcal{L}+\Theta)f = -k\Pi_{\mathcal{L},k} f = \zeta \Pi_{\mathcal{L},k} f,$$

where we used $\Theta f \in \mathcal{E}_{k+1}$ (cf. Lemma 3.4). Hence, the eigenvalue corresponding to f satisfies $\zeta = -k$. Thus $\sigma(\mathcal{L}+\Theta) \subseteq -\mathbb{N}_0$. If now f_k is an eigenfunction with eigenvalue $-k$, we can apply k times the continuous operator α^- to f_k, and create eigenfunctions to all eigenvalues $\{-k+1, \dots, 0\}$. So either $\sigma(\mathcal{L}+\Theta) = -\mathbb{N}_0$ or $\sigma(\mathcal{L}+\Theta) = \{-k_0, \dots, 0\}$, i.e. there exists some minimal eigenvalue $-k_0$. But the latter scenario is actually not possible, because then the operator $(\mathcal{L}+\Theta)|_{\mathcal{E}_{k_0+1}}$ would have empty spectrum in $\mathcal{E}_{k_0+1}$, which contradicts the fact that it still has a compact resolvent in $\mathcal{E}_{k_0+1}$.

In order to verify (ii) we recall from the first part of the proof that if f is an eigenfunction of $\mathcal{L}+\Theta$ to the eigenvalue $-k$, then $k = \operatorname{argmin}\{\Pi_{\mathcal{L},j} f \neq 0 : j \in \mathbb{N}_0\}$. In particular,

$$\Pi_{\mathcal{L},k} f \neq 0 \tag{3.4}$$

for such an eigenfunction. Assume that $\dim\ker(\mathcal{L}+\Theta+k) > 1$ for some $k \in \mathbb{N}_0$. Thus we may choose two linearly independent eigenfunctions to the eigenvalue $-k$. Since $\dim \operatorname{ran} \Pi_{\mathcal{L},k} = 1$, we can find a linear combination of these two eigenfunctions, yielding an eigenfunction f which satisfies $\Pi_{\mathcal{L},k} f = 0$. But this contradicts (3.4) and hence $\dim\ker(\mathcal{L}+\Theta+k) = 1$.

For the third result (iii) we consider the Fourier transform of the eigenvalue equation $(\mathcal{L}+\Theta)f_k = -kf_k$ for $k \in \mathbb{N}_0$. This yields the following differential equation for $\hat{f}_k$:

$$\xi \hat{f}_k'(\xi) = (\hat{\vartheta}(\xi) + k - \xi^2)\hat{f}_k(\xi).$$

Its general solution reads

$$\hat{f}_k(\xi) = c_k \xi^k q(\xi), \quad \text{with } q(\xi) := \exp\Big(-\frac{\xi^2}{2} + \int_0^1 \frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\Big),$$

for all $k \in \mathbb{N}_0$, with $c_k \in \mathbb{C}$. We may now fix $c_k := \mathrm{i}^k$, which completes the proof. □

Remark 3.10. According to the results of Proposition 2.12 (ii) we may formally write Θ and $\mathcal{L}$ as infinite-dimensional matrices with respect to the eigenfunctions μ_k, $k \in \mathbb{N}_0$. Due to the property $\Theta : \mathcal{E}_k \to \mathcal{E}_{k+1}$ shown in Lemma 3.4 this representation of Θ is *strictly lower triangular*. Furthermore, due to Theorem 2.18 (iii), $\mathcal{L}$ is formally diagonal. And according to Proposition A.2 $\sigma(\mathcal{L}) = \sigma(\mathcal{L}+\Theta)$. This situation resembles the finite-dimensional case, in which adding a strictly triangular matrix does not change the spectrum of a diagonal matrix.

Lemma 3.11. *The spectral projection $\mathcal{P}_k$ of $\mathcal{L}+\Theta$ corresponding to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills*

$$\operatorname{ran} \mathcal{P}_k = \operatorname{span}\{f_k\}, \quad \ker \mathcal{P}_k = \mathcal{E}_{k+1} \oplus \operatorname{span}\{f_{k-1}, \dots, f_0\},$$

with the eigenfunctions $f_k, \dots, f_0$ given in (3.3). *Therefore, all singularities of the resolvent are of order one, and for all $k \in \mathbb{N}_0$ there holds $M(\mathcal{L}+\Theta+k) = \ker(\mathcal{L}+\Theta+k)$.*

Proof. The set $\mathcal{K}_k := \mathcal{E}_{k+1} \oplus \operatorname{span}\{f_{k-1}, \dots, f_0\}$ is invariant under $\mathcal{L}+\Theta$, cf. Corollary 3.5. Therefore the algebraic eigenspace satisfies $M(\mathcal{L}+\Theta+k) = \ker(\mathcal{L}+\Theta+k) = \operatorname{span}\{f_k\}$, being the complement of $\mathcal{K}_k$. In particular we obtain the $(\mathcal{L}+\Theta)$-invariant decomposition $\mathcal{E} = \mathcal{K}_k \oplus M(\mathcal{L}+\Theta+k)$, and $\sigma((\mathcal{L}+\Theta)|_{\mathcal{K}_k}) = -\mathbb{N}_0 \backslash \{-k\}$. So we can apply Lemma A.3 from the appendix, which yields the properties of the spectral projections.

Since $\dim \mathcal{P}_k = 1$ and $M(\mathcal{L}+\Theta+k) = \ker(\mathcal{L}+\Theta+k)$, the singularity of $R_{\mathcal{L}+\Theta}(\zeta)$ at $\zeta = -k$ is a pole of order one, see Proposition A.2 (iv)–(v). □

Having explicitly determined the spectrum of the perturbed Fokker-Planck operator, we now turn to the generated semigroup and the corresponding decay rates. We start with the fact that $\mathcal{L}+\Theta$ generates a C_0-semigroup:

Proposition 3.12. *For each $k \in \mathbb{N}_0$ the operator $(\mathcal{L}+\Theta)|_{\mathcal{E}_k}$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{E}_k$. The semigroup on $\mathcal{E}$ preserves mass, i.e.*

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} [\mathrm{e}^{t(\mathcal{L}+\Theta)} f](x)\,\mathrm{d}x, \quad \forall t \geq 0.$$

PROOF. According to Theorem 2.18 the operator $\mathcal{L}$ generates a C_0-semigroup on $\mathcal{E}_k$ for every $k \in \mathbb{N}_0$, and due to Lemma 3.4 and Corollary 3.3 we have $\Theta|_{\mathcal{E}_k} \in \mathscr{B}(\mathcal{E}_k)$. Now a bounded perturbation of the infinitesimal generator of a C_0-semigroup is again infinitesimal generator, see [10, Theorem III.1.3], and so the first result follows.

To show the conservation of mass we use the decomposition of $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ by $\mathcal{P}_0$ corresponding to $\mathcal{E} = \mathcal{E}_1 \oplus \operatorname{span}\{f_0\}$. The space $\mathcal{E}_1$ consists of all massless functions, so the part $\mathcal{P}_0 f$ alone determines the mass of any $f \in \mathcal{E}$. Since $\mathcal{E}_1$ and $\operatorname{span}\{f_0\}$ are both invariant under the semigroup, $\mathcal{P}_0$ and $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ commute. Furthermore we have $\mathcal{P}_0 f \in \ker(\mathcal{L}+\Theta)$, and hence $\mathrm{e}^{t(\mathcal{L}+\Theta)}\mathcal{P}_0 f = \mathcal{P}_0 f$ for all $t \geq 0$. Altogether we obtain $\mathcal{P}_0 \mathrm{e}^{t(\mathcal{L}+\Theta)} f = \mathcal{P}_0 f$ for all $f \in \mathcal{E}$, $t \geq 0$, i.e. the semigroup preserves mass. □

Next we investigate the decay rate of $(\mathrm{e}^{t(\mathcal{L}+\Theta)})_{t\geq 0}$ on the subspaces $\mathcal{E}_k$. To this end we define:

$$\hat{\psi}(\xi) := \exp\Big(\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s}\,\mathrm{d}s\Big), \quad \xi \in \Omega_{\beta/2}, \tag{3.5}$$

which is analytic in $\Omega_{\beta/2}$ according to Remark 3.1.

LEMMA 3.13. *The map* $\Psi : f \mapsto f * \psi$ *has the properties:*

(i) *For each* $k \in \mathbb{N}_0$, $\Psi : \mathcal{E}_k \to \mathcal{E}_k$ *is a bijection, with inverse* $\Psi^{-1} : f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$.
(ii) $\Psi, \Psi^{-1} \in \mathscr{B}(\mathcal{E})$.

