



Partial Differential Equations — *Existence and multiplicity results for anisotropic stationary Schrödinger equations*, by GHASEM A. AFROUZI, M. MIRZAPOUR and VICENȚIU D. RĂDULESCU, communicated on 13 December 2013.

ABSTRACT. — In this paper we are concerned with a new class of quasilinear elliptic equations with a power-like reaction term and a differential operator that involves partial derivatives with different powers. The functional-analytic framework relies on anisotropic Sobolev spaces. By means of combined variational arguments, we obtain the existence of weak solutions and, in case of symmetric settings, the existence of large or small energy solutions. In particular, we establish some results that extend the classical theory of combined effects of concave and convex nonlinearities.

KEY WORDS: Anisotropic Sobolev spaces, variable exponent, mountain pass theorem, fountain theorem, dual fountain theorem.

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1. HISTORICAL PERSPECTIVES

Schrödinger gave the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to Louis de Broglie's ideas. He also developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the equivalence between his wave mechanics and Heisenberg's matrix, and introduced the time dependent Schrödinger's equation. It is striking to point out that talking about his celebrating equation, Erwin Schrödinger said: "I don't like it, and I'm sorry I ever had anything to do with it".

The Schrödinger equation is central in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear form of this equation provides a thorough description of a particle in a non-relativistic setting. The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades now to a variety of areas in mathematical physics. The relevant fields of application vary from Bose-Einstein condensates and nonlinear optics, propagation of the electric field in optical fibers to the self-focusing and collapse of Langmuir waves in plasma physics and the behaviour of deep water waves and freak waves (the so-called rogue waves) in the ocean. The nonlinear Schrödinger equation also describes various phenomena

arising in the theory of Heisenberg ferromagnets and magnons, self-channelling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields, plasma physics (e.g., the Kurihara superfluid film equation). We refer to Ablowitz, Prinari and Trubatch [1], Sullem [21] for a modern overview, including applications.

Our purpose in the present paper is to establish some multiplicity results for a Schrödinger-type equation in the framework of Sobolev spaces with variable exponents. The study is motivated by the fact that in the last few years these function spaces have described several important topics of nonlinear partial differential equations. More precisely, problems involving the $p(\cdot)$ -Laplace operator

$$\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$$

have been intensively studied. Lebesgue and Sobolev spaces with variable exponent have been used in the last decades to model various phenomena. Chen, Levine and Rao [5] proposed a framework for image restoration based on a variable exponent Laplacian. Another application that uses nonhomogeneous Laplace operators is related to the modeling of electrorheological fluids. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent space to the fluids, we refer to Diening [7], Rajagopal and Ruzicka [18] and Ruzicka [19]. For an excellent overview of the most significant mathematical methods employed in this paper we refer to Ciarlet [6].

2. STATEMENT OF THE PROBLEM

In a celebrated paper, Rabinowitz [16] proved that the nonlinear Schrödinger equation has a ground-state solution (mountain-pass solution) for small positive perturbations and in the case of positive potentials. After making a standing wave ansatz, Rabinowitz reduces the problem to that of studying the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x, u) \quad \text{in } \mathbb{R}^N,$$

under suitable conditions on a and assuming that f is smooth, superlinear and has a subcritical growth. Motivated by the paper [16], the goal of this work is to study the existence and multiplicity of weak solutions of problem (2.1). A central role in our arguments will be played by the fountain theorem, which is due to Bartsch [3]. This result is nicely presented in Willem [22] by using the quantitative deformation lemma. We also point out that the dual version of the fountain the-

orem is due to Bartsch and Willem, see [22]. Both the fountain theorem and its dual form are effective tools for studying the existence of infinitely many large or small energy solutions. It should be noted that the Palais-Smale condition plays an important role for these theorems and their applications.

The purpose of this paper is to analyze the existence and multiplicity of weak solutions of the anisotropic quasilinear elliptic problem

$$(2.1) \quad \begin{cases} -\sum_{i=1}^N \partial_{x_i} (|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u) + b(x)|u|^{p_+^+} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, p_i , $i \in \{1, \dots, N\}$ are continuous functions on $\bar{\Omega}$ such that $p_i(x) > 1$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions.

In this paper, the operator involved in equation (2.1) is more general than the $p(\cdot)$ -Laplace operator. Thus, the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is not adequate to study nonlinear problems of this type. This lead us to seek weak solutions for problem (2.1) in a more general variable exponent Sobolev space which was introduced for the first time in [12].

In [4, 14] (see also [11, 15, 17]) the authors studied the anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i} (|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary. They have established both existence and multiplicity results. in [4] it is applied the symmetric fountain pass theorem of Ambrosetti and Rabinowitz [2], while in [14] the authors combine the minimum principle, the mountain pass theorem and the Ekeland variational principle, where $f(x, u) = \lambda|u|^{q(x)-2}u$ is assumed.

Our paper is organized as follows. We first introduce the theory of generalized Lebesgue–Sobolev spaces and the generalized anisotropic Sobolev spaces, in which we seek the solutions of (2.1). Next, we state and prove the main results. The final part of the paper is concerned with combined effects of concave and convex nonlinearities.

