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Partial Differential Equations — *A degenerate elliptic operator with unbounded diffusion coefficients*, by G. Metafune and C. Spina, communicated on 14 February 2014.

Abstract. — We prove that, for $N \geq 3$, the operator $L = |x|^{\alpha}\Delta$ generates an analytic semigroup in $L^{p}(\mathbb{R}^{N})$ if $\alpha = 2$ and $1 < p < \infty$ or $\alpha < 2$ and $\frac{N}{N-\alpha} < p < \infty$ or $\alpha > 2$ and $\frac{N}{N-2} < p < \infty$. The above bounds are shown to be sharp.

Key words: Degenerate elliptic operators, unbounded coefficients, analytic semigroups.

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1. Introduction and notation

In this paper we deal with the operator $Lu = |x|^{\alpha}\Delta u$ for $\alpha \in \mathbb{R}$, on $L^p = L^p(\mathbb{R}^N, dx)$, $N \geq 3$, with respect to the Lebesgue measure. We are interested both in parabolic problems $u_t - Lu = 0$, $u(0) = f$ and in the solvability of the elliptic equation $\lambda u - Lu = f$ for $\lambda \in \mathbb{C}$ anf f in L^p.

We recall that a (minimal) resolvent and a (minimal) semigroup can be constructed in spaces of continuous functions as in [13], by solving elliptic and parabolic problems associated with L in a sequence of annuli filling the whole space, see also the next Section.

On the other hand related results concerning the L^p-theory for some second-order elliptic divergence-type operators with measurable coefficients have been developped in [8] and [18].

Since the operator is degenerate both at 0 and ∞, we study separately the operators $L_1 = |x|^{\alpha}\Delta$ in the ball B_R and $L_2 = |x|^{\alpha}\Delta$ in the exterior domain B_R^c, both with Dirichlet boundary conditions.

Concerning the operator L_2, we observe that it can be treated as the operator $(1 + |x|^{\alpha})\Delta$ in the whole space $\mathbb{R}^N$. Generation results and domain description for this last operator are already known by [14] in the case $\alpha > 2$ and by [5] in the case $\alpha \leq 2$. It follows that L_2 generates an analytic semigroup for $1 < p < \infty$ when $\alpha \leq 2$ and for $\frac{N}{N-2} < p < \infty$ when $\alpha > 2$, the restriction on p being sharp.

The operator L_1 is singular near the origin. However a generalization of the results of [5] allows to prove generation of analytic semigroup when $\alpha \geq 2$, together with an explicit desciption of the domain.

The case $\alpha < 2$ requires several steps. We first prove that L_1 is invertible and that its resolvent is positive. Then the bound on the resolvent norm

$\|(\lambda - L_1)^{-1}\| \le \|L_1^{-1}\|$ follows for $\lambda > 0$. This however is not enough to obtain generation results by the classical Hille-Yosida Theorem. The operator $L_1 = |x|^{\alpha}\Delta$ is similar in $L^{\frac{2N}{N-2}}$, via the Kelvin transform, to the operator $|x|^{4-\alpha}\Delta$ defined on the exterior domain B_R^c. Since the operator $|x|^{4-\alpha}\Delta$ generates an analytic semigroup in $L^p(B_R^c)$, $p = (2N)/(N-2)$, consequently, L_1 generates an analytic semigroup in $L^p(B)$ for the same p. By interpolation we deduce analiticity for $p \ge \frac{2N}{N-2}$. To conclude, an extrapolation procedure based on the boundedness of the resovent, scaling arguments and the generation results for large p, allows to prove generation for every $p > \frac{N}{N-\alpha}$. We point out that the above restriction on p is sharp. Glueing togehther the resolvents of L_1 and L_2 we obtain the results for L.