PROOF. We define $\overline{\Psi} : f \mapsto f * \mathcal{F}^{-1}[1/\hat{\psi}]$. Due to the condition **(C)(iii)** there holds $\Psi f, \overline{\Psi} f \in \mathcal{E}$ for all $f \in \mathcal{E}$, which is shown analogously to Lemma 3.2. Let now $f \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$. Then $\hat{f}(\xi)$ has a zero of order greater or equal to k at $\xi = 0$, cf. Proposition 2.13. Since $\hat{\psi}$ and $1/\hat{\psi}$ are analytic in $\Omega_{\beta/2}$, the zero at $\xi = 0$ of $\mathcal{F}_{x\to\xi}\Psi f = \hat{f}(\xi)\hat{\psi}(\xi)$ and of $\mathcal{F}_{x\to\xi}\overline{\Psi} f = \hat{f}(\xi)/\hat{\psi}(\xi)$ is of the same order as of $\hat{f}$. So $\Psi, \overline{\Psi} : \mathcal{E}_k \to \mathcal{E}_k$ for all $k \in \mathbb{N}_0$.

By applying the Fourier transform, we see that $\Psi \circ \overline{\Psi} f = \overline{\Psi} \circ \Psi f = f$ for all $f \in \mathcal{E}$, i.e. $\overline{\Psi} = \Psi^{-1}$, and $\Psi, \Psi^{-1} : \mathcal{E}_k \to \mathcal{E}_k$ are bijections for all $k \in \mathbb{N}_0$.

Finally, as in Corollary 3.3 one proves the boundedness of Ψ and Ψ^{-1} by using the assumption **(C)(iii)**. □

The map Ψ plays a crucial role in the analysis of the perturbed Fokker-Planck operator $\mathcal{L}+\Theta$, because it relates the eigenspaces of $\mathcal{L}$ to the eigenspaces of $\mathcal{L}+\Theta$: According to Proposition 3.9 we have:

$$f_k = \Psi \mu_k, \quad k \in \mathbb{N}_0. \tag{3.6}$$

By using this property of Ψ we obtain the following result:

PROPOSITION 3.14. *Let $k \in \mathbb{N}_0$ and $\zeta \in \mathbb{C}\backslash\{-k, -k-1, \dots\}$. Then there holds*

$$R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta) \circ \Psi^{-1}|_{\mathcal{E}_k}. \tag{3.7}$$

In particular there exists a constant $\tilde{C}_k > 0$ such that

$$\|(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathscr{B}(\mathcal{E}_k)} \leq \frac{\tilde{C}_k}{(\mathrm{Re}\,\zeta + k)^n}, \quad \mathrm{Re}\,\zeta > -k,\, n \in \mathbb{N}. \tag{3.8}$$

PROOF. We fix $k \in \mathbb{N}_0$. Then for all $j \geq k$ and $\zeta \in \mathbb{C}\backslash\{-k, -k-1, \dots\}$ there holds due to (3.6):

$$R_{\mathcal{L}}(\zeta)\mu_j = \frac{\mu_j}{\zeta + j} = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) f_j = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi\mu_j.$$

So we have $R_{\mathcal{L}}(\zeta) = \Psi^{-1} \circ R_{\mathcal{L}+\Theta}(\zeta) \circ \Psi$ in the space $\mathrm{span}\{\mu_j : j \geq k\} \subset E_k$, which is dense in $\mathcal{E}_k$. Then this identity extends to $\mathcal{E}_k$ due to the continuity of the occurring operators.

In order to prove the resolvent estimate (3.8) we use

$$(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n = R_{\mathcal{L}+\Theta}(\zeta)^n|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta)^n \circ \Psi^{-1}|_{\mathcal{E}_k},$$

which follows from (3.7) and Lemma 3.13 (i). Because of $\Psi, \Psi^{-1} \in \mathscr{B}(\mathcal{E}_k)$ we conclude

$$\|(R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathscr{B}(\mathcal{E}_k)} \leq \|\Psi\|_{\mathscr{B}(\mathcal{E}_k)} \|(R_{\mathcal{L}}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathscr{B}(\mathcal{E}_k)} \|\Psi^{-1}\|_{\mathscr{B}(\mathcal{E}_k)}. \tag{3.9}$$

Due to the semigroup estimate in Theorem 2.18 (v) there holds

$$\|(R_{\mathcal{L}}(\zeta)|_{\mathcal{E}_k})^n\|_{\mathscr{B}(\mathcal{E}_k)} \leq \frac{C_k}{(\mathrm{Re}\,\zeta + k)^n}, \quad \mathrm{Re}\,\zeta > -k,\, n \in \mathbb{N},$$

according to the Hille-Yosida theorem. Inserting this estimate in (3.9) shows (3.8). □

REMARK 3.15. According to (3.7) the operators $\mathcal{L}$ and $\mathcal{L} + \Theta$ are similar:

$$\mathcal{L} + \Theta = \Psi \circ \mathcal{L} \circ \Psi^{-1}.$$

Now we consider the family of operators $(\mathcal{L}(\tau))_{\tau \in \mathbb{R}} := (\mathcal{L} + \tau\Theta)_{\tau \in \mathbb{R}}$. Clearly, for every $\tau \in \mathbb{R}$ the operators $\mathcal{L}(\tau)$ and $\mathcal{L}(0) = \mathcal{L}$ are similar with the transformation operator $\Psi(\tau)$ defined according to Lemma 3.13 (where we replace ϑ by $\tau\vartheta$ in (3.5)). Therefore, according to [19] there exists a family of operators $(B(\tau))_{\tau \in \mathbb{R}}$ such that $(\mathcal{L}(\tau), B(\tau))$ form a *Lax pair*, i.e. they obey

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{L}(\tau) = [B(\tau), \mathcal{L}(\tau)],$$

where the right hand side denotes the commutator. Since we explicitly know the transformation operator $\Psi(\tau)$ we can compute $B(\tau)$:

$$Bf := -\Psi(\tau) \circ \frac{\mathrm{d}[\Psi(\tau)]^{-1}}{\mathrm{d}\tau} f = \mathcal{F}^{-1}\left[\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} \mathrm{d}s \hat{f}\right],$$

which is independent of τ.

COROLLARY 3.16. *Let $k \in \mathbb{N}_0$. Then there exists a constant $\tilde{C}_k > 0$ such that*

$$\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k}\|_{\mathscr{B}(\mathcal{E}_k)} \leq \tilde{C}_k \mathrm{e}^{-kt}, \quad t \geq 0. \tag{3.10}$$

PROOF. The result immediately follows from (3.8) by application of the Hille-Yosida theorem. □

REMARK 3.17. The above result implies the exponential convergence of any solution of (1.1) towards the (appropriately scaled) stationary state: Choose any $f \in \mathcal{E}$. Then there exists a unique constant $m \in \mathbb{C}$ (the "mass" of f) such that $\mathcal{P}_0 f = mf_0$. So $f - mf_0 = (1 - \mathcal{P}_0)f \in \mathcal{E}_1$, cf. Lemma 3.11, which implies $\mathrm{e}^{t(\mathcal{L}+\Theta)}f - mf_0 = \mathrm{e}^{t(\mathcal{L}+\Theta)}(f - mf_0) \in \mathcal{E}_1$ for all $t \geq 0$, due to Proposition 3.12. With (3.10) and $k = 1$ this implies

$$\|\mathrm{e}^{t(\mathcal{L}+\Theta)}f - mf_0\|_\omega \leq \tilde{C}_1\|f - mf_0\|_\omega \mathrm{e}^{-t}, \quad t \geq 0.$$

REMARK 3.18. In the one dimensional case we can explicitly compute the Fourier transform of $R_{\mathcal{L}+\Theta}(\zeta)g$, see Proposition B.1: For any $k \in \mathbb{N}_0$, $\mathrm{Re}\,\zeta > -k$, and $g \in \mathcal{E}_k$, the unique solution $f \in \mathcal{E}_k$ of $(\zeta - \mathcal{L} - \Theta)f = g$ satisfies

$$\hat{f}(\xi) = \mathcal{F}_{x\to\xi}[R_{\mathcal{L}+\Theta}(\zeta)g] = \hat{f}_0(\xi)\int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1}\,\mathrm{d}s, \quad \xi \in \Omega_{\beta/2},$$

where $s^\zeta = \mathrm{e}^{\zeta \log s}$ and log is the natural logarithm on $\mathbb{R}^+$. One can use this representation for an alternative proof of the resolvent estimate (3.8). However, this becomes less convenient in higher dimensions, since it is then not clear how to properly compute the explicit Fourier transform of $R_{\mathcal{L}+\Theta}(\zeta)$.

Now we summarize our results in the final theorem:

THEOREM 3.19. *Let $\mathcal{E} = L^2(\omega)$, where $\omega(x) = \cosh \beta x$, for some $\beta > 0$, and let Θ fulfill the condition* **(C)** *for this $\beta > 0$. Then the perturbed operator $\mathcal{L} + \Theta$ has the following properties in $\mathcal{E}$:*

(i) *It has compact resolvent, and $\sigma(\mathcal{L}+\Theta) = \sigma_p(\mathcal{L}+\Theta) = -\mathbb{N}_0$.*
(ii) *There holds $M(\mathcal{L}+\Theta+k) = \ker(\mathcal{L}+\Theta+k) = \mathrm{span}\{f_k\}$, where f_k is the eigenfunction to the eigenvalue $-k$ given by* (3.3). *The eigenfunctions are related by $f_k = f_0^{(k)}$.*

(iii) *The spectral projection* $\mathcal{P}_k$ *corresponding to the eigenvalue* $-k \in -\mathbb{N}$ *fulfills*

$$\operatorname{ran}\mathcal{P}_k = \operatorname{span}\{f_k\}, \quad \ker \mathcal{P}_k = \mathcal{E}_{k+1} \oplus \operatorname{span}\{f_{k-1}, \dots, f_0\},$$

where the $(\mathcal{L}+\Theta)$*-invariant spaces* $\mathcal{E}_k$ *are explicitly given in* (2.9). *Moreover,* $\operatorname{ran}\mathcal{P}_0 = \operatorname{span}\{f_0\}$ *and* $\ker \mathcal{P}_0 = \mathcal{E}_1$.