3. FUNCTIONAL SETTING

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in \mathbb{R}^N . Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles.

Let Ω be a bounded domain of \mathbb{R}^N . Denote

$$C_+(\bar{\Omega}) = \{h(x); h(x) \in C(\bar{\Omega}), h(x) > 1, \forall x \in \bar{\Omega}\}.$$

For any $h \in C_+(\bar{\Omega})$, we define

$$h^+ = \max\{h(x); x \in \bar{\Omega}\}, \quad h^- = \min\{h(x); x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R}; u \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(\cdot)}(\Omega)} = |u|_{p(\cdot)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Then $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a Banach space, cf. [13].

PROPOSITION 3.1 (See [8]). (i) *The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(\cdot)} |v|_{q(\cdot)} \leq 2 |u|_{p(\cdot)} |v|_{q(\cdot)}.$$

(ii) *If $p_1(\cdot), p_2(\cdot) \in C_+(\bar{\Omega})$, $p_1(\cdot) \leq p_2(\cdot)$, $\forall x \in \bar{\Omega}$, then $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ and the embedding is continuous.*

An important role in manipulating the generalized Lebesgue space is played by the $p(\cdot)$ -modular of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

PROPOSITION 3.2 (See [9]). *For $u \in L^{p(\cdot)}(\Omega)$ and $u_n \in L^{p(\cdot)}(\Omega)$, we have*

- (1) $|u|_{p(\cdot)} < 1$ (respectively $= 1; > 1$) $\Leftrightarrow \rho_{p(\cdot)}(u) < 1$ (respectively $= 1; > 1$);
- (2) for $u \neq 0$, $|u|_{p(\cdot)} = \lambda \Leftrightarrow \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) = 1$;
- (3) if $|u|_{p(\cdot)} > 1$, then $|u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+}$;
- (4) if $|u|_{p(\cdot)} < 1$, then $|u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}$;
- (5) $|u_n - u|_{p(\cdot)} \rightarrow 0$ (respectively $\rightarrow \infty$) $\Leftrightarrow \rho_{p(\cdot)}(u_n - u) \rightarrow 0$ (respectively $\rightarrow \infty$), since $p^+ < \infty$.

The space $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u(x)|_{p(x)}.$$

The norm $\|u\| = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}$ is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$ (see [14]). Hence $W_0^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. Note that when $s \in C_+(\bar{\Omega})$ and $s(x) < p^*(x)$ for all $x \in \bar{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact.

Finally, we introduce a natural generalization of the Sobolev function space $W_0^{1,p(\cdot)}(\Omega)$, which will enable us to study with sufficient accuracy problem (2.1). For this purpose, let us denote by $\vec{p} : \bar{\Omega} \rightarrow \mathbb{R}^N$ the vectorial function $\vec{p}(\cdot) = (p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot))$ with $p_i(\cdot) \in C_+(\bar{\Omega})$, $i \in \{1, \dots, N\}$. We define $X = W_0^{1,\vec{p}(x)}(\Omega)$, the *anisotropic variable exponent space*, as the closure of $C_0^\infty(\Omega)$, with respect to the norm

$$\|u\| = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}.$$

As it was pointed out in [14], $W_0^{1,\vec{p}(x)}(\Omega)$ is a reflexive Banach space. In order to facilitate the manipulation of the space $W_0^{1,\vec{p}(x)}(\Omega)$ we introduce $\vec{P}_+, \vec{P}_- \in \mathbb{R}^N$ and $P_+, P_-, P_+, P_- \in \mathbb{R}^+$ as

$$\begin{aligned} \vec{P}_+ &= (p_1^+, p_2^+, \dots, p_N^+), & \vec{P}_- &= (p_1^-, p_2^-, \dots, p_N^-), \\ P_+ &= \max\{p_1^+, p_2^+, \dots, p_N^+\}, & P_- &= \max\{p_1^-, p_2^-, \dots, p_N^-\}, \\ P_+^- &= \min\{p_1^+, p_2^+, \dots, p_N^+\}, & P_-^- &= \min\{p_1^-, p_2^-, \dots, p_N^-\}. \end{aligned}$$

Throughout this paper, we assume that

$$(3.1) \quad \sum_{i=1}^N \frac{1}{P_i^-} > 1.$$

This condition ensures that the anisotropic space $W_0^{1,\vec{p}(x)}(\Omega)$ is embedded into some Lebesgue space $L^r(\Omega)$. If hypothesis (3.1) is no longer fulfilled, then one has embeddings into Orlicz or Hölder spaces.