The paper is organized as follows. In the first Section we recall the construction of the resolvent in spaces of continuous functions. In Section 3, we slightly generalize some results of [5] used throughout the paper. Section 4 is mainly devoted to understand the appropriate domains of L_1 and L_2. Moreover the invertibility, the positivity of the resolvent and the coherence of the resolvent in the L^p scale are proved. In Section 5 it is explained how to construct a resolvent for L by gluing the resolvent of L_1 and L_2 or to deduce some results for interior and exterior domains from results in the whole space. In Sections 6, 7 and 8 the main generation results are proved. Section 9 contains a variational proof of the dissipativity based on Hardy type inequalities. The precise values of p for which the semigroups is contractive are also obtained. Finally Section 10 contains kernel estimates for L. Even though some of our results are valid also for $N = 1, 2$ (e.g. the generation results for L_2 when $\alpha \le 2$), we keep the assumption $N \ge 3$ to unify the exposition.

Notation. Fix $R > 0$. Assume that $N \ge 3$, set $\Omega = \mathbb{R}^N \setminus \{0\}$, $\Omega_R = B_R \setminus \{0\}$,

$$C_0(\Omega) = \{u \in C_b(\Omega) : \lim_{|x| \to \infty} u(x) = 0\},$$

$$C_0(\Omega_R) = \{u \in C_b(\Omega_R) : u(x) = 0 \text{ on } \partial B_R\},$$

$$C_0(B_R^c) = \{u \in C_b(B_R^c) : u(x) = 0 \text{ on } \partial B_R, \lim_{|x| \to \infty} u(x) = 0\},$$

endowed with the sup-norm.

2. The operator in spaces of continuous functions

For a fixed real α (positive or negative) we consider the operators

$$L(x) = |x|^{\alpha}\Delta, \quad L_1(x) = |x|^{\alpha}\Delta, \quad L_2(x) = |x|^{\alpha}\Delta,$$

endowed with their maximal domain in the space of continuous functions respectively given by

$$D_{max}(L) = \{u \in C_0(\Omega) \cap W^{2,p}_{loc}(\Omega) \quad \text{for all } p < \infty : Lu \in C_0(\Omega)\},$$
$$D_{max}(L_1) = \{u \in C_0(\Omega_R) \cap W^{2,p}_{loc}(\Omega_R) \quad \text{for all } p < \infty : Lu \in C_0(\Omega_R)\},$$
$$D_{max}(L_2) = \{u \in C_0(B_R^c) \cap W^{2,p}_{loc}(B_R^c) \quad \text{for all } p < \infty : Lu \in C_0(B_R^c\}.$$

We start by studying existence and uniqueness of bounded solutions of the elliptic equation

$$\lambda u - Lu = f \tag{1}$$

for $\lambda > 0$ and $f \in C_0(\Omega)$. Due to the degeneracy at the origin and unboundedness at infinity (if $\alpha > 0$), the classical theory does not apply and existence and uniqueness are not clear. Existence is stated in the following result whise proof is identical to that in [13, Theorem 3.4].

Proposition 2.1. *For every $f \in C_0(\Omega)$, $\lambda > 0$, there exists $u \in D_{max}(L)$ solving equation* (1) *and satisfying the inequality $\|u\|_\infty \le \|f\|_\infty/\lambda$. Moreover, $u \ge 0$ whenever $f \ge 0$.*

Uniqueness follows from the existence of suitable Lyapunov functions for the operator L.

Definition 2.2. We say that V is a Lyapunov function for L if $V \in C^2(\Omega)$, $V \ge 1$, V goes to infinity as $|x| \to 0$ and $\lambda_0 V - LV \ge 0$ for some $\lambda_0 > 0$.

Proposition 2.3. *Suppose that there exists V Lyapunov function for the operator L. Then $\lambda - L$ is injective on $D_{max}(L)$ for every $\lambda > 0$.*