(iv) *For every* $k \in \mathbb{N}_0$, *the operator* $(\mathcal{L}+\Theta)|_{\mathcal{E}_k}$ *generates a* C_0*-semigroup in* $\mathcal{E}_k$, *denoted by* $(\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k})_{t\geq 0}$, *which satisfies the estimate*

$$\|\mathrm{e}^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k}\|_{\mathscr{B}(\mathcal{E}_k)} \leq \tilde{C}_k \mathrm{e}^{-kt}, \quad t \geq 0,$$

where the constant $\tilde{C}_k > 0$ *is independent of t.*

REMARK 3.20. Apparently, the particular choice of $\beta > 0$ has no influence on the above results, except possibly for the constants $\tilde{C}_k$. In practice, the constant β may therefore be chosen arbitrarily small, such that Θ satisfies **(C)** for this β.

4. THE HIGHER-DIMENSIONAL CASE

As already mentioned in the introduction, the preceding results can be generalized to higher dimensions without much additional effort. Most proofs are analogous to the ones in the one-dimensional case. Therefore we give here only an outline of the steps leading to the extension of Theorem 3.19 to higher dimensions.

In this section we consider the perturbed Fokker-Planck equation (1.1) on $\mathbb{R}^d$, where $d \in \mathbb{N}$ is the spatial dimension. Elements of $\mathbb{R}^d$ resp. $\mathbb{C}^d$ are represented by bold letters, e.g. $\mathbf{x} \in \mathbb{R}^d$, $\boldsymbol{\xi} \in \mathbb{C}^d$, and we write $\mathbf{x} = (x_1, \dots, x_d)$. For a multi-index $\mathbf{k} \in \mathbb{N}_0^d$ we define $|\mathbf{k}| := k_1 + \cdots + k_d$, $\mathbf{x}^{\mathbf{k}} := x_1^{k_1} \dots x_d^{k_d}$ and $\mathbf{k}! := k_1! \dots k_d!$. Furthermore

$$D^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial x_1^{k_1} \dots \partial x_d^{k_d}}.$$

We adopt the notation for weighted Sobolev spaces on $\mathbb{R}^d$ from Section 2, as well as the normalization of the Fourier transform.

We consider the Fokker-Planck operator on $\mathbb{R}^d$ given by

$$Lf := \nabla \cdot \left(\mu \nabla \left(\frac{f}{\mu}\right)\right) = \Delta f + \mathbf{x} \cdot \nabla f + df,$$

where $\mu(\mathbf{x}) := \exp(-\mathbf{x} \cdot \mathbf{x}/2)$. The natural space to consider L in is $E := L^2(1/\mu)$. Since it is isometrically equivalent to the harmonic oscillator $H := -\Delta - d/2 + |\mathbf{x}|^2/4$ in $L^2(\mathbb{R}^d)$, we transfer many results of H (see [23] and [26, Theorem XIII.67]) to L. In the following we summarize some properties of L in E (see also [21, 5, 16]):

THEOREM 4.1. *The Fokker-Planck operator L in E has the following properties:*

(i) *L with $D(L) = \{f \in E : Lf \in E\}$ is self-adjoint and has a compact resolvent.*
(ii) *The spectrum is $\sigma(L) = -\mathbb{N}_0$, and it consists only of eigenvalues.*
(iii) *For each eigenvalue $-k \in \sigma(L)$ the corresponding eigenspace has the dimension $\binom{k+d-1}{k}$, and it is spanned by the eigenfunctions*

$$\mu_{\mathbf{k}}(\mathbf{x}) := \prod_{\ell=1}^{d} \mu_{k_\ell}(x_\ell), \quad |\mathbf{k}| = k,$$

where the μ_j are defined in Theorem 2.1.
(iv) *The eigenfunctions $(\mu_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}_0^d}$ form an orthogonal basis of E.*
(v) *The spectral projection $\Pi_{L,k}$ onto the k-th eigenspace is given by*

$$\Pi_{L,k} = \sum_{|\mathbf{k}|=k} \Pi_{L,\mathbf{k}}, \quad \textit{where } \Pi_{L,\mathbf{k}} := \frac{(2\pi)^{d/2}}{\mathbf{k}!} \mu_{\mathbf{k}} \langle \cdot, \mu_{\mathbf{k}} \rangle_E.$$

There holds the spectral representation $L = \sum_{k\in\mathbb{N}_0} -k\Pi_{L,k}$.
(vi) *The operator L generates a C_0-semigroup of contractions on E_k for all $k \in \mathbb{N}_0$, where $E_k := \ker(\Pi_{L,0} + \cdots + \Pi_{L,k-1})$, $k \geq 1$, and $E_0 := E$. The semigroup satisfies the estimate*

$$\|e^{tL}|_{E_k}\|_{\mathscr{B}(E_k)} \leq e^{-kt}, \quad \forall k \in \mathbb{N}_0.$$

The next step is to properly define L in $\mathcal{E} := L^2(\omega)$ with a weight $\omega(\mathbf{x}) = \cosh\beta|\mathbf{x}|$ with $\beta > 0$. As in the one-dimensional case we have a characterization of $\mathcal{E}$ by the Fourier transform. Due to (a small variant of) [25, Theorem IX.13] we have: There holds $f \in \mathcal{E}$ iff $\hat{f}$ has an analytic extension (denoted by $\hat{f}$ as well) to the set $\Omega_{\beta/2} := \{\mathbf{z} \in \mathbb{C}^d : |\operatorname{Im} \mathbf{z}| < \beta/2\}$ and

$$\sup_{\substack{|\mathbf{b}|<\beta/2 \\ \mathbf{b}\in\mathbb{R}^d}} \|\hat{f}(\cdot + i\mathbf{b})\|_{L^2(\mathbb{R}^d)} < \infty. \tag{4.1}$$

For any $\mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{b}| < \beta/2$ we have $\hat{f}(\boldsymbol{\xi} + i\mathbf{b}) = \mathcal{F}_{\mathbf{x}\to\xi}(e^{\mathbf{b}\cdot\mathbf{x}} f(\mathbf{x}))$. The right hand side still makes sense for $|\mathbf{b}| = \beta/2$ as an $L^2(\mathbb{R}^d)$-function. And according to this identity and Plancherel's formula there holds $\mathbf{b} \mapsto \hat{f}(\cdot + i\mathbf{b}) \in C(\overline{B(\beta/2, 0)}; L^2(\mathbb{R}^d))$, where $B(\beta/2, 0) := \{\mathbf{b} \in \mathbb{R}^d : |\mathbf{b}| < \beta/2\}$. We can use this fact to define the norm

$$|||f|||_\omega^2 := \sum_{\ell=1}^{d} \left\| \hat{f}\left(\cdot + i\frac{\beta}{2}\boldsymbol{\delta}_\ell \right) \right\|_{L^2(\mathbb{R}^d)}^2 + \left\| \hat{f}\left(\cdot - i\frac{\beta}{2}\boldsymbol{\delta}_\ell \right) \right\|_{L^2(\mathbb{R}^d)}^2, \tag{4.2}$$

where $\boldsymbol{\delta}_\ell \in \mathbb{R}^d$ is the vector whose ℓ-th component is one, and all others are zero. The norm $|||\cdot|||_\omega$ is equivalent to $\|\cdot\|_\omega$.