Define $P_-^* \in \mathbb{R}^+$ and $P_{-, \infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{P_i^-} - 1}, \quad P_{-, \infty} = \max\{P_-^+, P_-^*\}.$$

In addition, for the Carathéodory function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, we consider the anti-derivative $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$,

$$F(x, s) = \int_0^s f(x, t) dt.$$

With the previous notations, we introduce the following conditions:

- (B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$ for all $x \in \Omega$.
- (f₀) There exist two constants $C_1 \geq 0, C_2 \geq 0$ and $\alpha(x) \in C_+(\overline{\Omega})$ and $P_+^+ < \alpha^- < \alpha(x) < P_-^*(x)$ such that

$$(3.2) \quad |f(x, t)| \leq C_1 + C_2|t|^{\alpha(x)-1}, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

- (f₁) There exist constants $M > 0, \theta > P_+^+$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M$,

$$0 < \theta F(x, t) \leq tf'(x, t).$$

- (f₂) $f(x, t) = o(|t|^{P_+^+-1})$ as $t \rightarrow 0$ uniformly with respect to $x \in \Omega$.

- (f₃) $f(x, -t) = -f(x, t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$.

PROPOSITION 3.3 (See [14]). *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary. Assume relation (3.1) is satisfied and $q \in C_+(\overline{\Omega})$ verifies*

$$1 < q(x) < P_{-\infty}, \quad \text{for all } x \in \overline{\Omega}.$$

Then the embedding

$$W_0^{1, \vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact.

It should be noticed that from the condition (f₀), we have $P_{-\infty} = \max\{P_-^+, P_-^*\} = P_-^*$.

DEFINITION 3.4. By a weak solution to problem (2.1), we mean a function $u \in X$ such that

$$\int_{\Omega} \left\{ \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v + b(x) |u|^{P_+^+-2} uv - f(x, u)v \right\} dx = 0$$

for all $v \in X$.

We associate to problem (2.1) the energy functional $J : X \rightarrow \mathbb{R}$ defined by

$$J(u) = \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |u|^{P_+^+} - F(x, u) \right\} dx.$$

REMARK 3.5. *Under the condition (f₀), the functional J is of class C^1 .*

REMARK 3.6. *For simplicity, we use $c, c', \bar{c}, C, C', M, M'$, to denote the general nonnegative or positive constant (the exact value may change from line to line).*

4. EXISTENCE OF SOLUTIONS

In this section we establish the existence of weak solutions to problem (2.1).

THEOREM 4.1. *Assume that hypothesis (3.2) holds with $\alpha(x) < P_{-, \infty}$, $\alpha^+ < P_-$. Then problem (2.1) has a weak solution.*

PROOF. From the assumption on f , using the Hölder's inequality and the Sobolev type embeddings, we deduce that the functional J is weakly lower semi-continuous in X . We will show that J is coercive.

Condition (3.2) implies

$$F(x, t) \leq C(1 + |t|^{\alpha(x)}), \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Without loss of generality, assume $\|u\| > 1$. Then

$$\begin{aligned} J(u) &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |u|^{P_+^+} - F(x, u) \right\} dx \\ &\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx + \frac{b_0}{P_+^+} \int_{\Omega} |u|^{P_+^+} dx - \int_{\Omega} C(1 + |u|^{\alpha(x)}) dx \\ &\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx + \frac{b_0}{P_+^+} |u|_{L^{P_+^+}(\Omega)}^{P_+^+} - C\|u\|^{\alpha^+} - M. \end{aligned}$$

Using (B), we have

$$(4.1) \quad \frac{1}{P_+^+} \int_{\Omega} b(x) |u|^{P_+^+} dx \geq \frac{b_0}{P_+^+} |u|_{L^{P_+^+}(\Omega)}^{P_+^+} \geq 0.$$

For each $i \in \{1, 2, \dots, N\}$ we define

$$\alpha_i = \begin{cases} P_+^+ & \text{if } |\partial_{x_i} u|_{p_i(x)} < 1, \\ P_-^- & \text{if } |\partial_{x_i} u|_{p_i(x)} > 1. \end{cases}$$

Using Proposition 3.2, and Jensen's inequality (applied to the convex function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $g(t) = t^{P_-^-}$, $P_-^- > 1$), we have

$$(4.2) \quad \begin{aligned} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx &\geq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{\alpha_i} \\ &\geq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_-^-} - \sum_{\{i: \alpha_i = P_+^+\}} (|\partial_{x_i} u|_{p_i(x)}^{P_-^-} - |\partial_{x_i} u|_{p_i(x)}^{P_+^+}) \end{aligned}$$

$$\begin{aligned} &\geq N \left(\frac{\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}}{N} \right)^{P_-^-} - N \\ &= \frac{\|u\|^{P_-^-}}{N^{P_-^- - 1}} - N. \end{aligned}$$

Thus, we obtain

$$J(u) \geq \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - C \|u\|^{\alpha^+} - M' \rightarrow \infty, \quad \text{as } \|u\| \rightarrow \infty,$$

so J is coercive, since $\alpha^+ < P_-^-$. Thus, J has a minimum point $u \in X$ and u is a weak solution (which may be trivial) of problem (2.1). \square

THEOREM 4.2. *Assume that conditions (f₀), (f₁) and (f₂) are fulfilled. Then problem (2.1) has a nontrivial weak solution.*

To prove Theorem 4.2, we apply the mountain pass theorem, see [2]. We need to verify the following auxiliary results.