Proof. Let $\lambda \ge \lambda_0$ where λ_0 is as in Definition 2.2. We show that if $u \in D_{max}(L)$ satisfies $\lambda u - Lu \le 0$ then $u \le 0$. For every $\varepsilon > 0$, introduce the function $u_\varepsilon = u - \varepsilon V$. Observe that, by assumption, u_ε satisfies $\lambda u_\varepsilon - Lu_\varepsilon \le 0$ in Ω. Suppose that $u_\varepsilon > 0$ somewhere. Since u_ε is negative near 0 and ∞, then u_ε attains its positive maximum at some $x_0 \in \Omega$. By Bony's maximum principle, see [13, Lemma 3.2], $\Delta u_\varepsilon(x_0) \le 0$, hence $Lu_\varepsilon(x_0) = |x_0|^\alpha \Delta u_\varepsilon(x_0) \le 0$ and $\lambda u_\varepsilon - Lu_\varepsilon > 0$ at x_0. Since this is a contraddiction, then $u_\varepsilon \le 0$ and, letting ε to 0, $u \le 0$ in Ω. Changing u with $-u$ we obtain that $\lambda u - Lu$ is injective on $D_{max}(L)$ for $\lambda \ge \lambda_0$. Combining the injectivity of $\lambda - L$ with the existence result stated in Proposition 2.1, it follows that if $u \in D_{max}(L)$ satisfies $\lambda u - Lu = f$, then $\|u\|_\infty \le \frac{1}{\lambda}\|f\|_\infty$. Let now $0 < \lambda < \lambda_0$ and $u \in D_{max}(L)$ such that $\lambda u - Lu = 0$. Clearly $\lambda_0 u - Lu = (\lambda_0 - \lambda)u$ and, as observed above, $\|u\|_\infty \le \frac{\lambda_0 - \lambda}{\lambda_0}\|u\|_\infty$. The last inequality yields $u = 0$ and the injectivity of $\lambda - L$ for $0 < \lambda < \lambda_0$. □

Remark 2.4. *Let $0 \le \phi \le 1$ be a smooth cut-off function such that $\phi(x) = 1$ for $|x| \le 1/2$ and $\phi(x) = 0$ for $|x| \ge 1$. By easy computations it follows that the function $V(x) = -\phi(x)\ln|x| + 1$ is a Lyapunov function for L. Therefore $\lambda - L$ is injective on $D_{max}(L)$ for every $\lambda > 0$.*

Since $D_{max}(L)$ need not to be dense in $C_0(\Omega)$, see Proposition 4.12, we cannot say that L generates a semigroup. In the sequel, however, we need resolvent estimates for complex values of λ.

PROPOSITION 2.5. *For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$, the operator $\lambda - L$ is invertible from $D_{max}(L)$ to $C_0(\Omega)$ and its resolvent $R(\lambda, L)$ satisfies $\|R(\lambda, L)\| \le \frac{1}{\mathrm{Re}\,\lambda}$.*

PROOF. For $\lambda > 0$ the statement follows from the previous two propositions and therefore the operator L is dissipative, see [7, Corollary I.3.5]. It follows that the inequality above holds whenever $\mathrm{Re}\,\lambda > 0$ and $\lambda - L$ is invertible. Since the resolvent set of L is open, contains $(0, \infty)$ and the norm of the resolvent operator $R(\lambda, L)$ explodes when λ approches the boundary of the resolvent set, the thesis follows. □

Existence and uniqueness of bounded solutions of the elliptic problems $\lambda u - L_i u = f$, $u \in D_{max}(L_i)$, for $i = 1, 2$, can be proved similarly.

PROPOSITION 2.6. *For every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$, the operators $\lambda - L_i$, $i = 1, 2$ are invertible from $D_{max}(L_i)$ to $C_0(\Omega_R)$, $C_0(B_R^c)$ respectively and their resolvents satisfy $\|R(\lambda, L_i)\| \le \frac{1}{\mathrm{Re}\,\lambda}$.*

3. A preliminary result in $L^p(\mathbb{R}^N)$

Here we consider the operator $A = a(x)\Delta$, with a satisfying

$$a : \mathbb{R}^N \to [0, +\infty[, \quad a^{\frac{1}{2}} \in W^{1,\infty}_{loc}(\mathbb{R}^N), \quad \|\nabla a^{\frac{1}{2}}\|_\infty \le c \tag{2}$$

for some positive constant c. Observe that if $a(x) = |x|^\alpha$ near the origin, then $\alpha \ge 2$ whereas if $a(x) = |x|^\alpha$ near infinity, then $\alpha \le 2$.