In $\mathcal{E}$ there hold Poincaré-type inequalities:

LEMMA 4.2. *For every* $\mathbf{k} \in \mathbb{N}_0^d$ *there exists a constant* $C_{\mathbf{k}} > 0$ *such that for all* $f \in C_0^\infty(\mathbb{R}^d)$*:*

$$\|f\|_\omega \le C_{\mathbf{k}} \|D^{\mathbf{k}} f\|_\omega. \tag{4.3}$$

For the proof see Appendix C. A similar statement is given in [14, Theorem 14.5]. By using this Poincaré inequality we can generalize Lemma 2.4: Let again $\mathfrak{L} = \Delta + \mathbf{x} \cdot \nabla + d$ be the distributional Fokker-Planck operator. For $f, g \in \mathcal{E} \subset \mathcal{S}'$ with $(\zeta - \mathfrak{L})f = g$ we have the estimate

$$\|f\|_\varpi + \|\nabla f\|_\omega \le C\|g\|_\omega, \tag{4.4}$$

where $\varpi(\mathbf{x}) = (2\operatorname{Re}\zeta - d)\omega(\mathbf{x}) + \mathbf{x} \cdot \nabla\omega(\mathbf{x}) - \Delta\omega(\mathbf{x})$, which is a weight function for $\operatorname{Re}\zeta$ sufficiently large. Now we may proceed analogously to the proof of Lemma 2.6 and show that $\mathfrak{L}|_{C_0^\infty(\mathbb{R}^d)}$ is closable in $\mathcal{E}$, and its closure $\mathcal{L}$ has the domain $D(\mathcal{L}) = \{f \in \mathcal{E} : \mathfrak{L}f \in \mathcal{E}\}$. From [22, Theorem 2.4] we get the compact embedding $W^{1,2}(\varpi, \omega) \hookrightarrow\hookrightarrow \mathcal{E}$, and together with the estimate (4.4) this implies the compactness of the resolvent of $\mathcal{L}$, analogously to Theorem 2.10. Hence, the spectrum of $\mathcal{L}$ consists only of eigenvalues, and there holds:

LEMMA 4.3. *In* $\mathcal{E}$ *we have* $\sigma(\mathcal{L}) = -\mathbb{N}_0$. *The eigenspaces are still spanned by the* $\mu_{\mathbf{k}}$.

PROOF (Sketch). We consider the Fourier transform of the eigenvalue equation $\mathfrak{L}f = \zeta f$, and by setting $\tilde{f}(\xi) := \hat{f}(\xi)/\hat{\mu}(\xi)$ we get analogously to the calculation in the Appendix B the equation

$$\xi \cdot \nabla\tilde{f}(\xi) = -\zeta\tilde{f}(\xi). \tag{4.5}$$

For each $j \in \{1, \dots, d\}$ the function $\tilde{f}(0, \dots, 0, \xi_j, 0, \dots, 0)$ needs to be analytic in $\Omega_{\beta/2}$, and satisfies (B.1) for $\tilde{g} = 0$. So, as in the Appendix B we find that it is necessary that $\zeta \in -\mathbb{N}_0$.

For $k := -\zeta \in \mathbb{N}_0$ and $\xi \in \mathbb{R}^d$ we obtain by differentiating (4.5) with respect to ξ_j:

$$\xi \cdot \nabla\left(\frac{\partial\tilde{f}(\xi)}{\partial\xi_j}\right) = (k-1)\left(\frac{\partial\tilde{f}(\xi)}{\partial\xi_j}\right).$$

Thus, for any $\mathbf{k} \in \mathbb{N}_0^d$ with $|\mathbf{k}| = k$ we get

$$\xi \cdot \nabla(D^{\mathbf{k}}\tilde{f}(\xi)) = 0,$$

and all characteristics meet at $\xi = 0$. $\hat{f}$ is analytic on $\mathbb{R}^d$. Hence, the continuity of $D^{\mathbf{k}}\tilde{f}(\xi)$ at $\xi = 0$ implies $D^{\mathbf{k}}\tilde{f}(\xi) = C$ for some constant $C \in \mathbb{C}$. This holds for any $|\mathbf{k}| = k$, so the general solution of (4.5) is a linear combination of all $\xi^{\mathbf{k}}$

with $|\mathbf{k}| = -\zeta = k$. Therefore, the Fourier transform of an eigenfunction f with $(\mathcal{L} + k)f = 0$ is a linear combination of the $\boldsymbol{\xi}^{\mathbf{k}}\mu(\boldsymbol{\xi})$ with $|\mathbf{k}| = k$ (and, equivalently, $f(\mathbf{x})$ is a linear combination of the $D^{\mathbf{k}}\mu(\mathbf{x})$). Then, according to Theorem 4.1 (iii) and Theorem 2.1 (iii), the eigenspace for $\zeta = -k$ is spanned by the $\mu_{\mathbf{k}}$. □

As in Proposition 2.12 we can define the $\mathcal{L}$-invariant subspaces $\mathcal{E}_k := \mathrm{cl}_{\mathcal{E}}\, E_k = \mathrm{cl}_{\mathcal{E}}\, \mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| \geq k\}$ for all $k \in \mathbb{N}_0$, and $\sigma(\mathcal{L}|_{\mathcal{E}_k}) = \{-k, -k-1, \ldots\}$. By applying Lemma C.2 we get by induction

$$\mathcal{E}_k = \left\{ f \in \mathcal{E} : \int_{\mathbb{R}^d} f(\mathbf{x})\mathbf{x}^{\mathbf{k}}\, \mathrm{d}\mathbf{x} = 0, |\mathbf{k}| \leq k-1 \right\} \tag{4.6}$$
$$= \{ f \in \mathcal{E} : D^{\mathbf{k}}\hat{f}(0) = 0, |\mathbf{k}| \leq k-1 \}.$$

Analogously to Proposition A.2 (ii) we can also characterize the spectral projections corresponding to the eigenvalues $-k \in -\mathbb{N}_0$, see the result of Theorem 4.4 (iii) below. Finally, as in the one-dimensional case, one shows that $\mathcal{L}$ generates a C_0-semigroup of bounded operators $(\mathrm{e}^{t\mathcal{L}})_{t\geq 0}$, which is given by the formula (cf. [11, Appendix A])

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\xi}}[\mathrm{e}^{t\mathcal{L}}f] = \exp\Big(-\frac{\boldsymbol{\xi}\cdot\boldsymbol{\xi}}{2}(1-\mathrm{e}^{-2t})\Big)\hat{f}(\boldsymbol{\xi}\mathrm{e}^{-t}), \quad t \geq 0.$$

The corresponding decay estimates on the subspaces $\mathcal{E}_k$ can be shown as in the proof of Proposition 2.17. Thereby one uses the norm (4.2) and the Poincaré inequality (4.3).

THEOREM 4.4. *In $\mathcal{E} := L^2(\omega)$, with $\omega(\mathbf{x}) = \cosh \beta|\mathbf{x}|$ and $\beta > 0$, the operator L is closable, and $\mathcal{L} := \mathrm{cl}_{\mathcal{E}}\, L$ has the following properties:*

(i) *The spectrum satisfies $\sigma(\mathcal{L}) = -\mathbb{N}_0$, and $M(\mathcal{L}+k) = \ker(\mathcal{L}+k) = \mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| = k\}$ for any $k \in \mathbb{N}_0$. The eigenfunctions satisfy $\mu_{\mathbf{k}} = D^{\mathbf{k}}\mu_{\mathbf{0}}$.*
(ii) *For any $k \in \mathbb{N}_0$ the closed subspace $\mathcal{E}_k := \mathrm{cl}_{\mathcal{E}}\, \mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| \geq k\}$ is an $\mathcal{L}$-invariant subspace of $\mathcal{E}$, and $\mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| \leq k-1\}$ is a complement. In particular $\mathcal{E}_0 = \mathcal{E}$.*
(iii) *The spectral projection $\Pi_{\mathcal{L},k}$ to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\mathrm{ran}\,\Pi_{\mathcal{L},k} = \mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| = k\}$ and $\ker \Pi_{\mathcal{L},k} = \mathcal{E}_{k+1} \oplus \mathrm{span}\{\mu_{\mathbf{k}} : |\mathbf{k}| \leq k-1\}$.*
(iv) *For any $k \in \mathbb{N}_0$ the operator $\mathcal{L}$ generates a C_0-semigroup on $\mathcal{E}_k$, and there exists a constant $C_k \geq 1$ such that we have the estimate*

$$\|\mathrm{e}^{t\mathcal{L}}|_{\mathcal{E}_k}\|_{\mathscr{B}(\mathcal{E}_k)} \leq C_k \mathrm{e}^{-kt}, \quad \forall t \geq 0.$$

Next we specify the conditions on the perturbation Θ.

($\mathbf{C_d}$) Conditions on $\boldsymbol{\Theta}$: We assume that $\Theta f = \vartheta * f$, for $f \in \mathcal{E}$, where ϑ is a tempered distribution that fulfills the following properties in $\Omega_{\beta/2}$ for some $\beta > 0$:

(i) The Fourier transform $\hat{\vartheta}$ can be extended to an analytic function in $\Omega_{\beta/2}$ (also denoted by $\hat{\vartheta}$), and $\hat{\vartheta} \in L^\infty(\Omega_{\beta/2})$.
(ii) It holds $\hat{\vartheta}(\mathbf{0}) = 0$, i.e. ϑ has zero mean.
(iii) The mapping $\xi \mapsto \operatorname{Re} \int_0^1 \hat{\vartheta}(\xi s)/s \, \mathrm{d}s$ is essentially bounded in $\Omega_{\beta/2}$.

Condition **($\mathbf{C_d}$)(i)** ensures that $\Theta \in \mathscr{B}(\mathcal{E})$, which is seen by using the norm $|||\cdot|||_\omega$. And due to **($\mathbf{C_d}$)(ii)** we have $\Theta : \mathcal{E}_k \to \mathcal{E}_{k+1}$ for all $k \in \mathbb{N}_0$. In the following we always assume that **($\mathbf{C_d}$)** holds.