LEMMA 4.3. *Let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional J . If the condition (f₁) is satisfied, then (u_n) is bounded.*

PROOF. Let (u_n) be a Palais-Smale sequence for the functional J . Thus, there exists $\bar{c} > 0$ such that

$$(4.3) \quad |I(u_n)| < \bar{c} \quad \text{and} \quad I'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By conditions (B) and (f₁), for all n we can write

$$\begin{aligned} \bar{c} + 1 &\geq J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle + \frac{1}{\theta} \langle J'(u_n), u_n \rangle \\ &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |u_n|^{P_+^+} - F(x, u_n) \right\} dx \\ &\quad - \frac{1}{\theta} \left[\int_{\Omega} \left\{ \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} + b(x) |u_n|^{P_+^+} - f(x, u_n) u_n \right\} dx \right] + \frac{1}{\theta} \langle J'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} + \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \int_{\Omega} b(x) |u_n|^{P_+^+} dx - \frac{1}{\theta} \|J'(u_n)\|_{X^*} \|u_n\| - c \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - \frac{1}{\theta} \|J'(u_n)\|_{X^*} - c. \end{aligned}$$

We have supposed, for convenience, that $\|u_n\| > 1$. From the inequality above, we know that (u_n) is bounded in X since $\theta > P_+^+$. The proof is complete. \square

In the following lemma, we establish that every bounded Palais-Smale sequence for the functional J contains a convergent subsequence.

LEMMA 4.4. *Let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional J . If the conditions (f_0) and (f_1) are satisfied then (u_n) contains a convergent subsequence.*

PROOF. Let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional J . By Lemma 4.3, (u_n) is bounded. Then there exists a subsequence, still denote by (u_n) , which converges weakly to a function u_0 in X . By relation (4.3) we deduce

$$\lim_{n \rightarrow \infty} \langle I'(u_n), u_n - u_0 \rangle = 0.$$

Therefore

$$(4.4) \quad \lim_{n \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u_0) dx \right. \\ \left. + \int_{\Omega} b(x) |u_n|^{P_+^+} (u_n - u_0) dx - \int_{\Omega} f(x, u_n) (u_n - u_0) dx \right) = 0.$$

Since $P_+^+ < \alpha(x) < P_-^* = P_{-, \infty}$, the embeddings $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ and $X \hookrightarrow L^{P_+^+}(\Omega)$ are compact, hence (u_n) converges strongly to u_0 in $L^{\alpha(\cdot)}(\Omega)$ and also in $L^{P_+^+}(\Omega)$.

From (B), (f_0) , Propositions 3.1 and 3.3, we obtain

$$\left| \int_{\Omega} f(x, u_n) (u_n - u_0) dx \right| \leq 2c |u_n|^{\alpha(x)-1} \Big|_{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}} |u_n - u_0|_{\alpha(\cdot)} \rightarrow 0,$$

and

$$\left| \int_{\Omega} b(x) |u_n|^{P_+^+-2} u_n (u_n - u_0) dx \right| \leq 2 \|b\|_{L^\infty(\Omega)} |u_n|^{P_+^+-1} \Big|_{\frac{P_+^+}{P_+^+-1}} |u_n - u_0|_{P_+^+} \rightarrow 0.$$

Taking into account the two above inequalities, relation (4.4) reduces to

$$\lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n (\partial_{x_i} u_n - \partial_{x_i} u_0) dx = 0.$$

We conclude that

$$(4.5) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} (|\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n - |\partial_{x_i} u_0|^{p_i(x)-2} \partial_{x_i} u_0) (\partial_{x_i} u_n - \partial_{x_i} u_0) dx = 0.$$

Next, we apply the following inequality (see [20])

$$(4.6) \quad (|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \geq 2^{-r}|\xi - \eta|^r, \quad \forall \xi, \eta \in \mathbb{R}^N,$$

valid for all $r \geq 2$. Relations (4.5) and (4.6) show that actually (u_n) converges strongly to u_0 in X . \square

LEMMA 4.5. *Assume that conditions (f_0) and (f_2) are fulfilled. Then there exist $r > 0$ and $\delta > 0$ such that $J(u) \geq \delta > 0$ for any $u \in X$ with $\|u\| = r$.*

PROOF. From (f_0) , we have $X \hookrightarrow L^{P_+^+}(\Omega)$. So there exist a constant $C > 0$ such that

$$\|u\|_{L^{P_+^+}(\Omega)} \leq C\|u\|,$$

for all $u \in X$. By (f_0) and (f_2) , there exist a constant $0 < \varepsilon < 1$ and a positive constant $C(\varepsilon)$ such that

$$(4.7) \quad |F(x, t)| \leq \varepsilon|t|^{P_+^+} + C(\varepsilon)|t|^{\alpha(x)}, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Next, we focus our attention on the case when $u \in X$ and $\|u\| < 1$. For such an element u , we have $|\partial_{x_i} u|_{p_i(x)} < 1$, $i \in \{1, \dots, N\}$ and by Proposition 3.2, we obtain

$$(4.8) \quad \begin{aligned} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx &\geq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_+^+} \geq \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_+^+} \\ &\geq N \left(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_+^+}}{N} \right)^{P_+^+} = \frac{\|u\|^{P_+^+}}{N^{P_+^+-1}}. \end{aligned}$$