Set $\Omega = \{x \in \mathbb{R}^N : a(x) > 0\}$ and $F = \{x \in \mathbb{R}^N : a(x) = 0\}$. The aim of this subsection consists in proving that, for any $p \in]1, \infty[$, the operator $A_p = (A, D_p)$, where $A = a(x)\Delta$ and

$$D_p = \{u \in W^{2,p}_{loc}(\Omega) \cap L^p(\mathbb{R}^N) : a^{\frac{1}{2}}\nabla u, aD^2u \in L^p(\mathbb{R}^N)\}, \tag{3}$$

generates an analytic semigroup in $L^p(\mathbb{R}^N)$. We point out that the present results are a slight generalization of those in [5, Section 2], where the additional assumption $a(x) > 0$ for every $x \in \mathbb{R}^N$ is required. Most of the proofs are unchanged. As a first step, we identify a core for A_p.

Set $D_F = \{u \in L^p(\mathbb{R}^N) : u = u\chi_F\}$ and observe that D_F, $C_c^\infty(\Omega) \subset D_p$.

LEMMA 3.1. *The space $D_F + C_c^\infty(\Omega)$ is dense in D_p, endowed with the norm*

$$\|u\|_{D_p} = \|u\|_p + \|a^{\frac{1}{2}}\nabla u\|_p + \|aD^2u\|_p.$$

PROOF. By the assumptions on a it follows that

$$a(x) \le c(1 + |x|^2), \quad a(x) \le c^2 d(x, F)^2. \tag{4}$$

Indeed, for every $x \in \Omega$ and $y \in F$,

$$\sqrt{a(x)} = |\sqrt{a(x)} - \sqrt{a(y)}| \le c|x-y|$$

and hence $a(x) \le c^2|x-y|^2$, as claimed. Let now $u \in D_p$, $u = u\chi_F + u\chi_\Omega$ and $u\chi_F \in D_F \subset D_p$. Setting $v = u\chi_\Omega$, we approximate v with functions in $W^{2,p}(\Omega)$ having compact support in Ω. Let

$$\Omega_n = \left\{x \in \Omega : d(x,F) \ge \frac{1}{n}\right\}, \qquad \xi_n = \chi_{\Omega_{2n}} * \phi_{\frac{1}{2n}}$$

where ϕ is a classical mollifier supported in B_1, with $\int_{\mathbb{R}^N} \phi = 1$ and $\phi_{\frac{1}{n}}(x) = n^N\phi(nx)$. It is easy to check that $\xi_n(x) = 1$ for $x \in \Omega_n$, ξ_n is supported in Ω and that $0 \le \xi_n \le 1$, $|\nabla \xi_n| \le Cn$, $|D^2\xi_n| \le Cn^2$. Consider also a smooth function η such that $\chi_{B_1} \le \eta \le \chi_{B_2}$ and, for every $n \in \mathbb{N}$, define $\eta_n(x) = \eta\left(\frac{x}{n}\right)$. Set $v_n = \xi_n\eta_n v$. It is immediate to check that v_n tends to v in $L^p(\mathbb{R}^N)$. Concerning the gradient term, we have

$$\|a^{\frac{1}{2}}(\nabla(\xi_n\eta_n v) - \nabla v)\|_p^p \le \int_{\mathbb{R}^N} a(x)^{\frac{p}{2}}|\xi_n\eta_n - 1|^p|\nabla v|^p + \frac{C}{n^p}\|\eta\|_\infty \sup_{n\le|x|\le 2n} a(x)^{\frac{p}{2}} \int_{\{n\le|x|\le 2n\}} |v|^p + n^p\int_{\Omega\setminus\Omega_n} a^{\frac{p}{2}}|v|^p.$$