Proposition 4.5. *We have the following spectral properties of $\mathcal{L} + \Theta$ in $\mathcal{E}$:*

(i) $\sigma(\mathcal{L} + \Theta) = -\mathbb{N}_0$.
(ii) *For each $k \in \mathbb{N}_0$, the eigenspace* $\ker(\mathcal{L} + \Theta + k)$ *has the dimension* $\binom{k+d-1}{k}$.
(iii) *Under appropriate scaling, the eigenfunctions $f_{\mathbf{k}}$ to the eigenvalue $-k \in \mathbb{N}_0$ are explicitly given by*

$$f_{\mathbf{k}} = D^{\mathbf{k}} f_{\mathbf{0}}, \quad |\mathbf{k}| = k, \tag{4.7}$$

where

$$\hat{f}_{\mathbf{0}}(\xi) := \exp\Big(-\frac{\xi \cdot \xi}{2} + \int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} \mathrm{d}s\Big), \quad \xi \in \Omega_{\beta/2} \subset \mathbb{C}^d. \tag{4.8}$$

Thereby $f_{\mathbf{0}}$ is the unique stationary solution of the perturbed Fokker-Planck equation (1.1) *with unit mass.*

Proof (Sketch). Since the resolvent is compact (see the discussion above), the spectrum consists only of eigenvalues. As in the one-dimensional case one shows $\sigma(\mathcal{L} + \Theta) \subseteq -\mathbb{N}_0$ by applying $\Pi_{\mathcal{L},k}$ to the eigenvalue equation. This also implies $\dim \ker(k + \mathcal{L} + \Theta) \le \dim \operatorname{ran} \Pi_{\mathcal{L},k} = \binom{k+d-1}{k}$. Then one verifies that the functions $f_{\mathbf{k}}$ given in (4.7) are eigenfunctions, and lie in $\mathcal{E}$, according to the condition (4.1). Since $\dim \operatorname{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\} = \binom{k+d-1}{k}$, there are no further eigenfunctions, due to the previous estimate on the dimension of the eigenspaces. So $\ker(k + \mathcal{L} + \Theta) = \operatorname{span}\{f_{\mathbf{k}} : |\mathbf{k}| = k\}$ for all $k \in \mathbb{N}_0$. □

Now we introduce

$$\hat{\psi}(\xi) := \exp\Big(\int_0^1 \frac{\hat{\vartheta}(\xi s)}{s} \mathrm{d}s\Big), \quad \xi \in \Omega_{\beta/2},$$

and the mapping $\Psi : f \mapsto f * \psi$. The results of Lemma 3.13 for Ψ still hold, and due to (4.8) we have for all $\mathbf{k} \in \mathbb{N}_0$:

$$f_{\mathbf{k}} = \Psi \mu_{\mathbf{k}}.$$

As in Proposition 3.14 we obtain $R_{\mathcal{L}+\Theta}(\zeta)|_{\mathcal{E}_k} = \Psi \circ R_{\mathcal{L}}(\zeta) \circ \Psi^{-1}|_{\mathcal{E}_k}$, for all $k \in \mathbb{N}_0$ and $\zeta \in \mathbb{C} \backslash \{-k, -k-1, \ldots\}$. The estimates (3.8) and (3.10) also hold here, and for the convergence of $f(t) = e^{t(\mathcal{L}+\Theta)}f$ to the stationary solution see Remark 3.17. As in Section 3 we finally have:

THEOREM 4.6. *Let $\mathcal{E} = L^2(\omega(\mathbf{x}))$, where $\omega(\mathbf{x}) = \cosh\beta|\mathbf{x}|$, for some $\beta > 0$ and $\mathbf{x} \in \mathbb{R}^d$, and let Θ fulfill the condition* **(C$_\mathbf{d}$)** *for this $\beta > 0$. Then the perturbed operator $\mathcal{L}+\Theta$ has the following properties in $\mathcal{E}$:*

(i) *It has compact resolvent, and $\sigma(\mathcal{L}+\Theta) = \sigma_p(\mathcal{L}+\Theta) = -\mathbb{N}_0$.*
(ii) *There holds $M(\mathcal{L}+\Theta+k) = \ker(\mathcal{L}+\Theta+k) = \operatorname{span}\{f_\mathbf{k} : |\mathbf{k}| = k\}$, where the $f_\mathbf{k}$ are the eigenfunctions given by* (4.7). *They are related by $f_\mathbf{k} = D^\mathbf{k} f_\mathbf{0}$.*
(iii) *The spectral projection $\mathcal{P}_k$ to the eigenvalue $-k \in -\mathbb{N}_0$ fulfills $\operatorname{ran}\mathcal{P}_k = \operatorname{span}\{f_\mathbf{k} : |\mathbf{k}| = k\}$ and $\ker\mathcal{P}_k = \mathcal{E}_{k+1} \oplus \operatorname{span}\{f_\mathbf{k} : |\mathbf{k}| \le k-1\}$, where the $(\mathcal{L}+\Theta)$-invariant spaces $\mathcal{E}_k$ are explicitly given in* (4.6).
(iv) *For every $k \in \mathbb{N}_0$, the operator $(\mathcal{L}+\Theta)|_{\mathcal{E}_k}$ generates a C_0-semigroup in $\mathcal{E}_k$, denoted by $(e^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k})_{t\ge 0}$, which satisfies the estimate*

$$\|e^{t(\mathcal{L}+\Theta)}|_{\mathcal{E}_k}\|_{\mathscr{B}(\mathcal{E}_k)} \le \tilde{C}_k e^{-kt}, \quad t \ge 0,$$

where the constant $\tilde{C}_k > 0$ is independent of t.

5. SIMULATION RESULTS

In this section we shall illustrate numerically the exponential convergence for the one-dimensional perturbed Fokker-Planck equation (1.1), with $\vartheta := \varepsilon(\delta_{-\alpha} - \delta_\alpha)$, i.e. $\Theta f(x) = \varepsilon(f(x+\alpha) - f(x-\alpha))$, for some $\varepsilon, \alpha \in \mathbb{R}$. The eigenfunctions f_k of the evolution operator $\mathcal{L}+\Theta$ can be obtained by an inverse Fourier transform, with $\hat{f}_k$ explicitly given in (3.3). If the initial condition φ is a (finite) linear combination of the f_k, the solution to (1.1) reads explicitly

$$f(t,x) = e^{t(\mathcal{L}+\Theta)}\left[\sum_{j=1}^{n} a_j f_{k_j}\right] = \sum_{j=1}^{n} a_j e^{-k_j t} f_{k_j}, \quad \forall t \ge 0.$$

In the simulation we use a mass conserving Crank-Nicolson finite difference scheme for (1.1). It is employed on the spatial interval $[-25, 25]$ (with 1500 gridpoints) along with zero-flux boundary conditions. Moreover, we choose $\alpha = \varepsilon = 2$ and $\beta = 1$, i.e. $\mathcal{E} = L^2(\cosh x)$.

The following numerical results verify the decaying behaviour of solutions to (1.1), and yield an estimate to the constants $\tilde{C}_k$ from Theorem 3.19. First we consider the initial condition $\varphi_1 = (f_1 - 1.32 f_2)/\|f_1 - 1.32 f_2\|_\omega$. For the corresponding solution we plot $\|f(t,\cdot)\|_\omega$ in Figure 1(a). Since the sequence $(f_k)_{k\in\mathbb{N}}$ is not orthogonal in $\mathcal{E}$, the initial decay rate is here smaller than the individual decay rate of f_1 (i.e. -1). But after some time, the f_1-term becomes dominant, and the decay rate approaches -1. For large times, the norm behaves approximately like $1.73\,e^{-t}$, so we have the lower bound $\tilde{C}_1 \ge 1.73$.

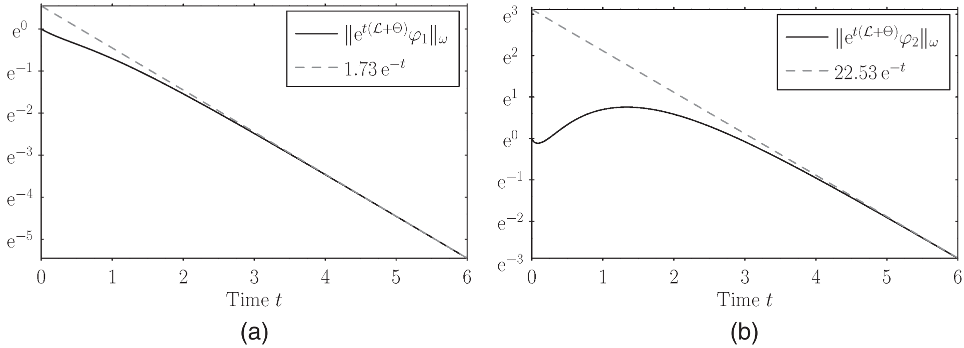

Figure 1: Evolution of the norm $\|\cdot\|_\omega$ of solutions of the perturbed equation for different initial conditions φ. (a) Initial condition $\varphi_1 = (f_1 - 1.32f_2)/\|f_1 - 1.32f_2\|_\omega$. (b) Initial condition $\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]})/\|\chi_{[-4,0]} - \chi_{[0,4]}\|_\omega \in \mathcal{E}_1$.