Relations (4.7), (4.8) and (4.1) yield

$$\begin{aligned} J(u) &\geq \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+-1}} - \int_{\Omega} (\varepsilon|u|^{P_+^+} + C(\varepsilon)|u|^{\alpha(x)}) dx \\ &\geq \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+-1}} - 2\varepsilon C\|u\|^{P_+^+} - \bar{C}(\varepsilon)\|u\|^{\alpha^-}. \end{aligned}$$

Choose $\varepsilon > 0$ so small that $0 < 2\varepsilon C < \frac{1}{2P_+^+ N^{P_+^+-1}}$. We obtain

$$J(u) \geq \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+-1}} - \bar{C}(\varepsilon)\|u\|^{\alpha^-}.$$

Since $\alpha^- > P_+^+$, there exist $r > 0$ small enough and $\delta > 0$ such that $J(u) \geq \delta > 0$ if $\|u\| = r$. \square

LEMMA 4.6. *Assume that condition (f₁) is fulfilled. Then there exists e ∈ X with ||e|| > r (where r is given in Lemma 4.5) such that J(e) < 0.*

PROOF. From (f₁), we have

$$(4.9) \quad F(x, t) \geq C|t|^\theta - 1, \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}.$$

Using (4.9), for ω ∈ X \ {0} and t > 1, we have

$$\begin{aligned} J(t\omega) &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} t\omega|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |t\omega|^{P_+^+} - F(x, t\omega) \right\} dx \\ &\leq \frac{t^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} \omega|^{p_i(x)} dx + \frac{t^{P_+^+}}{P_+^+} \int_{\Omega} b(x) |\omega|^{P_+^+} dx - Ct^\theta \int_{\Omega} |\omega|^\theta dx + C'. \end{aligned}$$

Since θ > P₊⁺, we deduce that for sufficiently large t > 1, we have J(tω) < 0. □

PROOF OF THEOREM 4.2 COMPLETED. Since J(0) = 0, considering Lemmas 4.3–4.6, we apply the mountain pass theorem [2] to obtain that problem (2.1) has a nontrivial weak solution. □

5. INFINITELY MANY SOLUTIONS

The following result establishes the existence of infinitely many solutions of problem (2.1), provided that the right-hand side is odd.

THEOREM 5.1. *Assume that the conditions (f₀), (f₁) and (f₃) hold. Then problem (2.1) has a sequence of solutions (±u_k) such that J(±u_k) → +∞ as k → +∞.*

The proof of Theorem 5.1 relies on the fountain theorem, see Willem [22].

Since X is a reflexive and separable Banach space, then X* is too. Thus, by [23], there exist {e_j} ⊂ X and {e_j^{*}} ⊂ X* such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \dots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where ⟨ ·, · ⟩ denotes the duality product between X and X*. We define

$$X_j = \text{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Then we have the following auxiliary result.

LEMMA 5.2 (See [10]). Assume that $\alpha, \beta \in C_+(\bar{\Omega})$, $\alpha(x), \beta(x) < P_{-, \infty}$, for all $x \in \bar{\Omega}$. Denote

$$\begin{aligned}\alpha_k &= \sup\{|u|_{L^{\alpha(x)}(\Omega)}; \|u\| = 1, u \in Z_k\} \\ \beta_k &= \sup\{|u|_{L^{\beta(x)}(\Omega)}; \|u\| = 1, u \in Z_k\}.\end{aligned}$$

Then $\lim_{k \rightarrow \infty} \alpha_k = 0$ and $\lim_{k \rightarrow \infty} \beta_k = 0$.

LEMMA 5.3 (Fountain Theorem, see [22]). Let $J \in C^1(X, \mathbb{R})$ be an even functional, where $(X, \|\cdot\|)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

- (A1) $\inf\{J(u) : u \in Z_k, \|u\| = r_k\} \rightarrow +\infty$ as $k \rightarrow +\infty$.
- (A2) $\max\{J(u) : u \in Y_k, \|u\| = \rho_k\} \leq 0$.
- (A3) J satisfies the Palais-Smale condition for every $c > 0$.

Then J has a sequence of critical values tending to $+\infty$.

5.1. Proof of Theorem 5.1. According to (f₃), Lemmas 4.3 and 4.4, J is an even functional and satisfies the Palais-Smale condition. We prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A1) and (A2) hold.

- (A1) For any $u \in Z_k$, $\|u\| = r_k > 1$ (r_k will be specified below), using (4.1) and (f₀) we have

$$\begin{aligned}J(u) &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |u|^{P_+^+} - F(x, u) \right\} dx \\ &\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx - C \int_{\Omega} (1 + |u|^{\alpha(x)}) dx \\ &\geq \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - C \max\{|u|_{L^{\alpha(x)}(\Omega)}^{\alpha_+^+}, |u|_{L^{\alpha(x)}(\Omega)}^{\alpha_-^-}\} - M.\end{aligned}$$

If $\max\{|u|_{L^{\alpha(x)}(\Omega)}^{\alpha_+^+}, |u|_{L^{\alpha(x)}(\Omega)}^{\alpha_-^-}\} = |u|_{L^{\alpha(x)}(\Omega)}^{\alpha_+^+}$, we have

$$J(u) \geq \frac{\|u\|^{P_-^-}}{P_+^+} - C \alpha_k^{\alpha_+^+} \|u\|^{\alpha_+^+} - M.$$

At this stage, we fix r_k as follows:

$$r_k = (C \alpha_k^{\alpha_+^+})^{\frac{1}{P_-^- - \alpha_+^+}} \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

Consequently, if $\|u\| = r_k$ then

$$J(u) \geq \left(\frac{1}{P_+^+} - \frac{1}{\alpha_+^+} \right) r_k^{P_-^-} - M \rightarrow +\infty \quad \text{as } k \rightarrow +\infty,$$

due to $\alpha^+ > \alpha^- > P_+^+$ and $\alpha_k \rightarrow 0$ as $k \rightarrow +\infty$.

(A2) Using relation (4.6) for any $u \in Y_k \setminus \{0\}$ with $\|u\| = 1$ and $1 < \rho_k = t_k$ with $t_k \rightarrow +\infty$, we have

$$\begin{aligned} J(t_k u) &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} t_k u|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |t_k u|^{P_+^+} - \int_{\Omega} F(x, t_k u) \right\} dx \\ &\leq \frac{t_k^{P_+^+}}{P_-^-} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u|^{p_i(x)} dx + \frac{t_k^{P_+^+}}{P_+^+} \int_{\Omega} b(x) |u|^{P_+^+} dx - C t_k^{\theta} \int_{\Omega} |u|^{\theta} dx + M. \end{aligned}$$

Since $\theta > P_+^+$ and $\dim Y_k < \infty$, we observe that $J(t_k u) \rightarrow -\infty$ as $k \rightarrow +\infty$ for $u \in Y_k$. This implies that

$$\max\{J(u) : \|u\| = \rho_k, u \in Y_k\} \leq 0,$$

for every ρ_k large enough. Applying the fountain theorem, we complete the proof.

6. THE CASE OF CONCAVE-CONVEX NONLINEARITY

In this section, similarly to the result named ‘‘concave and convex nonlinearities’’ for the Laplace operator in [22], we establish the following qualitative property.

THEOREM 6.1. *Let $\gamma(x), \beta(x) \in C_+(\bar{\Omega})$, $\gamma(x), \beta(x) < P_-^*(x)$ for any $x \in \bar{\Omega}$ with $\gamma^- > P_+^+$, $\beta^+ < P_-^-$ and $f(x, t) = \lambda|t|^{\gamma(x)-2}t + \mu|t|^{\beta(x)-2}t$. Then the following properties hold:*

- (i) *For every $\lambda > 0$, $\mu \in \mathbb{R}$, problem (2.1) has a sequence of weak solutions $(\pm u_k)$ such that $J(\pm u_k) \rightarrow +\infty$ as $k \rightarrow +\infty$.*
- (ii) *For every $\mu > 0$, $\lambda \in \mathbb{R}$, problem (2.1) has a sequence of weak solutions $(\pm v_k)$ such that $J(\pm v_k) \rightarrow 0$ as $k \rightarrow +\infty$.*

We will use Lemma 5.3 to prove Theorem 6.1 (i) and the following dual fountain theorem to prove Theorem 6.1 (ii), respectively.

LEMMA 6.2 (Dual Fountain Theorem, see [22]). *Assume (A1) is satisfied and there is $k_0 > 0$ so that, for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

- (B1) $a_k = \inf\{J(u) : u \in Z_k, \|u\| = \rho_k\} \geq 0$.
- (B2) $b_k = \max\{J(u) : u \in Y_k, \|u\| = r_k\} < 0$.
- (B3) $d_k = \inf\{J(u) : u \in Z_k, \|u\| \leq \rho_k\} \rightarrow 0$ as $k \rightarrow +\infty$.
- (B4) J satisfies the $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0)$.

Then J has a sequence of negative critical values converging to 0.

DEFINITION 6.3. We say that J satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $(u_{n_j}) \subset X$ such that $n_j \rightarrow +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \rightarrow c$

and $(J|_{Y_{n_j}})'(u_{n_j}) \rightarrow 0$, contain a subsequence converging to a critical point of J .