As a second example we choose the initial condition $\varphi_2 = (\chi_{[-4,0]} - \chi_{[0,4]})/\|\chi_{[-4,0]} - \chi_{[0,4]}\|_\omega$. It lies in $\mathcal{E}_1$ since it is massless. The evolution of $\|f(t,\cdot)\|_\omega$ is displayed in Figure 1(b). Here, the norm even increases initially. Only after some time, the norm begins to decay with a rate tending to -1. For large times t, the norm behaves approximately like $22.53\,e^{-t}$, which shows $\tilde{C}_1 \geq 22.53$.

A. Spectral Projections

In this section we review some properties of spectral projections and resolvents, cf. [29, Chapters V.9–10], [30, Chapter VIII.8] and [17, Sections III.6.4–5].

Here, X is a Hilbert space, $A \in \mathscr{C}(X)$, and we assume $\lambda \in \sigma(A)$ to be an isolated point of the spectrum. Then the corresponding spectral projection $\mathrm{P}_{A,\lambda}$ is defined by (2.1), and λ is an isolated singularity of the resolvent $R_A(\zeta)$.

PROPOSITION A.1. *For every $n \in \mathbb{N}$ we have*

$$\operatorname{ran}(\lambda - A)^n \supseteq \ker \mathrm{P}_{A,\lambda},$$
$$\ker(\lambda - A)^n \subseteq \operatorname{ran} \mathrm{P}_{A,\lambda}.$$

There exists some $n \in \mathbb{N}$ such that both inclusion relations become equalities iff λ is a pole of $R_A(\zeta)$. In this case $\lambda \in \sigma_p(A)$, i.e. an eigenvalue.

PROPOSITION A.2. *For the reduction of A by a fixed spectral projection $\mathrm{P}_{A,\lambda}$ we have:*

(i) *There holds* $\mathrm{P}_{A,\lambda} D(A) \subset D(A)$, *and* $\ker \mathrm{P}_{A,\lambda}$ *and* $\operatorname{ran} \mathrm{P}_{A,\lambda}$ *are A-invariant subspaces of X.*
(ii) $A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}} \in \mathscr{C}(\operatorname{ran} \mathrm{P}_{A,\lambda})$ *and* $A|_{\ker \mathrm{P}_{A,\lambda}} \in \mathscr{C}(\ker \mathrm{P}_{A,\lambda})$.

(iii) *There holds* $\sigma(A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}}) = \{\lambda\}$ *and* $\sigma(A|_{\ker \mathrm{P}_{A,\lambda}}) = \sigma(A)\backslash\{\lambda\}$. *Furthermore* $A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}} \in \mathscr{B}(\operatorname{ran} \mathrm{P}_{A,\lambda})$.
(iv) *If* $\dim \operatorname{ran} \mathrm{P}_{A,\lambda} < \infty$, *then* $\lambda - A|_{\operatorname{ran} \mathrm{P}_{A,\lambda}}$ *is nilpotent,* λ *is a pole of* $R_A(\zeta)$, *and* $\lambda \in \sigma_p(A)$.
(v) *If* λ *is a pole of* $R_A(\zeta)$, *then* $M(\lambda - A) = \ker(\lambda - A)$ *iff the pole has order one.*

For a finite number of isolated points of the spectrum we have:

Lemma A.3. *For* $N \in \mathbb{N}_0$, *let* A *have isolated points of the spectrum* $\zeta_0, \ldots, \zeta_{N-1}$, *which are eigenvalues with* $\dim M(\zeta_k - A) < \infty$ *for all* $0 \le k \le N-1$. *Assume there exists a closed subspace* $Y \subset X$, *such that*

(i) Y *is* A*-invariant, and* $\sigma(A|_Y) \cap \{\zeta_0, \ldots, \zeta_{N-1}\} = \emptyset$.
(ii) X *can be decomposed as* $X = Y \oplus M(\zeta_0 - A) \oplus \cdots \oplus M(\zeta_{N-1} - A)$.

Then $Y = \ker \Pi_A$, *where* $\Pi_A := \Pi_{A,0} + \cdots + \Pi_{A,N-1}$ *is the sum of the spectral projections* $\Pi_{A,k}$ *corresponding to the* ζ_k, *and* $M(\zeta_k - A) = \operatorname{ran} \Pi_{A,k}$ *for all* $0 \le k \le N-1$.

Proof. According to the assumptions there holds $\sigma(A|_Y) = \sigma(A)\backslash\{\zeta_0, \ldots, \zeta_{N-1}\}$, and therefore the map $\zeta \mapsto R_A(\zeta)|_Y$ is analytic in $\rho(A) \cup \{\zeta_0, \ldots, \zeta_{N-1}\}$. Due to the definition (2.1) of spectral projections this implies that $\Pi_{A,k} Y \equiv 0$ for every $\Pi_{A,k}$, and therefore $Y \subseteq \ker \Pi_A$. On the other hand we have $M(\zeta_k - A) \subseteq \operatorname{ran} \Pi_{A,k}$ for all $0 \le k \le N-1$, according to Proposition A.1. From (ii) we conclude that the inclusions have to be equalities, otherwise $\ker \Pi_A \cap \operatorname{ran} \Pi_A \ne \{0\}$, which is impossible. □

B. Fourier Transform of the Resolvent

This section deals with the explicit computation of the Fourier transform of the resolvent $R_{\mathcal{L}+\Theta}(\zeta)$ of the (one-dimensional) perturbed Fokker-Planck operator $\mathcal{L} + \Theta$ in $\mathcal{E}$, where Θ fulfills the condition **(C)**. We begin by considering the resolvent equation

$$(\zeta - \mathcal{L} - \Theta) f = g$$

on $\mathbb{R}$, where we assume $\operatorname{Re} \zeta > -k$ and $f, g \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$. We apply the Fourier transform, which yields the following differential equation:

$$\xi \left[\hat{f}'(\xi) + \left(\xi + \frac{\zeta - \hat{\vartheta}(\xi)}{\xi} \right) \hat{f}(\xi) \right] = \hat{g}(\xi).$$

By defining $\tilde{f} := \hat{f}/\hat{f}_0$ and $\tilde{g} := \hat{g}/\hat{f}_0$ we obtain the equivalent equation

$$\xi \tilde{f}'(\xi) + \zeta \tilde{f}(\xi) = \tilde{g}(\xi). \tag{B.1}$$

The general solution for $\xi \in \mathbb{R}^{\pm}$ reads

$$\tilde{f}(\xi) = \int_0^1 \tilde{g}(\xi s) s^{\zeta-1}\, \mathrm{d}s + C_{\pm}\xi^{-\zeta} =: I(\xi) + C_{\pm}\xi^{-\zeta}, \tag{B.2}$$

where the $C_{\pm} \in \mathbb{C}$ are integration constants to be determined.

First we shall show that the integral $I(\xi)$ is an analytic function on $\Omega_{\beta/2}$: If $g \in \mathcal{E}_k$, then $\tilde{g}$ is analytic in $\Omega_{\beta/2}$ and has a zero at $\xi = 0$ of order not less than k, see (2.10). Therefore, for any fixed $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > -k$,

$$\tilde{g}(\xi s) s^{\zeta-1} = \frac{\tilde{g}(\xi s)}{s^k} s^{\zeta+k-1}, \quad s \in (0, 1],$$

is locally integrable at $s = 0$, and $I(\xi)$ is well defined for all $\xi \in \Omega_{\beta/2}$. To see that it is actually analytic, we define $I_\varepsilon(\xi) := \int_\varepsilon^1 G_k(\xi, s) s^{\zeta+k-1}\, \mathrm{d}s$ for $\varepsilon \in [0, 1)$, where

$$G_k(\xi, s) := \begin{cases} \dfrac{\tilde{g}(\xi s)}{s^k}, & s \in (0, 1], \\[2mm] \dfrac{\tilde{g}^{(k)}(0)\xi^k}{k!}, & s = 0, \end{cases}$$

for $\xi \in \Omega_{\beta/2}$. The function $G_k(\cdot, s)$ is analytic in $\Omega_{\beta/2}$ for all (fixed) $s \in [0, 1]$, and G_k is continuous in $\Omega_{\beta/2} \times [0, 1]$. According to [9, Theorem 4.9.1], the functions $I_\varepsilon(\xi)$ are analytic in $\Omega_{\beta/2}$ for all $\varepsilon \in (0, 1)$. Now we show that $(I_\varepsilon)_{\varepsilon \in (0,1)}$ converges normally to I in $\Omega_{\beta/2}$ as $\varepsilon \to 0$: Let $K \subset \Omega_{\beta/2}$ be compact. Then we have

$$\begin{aligned} \sup_{\substack{\xi \in K \\ s \in [0,1]}} |G_k(\xi, s)| \le \sup_{\substack{\xi \in K_0 \\ s \in [0,1]}} |G_k(\xi, s)| &= \sup_{\substack{\xi \in K_0 \setminus \{0\} \\ s \in (0,1]}} \left| \frac{\tilde{g}(\xi s)}{(\xi s)^k} \xi^k \right| \\ &\le \sup_{\xi \in K_0 \setminus \{0\}} \left| \frac{\tilde{g}(\xi)}{\xi^k} \right| \cdot \sup_{\xi \in K_0} |\xi^k| =: C_K < \infty, \end{aligned} \tag{B.3}$$

since $\tilde{g}(\xi)/\xi^k$ is analytic in $\Omega_{\beta/2}$ (the singularity at $\xi = 0$ is removable). Thereby, K_0 is an appropriate convex, compact set with $\{0\} \cup K \subseteq K_0 \subset \Omega_{\beta/2}$, and $C_K > 0$ is a constant. With (B.3) we obtain the following estimate for $\xi \in K$ and $0 < \varepsilon \le 1$:

$$|I(\xi) - I_\varepsilon(\xi)| = \left| \int_0^\varepsilon G_k(\xi, s) s^{\zeta+k-1}\, \mathrm{d}s \right| \le C_K \frac{\varepsilon^{\operatorname{Re}\zeta + k}}{\operatorname{Re}\zeta + k}.$$