6.1. *Proof of Theorem 6.1.* (i) The proof is similar to that of Theorem 5.1 if we use the fountain theorem, so we only verify the Palais-Smale condition. According to Lemma 4.3 it is sufficient to verify that the Palais-Smale sequence (u_n) is bounded in X . We assume there exists a constant $\bar{c} > 0$ such that

$$|I(u_n)| < \bar{c} \quad \text{and} \quad I'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Assume $\|u_n\| > 1$. We have for n large enough

$$\begin{aligned} \bar{c} + 1 &\geq J(u_n) - \frac{1}{\gamma^-} \langle J'(u_n), u_n \rangle + \frac{1}{\gamma^-} \langle J'(u_n), u_n \rangle \\ &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |u_n|^{P_+^+} - \frac{\lambda}{\gamma(x)} |u_n|^{\gamma(x)} - \frac{\mu}{\beta(x)} |u_n|^{\beta(x)} \right\} dx \\ &\quad - \frac{1}{\gamma^-} \left[\int_{\Omega} \left\{ \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} + b(x) |u_n|^{P_+^+} - \lambda |u_n|^{\gamma(x)} - \mu |u_n|^{\beta(x)} \right\} dx \right] \\ &\quad + \frac{1}{\gamma^-} \langle J'(u_n), u_n \rangle \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\gamma^-} \right) \frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} + \left(\frac{1}{P_+^+} - \frac{1}{\gamma^-} \right) \int_{\Omega} b(x) |u_n|^{P_+^+} dx \\ &\quad + \mu \int_{\Omega} \left(\frac{1}{\gamma^-} - \frac{1}{\beta(x)} \right) |u_n|^{\beta(x)} dx - \frac{1}{\gamma^-} \|J'(u_n)\|_{X^*} \|u_n\| \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\gamma^-} \right) \frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - C \|u_n\|^{\beta^+} - \frac{1}{\gamma^-} \|u_n\|. \end{aligned}$$

Since $P_-^- > \beta^+$ and $\gamma^- > P_+^+$, we deduce that (u_n) is bounded in X .

(ii) We know that J satisfies (A1), the assertion of conclusion can be obtained from the dual fountain theorem. Now, it remains to prove that J satisfies the $(PS)_c^*$ condition and there exist $\rho_k > r_k > 0$ such that if k is large enough (B1), (B2) and (B3) are satisfied.

(B1) Let $u \in Z_k$, then

$$\begin{aligned} J(u) &\geq \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{|\lambda|}{\gamma^-} \int_{\Omega} |u|^{\gamma(x)} dx - \frac{\mu}{\beta^-} \int_{\Omega} |u|^{\beta(x)} dx \\ &\geq \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{C|\lambda|}{\gamma^-} \|u\|^{\gamma^-} - \frac{\mu}{\beta^-} \max\{|u|_{L^{\beta(x)}(\Omega)}^{\beta^+}, |u|_{L^{\beta(x)}(\Omega)}^{\beta^-}\}. \end{aligned}$$

There exists $0 < \rho_1 < 1$ small enough such that $\frac{C|\lambda|}{\gamma^-} \|u\|^{\gamma^-} \leq \frac{1}{2P_+^+ N^{P_+^+-1}} \|u\|^{P_+^+}$ as $0 < \rho = \|u\| \leq \rho_1$. Then we have

$$= J(u) \geq \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+-1}} - \frac{\mu}{\beta^-} \max\{|u|_{L^{\beta(x)}(\Omega)}^{\beta^+}, |u|_{L^{\beta(x)}(\Omega)}^{\beta^-}\}.$$

If $\max\{|u|_{L^{\beta(x)}(\Omega)}^{\beta^+}, |u|_{L^{\beta(x)}(\Omega)}^{\beta^-}\} = |u|_{L^{\beta(x)}(\Omega)}^{\beta^+}$, then

$$J(u) \geq \frac{1}{2P_+^+ N^{P_+^+-1}} \|u\|^{P_+^+} - \frac{\mu}{\beta^-} \beta_k^{\beta^+} \|u\|^{\beta^+}.$$

Choose $\rho_k = \left(\frac{2P_+^+ N^{P_+^+-1} \mu \beta_k^{\beta^+}}{\beta^-}\right)^{\frac{1}{P_+^+ - \beta^+}}$, then

$$J(u) \geq \frac{1}{2P_+^+ N^{P_+^+-1}} (\rho_k)^{P_+^+} - \frac{1}{2P_+^+ N^{P_+^+-1}} (\rho_k)^{P_+^+} = 0.$$

Since $P_-^- > \beta^+$, $\beta_k \rightarrow 0$, we know that $\rho_k \rightarrow 0$ as $k \rightarrow +\infty$.

If $\max\{|u|_{L^{\beta(x)}(\Omega)}^{\beta^+}, |u|_{L^{\beta(x)}(\Omega)}^{\beta^-}\} = |u|_{L^{\beta(x)}(\Omega)}^{\beta^-}$, we can do the same work as the case above. So (B1) is satisfied.