Since $\operatorname{Re}\zeta + k > 0$, this shows the normal convergence of the analytic functions I_ε towards I. According to [9, Theorem 4.2.3] this implies that $I(\xi)$ is analytic in $\Omega_{\beta/2}$.

Now it remains to determine the constants $C_\pm$ in (B.2). If we require $f \in \mathcal{E}_k$, it is necessary that $\tilde{f}$ is analytic in $\Omega_{\beta/2}$ and has a zero of order not less than k at $\xi = 0$. As already shown, $I(\xi)$ is analytic in $\Omega_{\beta/2}$. Furthermore, for $g \in \mathcal{E}_k$ and all (fixed) $s \in [0,1]$, $\xi \mapsto G_k(\xi, s)$ has a zero of order not less than k at $\xi = 0$. Therefore $I(\xi) = \int_0^1 G_k(\xi, s) s^{\zeta+k-1}\,\mathrm{d}s$ has the same property, so $\mathcal{F}^{-1} I \in \mathcal{E}_k$. Thus, it is sufficient to consider the term $C_\pm \xi^{-\zeta}$. If $\zeta \notin -\mathbb{N}_0$, then $\xi^{-\zeta}$ is not analytic in $\Omega_{\beta/2}$ anyway, hence $C_+ = C_- = 0$. If $\zeta \in \{-k+1, \dots, -1\}$ for $g \in \mathcal{E}_k$, $\xi^{-\zeta}$ is analytic, and we obtain $C_+ = C_-$ because we require continuity of the solution. But the order of the zero of $\xi^{-\zeta}$ is at most $k-1$. Since we need a zero of at least order k, we again obtain $C_+ = C_- = 0$. The conclusion of the above analysis is summarized in the following proposition:

Proposition B.1. *Let $g \in \mathcal{E}_k$ for some $k \in \mathbb{N}_0$, and $\operatorname{Re}\zeta > -k$. Then the unique $f \in \mathcal{E}_k$ with $f = R_{\mathcal{L}+\Theta}(\zeta) g$ satisfies*

$$\hat{f}(\xi) = \hat{f}_0(\xi) \int_0^1 \frac{\hat{g}(s\xi)}{\hat{f}_0(s\xi)} s^{\zeta-1}\,\mathrm{d}s, \quad \xi \in \Omega_{\beta/2}.$$

C. Deferred Proofs and Lemmata

Proof of Lemma 2.2. For $f \in \mathcal{E}$ there holds $f(x)\mathrm{e}^{bx} \in L^2(\mathbb{R})$ for all $b \in \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$. Therefore $\hat{f}$ is analytic in $\Omega_{\beta/2}$ according to [25, Theorem IX.13]. Due to part (b) of the proof of that theorem (see page 132 in [25]), Result (ii) follows. We proceed to the proof of (i). If $f \in \mathcal{E}$ and $b \in \left[-\frac{\beta}{2}, \frac{\beta}{2}\right]$, we clearly have

$$\|f(x)\mathrm{e}^{bx}\|_{L^2(\mathbb{R})} \leq \|f(x)\mathrm{e}^{\frac{\beta}{2}|x|}\|_{L^2(\mathbb{R})} \leq \sqrt{2}\|f\|_\omega.$$

On the left hand side we insert the identity from (ii) and use Plancherel's identity, which shows (2.2). Conversely, let us now assume that $\hat{f}$ is analytic in $\Omega_{\beta/2}$ and that (2.2) holds. We shall now show that $f \in \mathcal{E}$. Due to these assumptions we conclude from [25, Theorem IX.13] that $f(x)\mathrm{e}^{bx} \in L^2(\mathbb{R})$ for all $b \in \left(-\frac{\beta}{2}, \frac{\beta}{2}\right)$. For these values of b we may therefore use the representation from (ii). We insert it in (2.2) and after applying Plancherel's identity we get

$$\sup_{\substack{|b|<\beta/2 \\ b \in \mathbb{R}}} \|f(x)\mathrm{e}^{b|x|}\|_{L^2(\mathbb{R})} < \infty. \tag{C.1}$$

But this is only possible if $f \in \mathcal{E}$, otherwise the supremum in (C.1) would not be finite.

Finally we show (iii). For $f \in \mathcal{E}$ there holds $f(x)\mathrm{e}^{\pm\frac{\beta}{2}x} \in L^2(\mathbb{R})$, and therefore $\xi \mapsto \hat{f}(\xi \pm \mathrm{i}\beta/2)$, as defined in (2.3), is again an element of $L^2(\mathbb{R})$. With this definition we now show $b \mapsto \hat{f}(\cdot + \mathrm{i}b) \in C([-\beta/2, \beta/2]; L^2(\mathbb{R}))$. Due to Plancherel's identity we may show equivalently that $b \mapsto f(x)\mathrm{e}^{bx}$ is continuous in $L^2(\mathbb{R})$. To this end we fix $b, b_0 \in [-\beta/2, \beta/2]$, and we split the integral for any $R > 0$:

$$\text{(C.2)} \qquad \|f(x)e^{bx} - f(x)e^{b_0 x}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}\backslash[-R,R]} |f(x)|^2 (e^{b_0 x} - e^{bx})^2 \, dx + \int_{-R}^{R} |f(x)|^2 (e^{b_0 x} - e^{bx})^2 \, dx$$

Now, for any $\varepsilon > 0$ we can find some $R = R(\varepsilon) > 0$ so that $\int_{\mathbb{R}\backslash[-R,R]} |f(x)|^2 e^{\beta|x|} \, dx < \varepsilon$. So we get for the first integral (independent of b, b_0)

$$\int_{\mathbb{R}\backslash[-R,R]} |f(x)|^2 (e^{b_0 x} - e^{bx})^2 \, dx \leq \int_{\mathbb{R}\backslash[-R,R]} |f(x)|^2 e^{2|x| \max\{|b|,|b_0|\}} \, dx \leq \int_{\mathbb{R}\backslash[-R,R]} |f(x)|^2 e^{\beta|x|} \, dx < \varepsilon.$$

The second integral in (C.2) converges to zero, for any fixed $R > 0$, as $b \to b_0$. Altogether

$$\lim_{b \to b_0} \|f(x)e^{bx} - f(x)e^{b_0 x}\|^2_{L^2(\mathbb{R})} < \varepsilon. \qquad \square$$

Proof of Lemma 2.6. According to Corollary 2.5 the operator $(L - 1 - \beta^2/2)|_{C_0^\infty(\mathbb{R})}$ is dissipative, so it is closable (cf. [24, Theorem 1.4.5 (c)]), and so is $L|_{C_0^\infty(\mathbb{R})}$. We define $\mathcal{L} := \mathrm{cl}_{\mathcal{E}} \, L|_{C_0^\infty(\mathbb{R})}$, and the domain $D(\mathcal{L})$ consists of all $f \in \mathcal{E}$ such that there exists some $h \in \mathcal{E}$ such that (for some $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$)

$$\begin{cases} \lim_{n\to\infty} \|f_n - f\|_\omega = 0, \\ \lim_{n\to\infty} \|Lf_n - h\|_\omega = 0. \end{cases}$$

For such f we have $\mathcal{L}f := h = \mathfrak{L}f$. Therefore $D(\mathcal{L}) \subseteq \{f \in \mathcal{E} : \mathfrak{L}f \in \mathcal{E}\}$. Since $\|\cdot\|_E$ is stronger than $\|\cdot\|_\omega$ we also have $D(L) \subset D(\mathcal{L})$.

Finally we need to show that the above inclusion for the domain indeed is an equality. We take $\zeta \in \mathbb{C}$ with $\mathrm{Re}\,\zeta \geq 1 + \beta^2/2$. From Theorem 2.1 and the dissipativity of $\zeta - \mathcal{L}$ we know that $(\zeta - \mathcal{L})^{-1}|_E = (\zeta - L)^{-1}$ is a well-defined operator on E. And from (2.6) we conclude that this is even a bounded operator in $\mathcal{E}$ with dense domain E. Therefore, also its closure $\mathrm{cl}_{\mathcal{E}}((\zeta - \mathcal{L})^{-1}|_E) = (\zeta - \mathcal{L})^{-1}$ is bounded in $\mathcal{E}$, and therefore $\zeta \in \rho(\mathcal{L})$. Now assume that there is some $f \in \mathcal{E}\backslash D(\mathcal{L})$ such that $\mathfrak{L}f \in \mathcal{E}$. Because $\zeta \in \rho(\mathcal{L})$, $\zeta - \mathcal{L} : D(\mathcal{L}) \to \mathcal{E}$ is a bijection, and therefore there exists a unique $\mathfrak{f} \in D(\mathcal{L})$ with $(\zeta - \mathcal{L})\mathfrak{f} = (\zeta - \mathfrak{L})f$, which is equivalent to the existence of $\mathfrak{f}^\star \in \mathcal{E}$ with $\mathfrak{f}^\star \neq 0$ such that $(\zeta - \mathfrak{L})\mathfrak{f}^\star = 0$. But according to (2.6) this is impossible. $\square$

LEMMA C.1. *Consider two Hilbert spaces* $X \hookrightarrow \mathcal{X}$, *and a projection* $\mathrm{P}_{\mathcal{X}} \in \mathscr{B}(\mathcal{X})$, *such that* $\mathrm{P}_X := \mathrm{P}_{\mathcal{X}}|_X \in \mathscr{B}(X)$. *Then* $\operatorname{ran} \mathrm{P}_{\mathcal{X}} = \operatorname{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X$ *and* $\ker \mathrm{P}_{\mathcal{X}} = \operatorname{cl}_{\mathcal{X}} \ker \mathrm{P}_X$.

PROOF. We give here the proof of the equality of the ranges, the other identity can be shown analogously, using the complementary projections instead. On the one hand we have $\operatorname{ran} \mathrm{P}_X \subseteq \operatorname{ran} \mathrm{P}_{\mathcal{X}}$, and so $\operatorname{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X \subseteq \operatorname{ran} \mathrm{P}_{\mathcal{X}}$, since $\operatorname{ran} \mathrm{P}_{\mathcal{X}}$ is closed in $\mathcal{X}$ due to the boundedness of $\mathrm{P}_{\mathcal{X}}$. On the other hand $\mathrm{P}_{\mathcal{X}} = \operatorname{cl}_{\mathcal{X}} \mathrm{P}_X$, which implies $\operatorname{ran} \mathrm{P}_{\mathcal{X}} \subseteq \operatorname{cl}_{\mathcal{X}} \operatorname{ran} \mathrm{P}_X$. □

LEMMA C.2. *Let* $X \hookrightarrow \mathcal{X}$ *be Hilbert spaces, and* $\psi_0, \dots, \psi_{k-1} \in \mathscr{B}(\mathcal{X}, \mathbb{C})$, $k \in \mathbb{N}$, *be linearly independent functionals. Then* $\tilde{\psi}_j := \psi_j|_X \in \mathscr{B}(X, \mathbb{C})$ *for all* $0 \leq j \leq k-1$, *and*

$$\bigcap_{j=0}^{k-1} \ker \psi_j = \operatorname{cl}_{\mathcal{X}} \bigcap_{j=0}^{k-1} \ker \tilde{\psi}_j.$$

PROOF. The boundedness of the $\tilde{\psi}_j$ is an immediate consequence of $X \hookrightarrow \mathcal{X}$. In order to show the second statement, we notice that according to the Riesz representation theorem there exists a unique $x_j \in X$ such that $\tilde{\psi}_j(\cdot) = \langle \cdot, x_j \rangle_X$ for every $0 \leq j \leq k-1$, where $\langle \cdot, \cdot \rangle_X$ denotes the inner product in X. The set $\{x_0, \dots, x_{k-1}\}$ is linearly independent, because the corresponding functionals are. We now apply the Gram-Schmidt process to $\{x_0, \dots, x_{k-1}\}$ to obtain the orthonormal family $\{\hat{x}_0, \dots, \hat{x}_{k-1}\}$ with same linear hull. As a consequence, there exists a regular matrix $\Lambda := (\lambda_\ell^j)_{\ell,j} \in \mathbb{C}^{k \times k}$ such that $\hat{x}_\ell = \sum_{j=0}^{k-1} \lambda_\ell^j x_j$. With this we get

$$\hat{x}_\ell \langle \cdot, \hat{x}_\ell \rangle_X = \sum_{i,j=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \langle \cdot, x_j \rangle_X = \sum_{i,j=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \tilde{\psi}_j(\cdot), \quad 0 \leq \ell \leq k-1.$$

We may now define the orthogonal projection

$$\mathrm{P}_X := \sum_{\ell=0}^{k-1} \hat{x}_\ell \langle \cdot, \hat{x}_\ell \rangle_X = \sum_{i,j,\ell=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j x_i \tilde{\psi}_j(\cdot). \tag{C.3}$$

It can naturally be extended to a projection $\mathrm{P}_{\mathcal{X}}$ in $\mathcal{X}$ by replacing the $\tilde{\psi}_j$ by ψ_j. Since $\psi_j \in \mathscr{B}(\mathcal{X}, \mathbb{C})$ for all $0 \leq j \leq k-1$, there follows $\mathrm{P}_{\mathcal{X}} \in \mathscr{B}(\mathcal{X})$ from (C.3). Now we apply Lemma C.1 to $\mathrm{P}_X \subset \mathrm{P}_{\mathcal{X}}$ to obtain $\ker \mathrm{P}_{\mathcal{X}} = \operatorname{cl}_{\mathcal{X}} \ker \mathrm{P}_X$.

Now it remains to characterize the kernels of the projections. Due to (C.3) we have $\mathrm{P}_X f = 0$ in X iff

$$\sum_{j=0}^{k-1} \tilde{\psi}_j(f) \sum_{\ell=0}^{k-1} \lambda_\ell^i \bar{\lambda}_\ell^j = 0, \quad 0 \leq i \leq k-1, \tag{C.4}$$

since the vectors x_i are linearly independent. We note that the sums $\sum_{\ell=0}^{k-1} \lambda_\ell^i \overline{\lambda}_\ell^j$ for $0 \le i, j \le k-1$ are the elements of the matrix $\Lambda_2 := \Lambda\Lambda^*$, where Λ^* is the Hermitian conjugate of Λ. Since Λ_2 is regular, it follows that (C.4) holds iff $\tilde{\psi}_j(f) = 0$ for all $0 \le j \le k-1$. The proof of $\mathrm{P}_\mathcal{X} f = 0$ iff $\psi_j(f) = 0$ for all $0 \le j \le k-1$ is analogous. □

Proof of Lemma 4.2. We only consider the situation $|\mathbf{k}| = 1$, the estimate for higher derivatives follows by repeated application of that result. Without loss of generality we assume $\mathbf{k} = (1, 0, \dots, 0)$ for the proof. For the norm we use the equivalent weight $\omega_\star(\mathbf{x}) = \cosh \beta x_1 \cosh \beta x_2 \dots \cosh \beta x_d$. In this context we write $\hat{\mathbf{x}}_1 := (x_2, \dots, x_d)$ and $\omega_\star(\hat{\mathbf{x}}_1) := \omega_\star(\mathbf{x})/\cosh \beta x_1$. By applying the one-dimensional Poincaré inequality (2.5) we obtain for $f \in C_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \text{(C.5)} \qquad \|f\|_{\omega_\star}^2 &= \int_{\mathbb{R}^d} |f(\mathbf{x})|^2 \omega_\star(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}} \right]^2 \cosh \beta x_1 \, \mathrm{d}x_1 \\ &\le C_\beta \int_{\mathbb{R}} \left[\frac{\partial}{\partial x_1} \left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}} \right]^2 \cosh \beta x_1 \, \mathrm{d}x_1. \end{aligned}$$

For the inner integral we compute

$$\begin{aligned} &\frac{\partial}{\partial x_1} \left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}} \\ &\quad = \frac{1}{2} \cdot \frac{\int_{\mathbb{R}^{d-1}} \frac{\partial}{\partial x_1} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1}{\left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}}} \\ &\quad \le \frac{\left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d-1}} |\frac{\partial}{\partial x_1} f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^{d-1}} |f(\mathbf{x})|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}}} \\ &\quad = \left(\int_{\mathbb{R}^{d-1}} \left| \frac{\partial}{\partial x_1} f(\mathbf{x}) \right|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right)^{\frac{1}{2}}. \end{aligned}$$

Inserting this in (C.5) we conclude

$$\|f\|_{\omega_\star}^2 \le C_\beta \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{d-1}} \left| \frac{\partial}{\partial x_1} f(\mathbf{x}) \right|^2 \omega_\star(\hat{\mathbf{x}}_1)\, \mathrm{d}\hat{\mathbf{x}}_1 \right) \cosh \beta x_1 \, \mathrm{d}x_1 = C_\beta \left\| \frac{\partial f}{\partial x_1} \right\|_{\omega_\star}^2. \qquad \square$$

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