(B2) For $v \in Y_k$ with $\|v\| = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned} J(tv) &= \int_{\Omega} \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} tv|^{p_i(x)}}{p_i(x)} + \frac{b(x)}{P_+^+} |tv|^{P_+^+} - \frac{\lambda}{\gamma(x)} |tv|^{\gamma(x)} - \frac{\mu}{\beta(x)} |tv|^{\beta(x)} \right\} dx \\ &\leq \sum_{i=1}^N \int_{\Omega} \frac{t^{p_i(x)}}{p_i(x)} |\partial_{x_i} v|^{p_i(x)} + \frac{t^{P_+^+}}{P_+^+} \int_{\Omega} b(x) |v|^{P_+^+} dx + \frac{|\lambda|}{\gamma^-} \int_{\Omega} t^{\gamma(x)} |v|^{\gamma(x)} dx \\ &\quad - \frac{\mu}{\beta^+} \int_{\Omega} t^{\beta(x)} |v|^{\beta(x)} dx \\ &\leq \frac{t^{P_-^-}}{P_-^-} \sum_{i=1}^N \int_{\Omega} |\partial_{x_i} v|^{p_i(x)} + \frac{t^{P_+^+}}{P_+^+} \int_{\Omega} b(x) |v|^{P_+^+} dx + \frac{|\lambda| t^{\gamma^-}}{\gamma^-} \int_{\Omega} |v|^{\gamma(x)} dx \\ &\quad - \frac{\mu t^{\beta^+}}{\beta^+} \int_{\Omega} |v|^{\beta(x)} dx. \end{aligned}$$

Since $\dim Y_k = k$, conditions $\beta^+ < P_-^-$ and $P_+^+ < \gamma^-$ imply that there exists a $r_k \in (0, \rho_k)$ such that $J(u) < 0$ when $\|u\| = r_k$. Hence $b_k = \max\{J(u) : u \in Y_k, \|u\| = r_k\} < 0$, so (B2) is satisfied.

(B3) Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf\{J(u) : u \in Z_k, \|u\| \leq \rho_k\} \leq b_k = \max\{J(u) : u \in Y_k, \|u\| = r_k\} < 0.$$

In view of the proof of (B1), we have

$$J(u) \geq -\frac{\mu}{\beta^-} \beta_k^{\beta^+} \|u\|^{\beta^+} \quad \text{or} \quad -\frac{\mu}{\beta^-} \beta_k^{\beta^-} \|u\|^{\beta^-}$$

Since $\beta_k \rightarrow 0$ and $\rho_k \rightarrow 0$ as $k \rightarrow +\infty$, (B3) is satisfied.

Finally, we verify the $(PS)_c^*$ condition. Suppose $(u_{n_j}) \subset X$ such that $n_j \rightarrow +\infty$, $u_{n_j} \in Y_{n_j}$ and $(J|_{Y_{n_j}})'(u_{n_j}) \rightarrow 0$. Assume $\|u_{n_j}\| > 1$ for convenience. If $\lambda \geq 0$, for n large enough, we have

$$\begin{aligned} \bar{c} + 1 &\geq J(u_{n_j}) - \frac{1}{\gamma^-} \langle J'(u_{n_j}), u_{n_j} \rangle + \frac{1}{\gamma^-} \langle J'(u_{n_j}), u_{n_j} \rangle \\ &\geq \left(\frac{1}{P_+^+} - \frac{1}{\gamma^-} \right) \frac{\|u_{n_j}\|^{P_-^-}}{N^{P_-^- - 1}} - C \|u_{n_j}\|^{\beta^+} - \frac{1}{\gamma^-} \|u_{n_j}\|. \end{aligned}$$

Since $P_-^- > \beta^+$ and $\gamma^- > P_+^+$, we deduce that (u_{n_j}) is bounded in X .

If $\lambda < 0$, for n large enough, we can consider the inequality below to get the boundedness of (u_{n_j}) .

$$\bar{c} + 1 \geq J(u_{n_j}) - \frac{1}{\gamma^+} \langle J'(u_{n_j}), u_{n_j} \rangle + \frac{1}{\gamma^+} \langle J'(u_{n_j}), u_{n_j} \rangle.$$

Going if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X . As $X = \bigcup_{n_j} Y_{n_j}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \rightarrow u$. Hence

$$\begin{aligned} \lim_{n_j \rightarrow +\infty} \langle J'(u_{n_j}), u_{n_j} - u \rangle &= \lim_{n_j \rightarrow +\infty} \langle J'(u_{n_j}), u_{n_j} - v_{n_j} \rangle + \lim_{n_j \rightarrow +\infty} \langle J'(u_{n_j}), v_{n_j} - u \rangle \\ &= \lim_{n_j \rightarrow +\infty} \langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j} \rangle \\ &= 0. \end{aligned}$$

Similar to the process of verifying the Palais-Smale condition in the proof of Lemma 4.3, we conclude $u_{n_j} \rightarrow u$, furthermore we have $J'(u_{n_j}) \rightarrow J'(u)$. Let us prove $J'(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned} \langle J'(u), \omega_k \rangle &= \langle J'(u) - J'(u_{n_j}), \omega_k \rangle + \langle J'(u_{n_j}), \omega_k \rangle \\ &= \langle J'(u) - J'(u_{n_j}), \omega_k \rangle + \langle (J|_{Y_{n_j}})'(u_{n_j}), \omega_k \rangle. \end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle J'(u), \omega_k \rangle = 0, \quad \forall \omega_k \in Y_k,$$

so $J'(u) = 0$, this show that J satisfies the $(PS)_c^*$ condition for every $c \in \mathbb{R}$. The conclusion of Theorem 6.1 (ii) is reached by the dual fountain theorem. \square

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