By (4)

$$\|a^{\frac{1}{2}}(\nabla(\xi_n\eta_n v) - \nabla v)\|_p^p \le \int_{\mathbb{R}^N} a(x)^{\frac{p}{2}}|\xi_n\eta_n - 1|^p|\nabla v|^p + C\|\eta\|_\infty \int_{\{n\le|x|\le 2n\}} |v|^p + \int_{\Omega\setminus\Omega_n} |v|^p$$

which tends to 0, by dominated convergence. Using a similar argument one shows that aD^2v_n tends to aD^2v in $L^p(\mathbb{R}^N)$. Finally we can use a standard convolution argument to approximate functions with compact support in $W^{2,p}(\Omega)$ with $C_c^\infty(\Omega)$ functions. □

In the next lemma we state the main a-priori estimates.

LEMMA 3.2. *There exist ε_0, $C > 0$ depending only on c, N such that for every $0 < \varepsilon < \varepsilon_0$ and any $u \in D_p$,*

$$\|a^{\frac{1}{2}}\nabla u\|_p \le \varepsilon\|a\Delta u\|_p + \frac{C}{\varepsilon}\|u\|_p \tag{5}$$

$$\|aD^2u\|_p \le C(\|a\Delta u\|_p + \|u\|_p). \tag{6}$$

Proof. In view of Lemma 3.1, it is enough to prove (5) and (6) for functions in $D_F + C_c^\infty(\Omega)$. The claimed inequalities are obvious for $u \in D_F$. If $u \in C_c^\infty(\Omega)$ we consider, for $\delta > 0$, $a_\delta = a + \delta$ which is positive and satisfies (2) with c independent of δ. From [5, Lemma 2.4] we deduce that (5) and (6) hold for $a_\delta \Delta$ with ε_0, C independent of δ. Letting $\delta \to 0$ the thesis follows. □

Theorem 3.3. *For any $1 < p < \infty$, the operator A_p generates a strongly continuous analytic semigroup. Moreover, such a semigroup is positive and consistent with respect to p.*

The proof follows from the previous two lemmas, as in [5, Theorem 2.5, 2.7]. The following a-priori estimates will be useful in the next section.

Corollary 3.4. *There exist two constants $\Lambda_p > 0$ and $C > 0$ such that, for every $u \in D_p$ and every $\operatorname{Re}\lambda \geq \Lambda_p$*

$$|\lambda|\,\|u\|_p + |\lambda|^{\frac{1}{2}}\|a^{\frac{1}{2}}\nabla u\|_p + \|aD^2u\|_p \leq C\|\lambda u - Au\|_p.$$

Proof. The estimate $|\lambda|\,\|u\|_p \leq C\|\lambda u - Au\|_p$ is nothing but sectoriality. The gradient estimate follows from it, using (5) with $\varepsilon = |\lambda|^{-\frac{1}{2}}$. Similarly, the Hessian estimate follows from sectoriality and (6). □

In the next proposition we prove that D_p given by (3) coincides with the maximal domain.

Proposition 3.5. *The domain D_p given by (3) coincides with the maximal domain*

$$D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega) : Au \in L^p(\mathbb{R}^N)\}.$$

Proof. The inclusion $D_p \subset D_{p,max}(A)$ is obvious. Conversely, let $u \in D_{p,max}(A)$ and let $\lambda > 0$ be in the resolvent set of (A, D_p). Set $f = \lambda u - Au$ and $v = u - R(\lambda, A)f$. Then v belongs to $D_{p,max}(A)$ and satisfies $\lambda v - Av = 0$. We prove that $v \equiv 0$ if λ is large enough. Let ξ_n, η_n be as in the proof of Lemma 3.1 and set $\zeta_n = \xi_n\eta_n$ and recall that $\xi_n(x) = 1$ for $x \in \Omega_n$ and has support in Ω_{2n}. Then $|\nabla\xi_n| \leq Cn\chi_{\Omega_{2n}\setminus\Omega_n}$. Similarly, $|\nabla\eta_n| \leq Cn^{-1}\chi_{B_{2n}\setminus B_n}$. Using (4), we see that $a|\nabla\zeta_n|^2 \leq C$, with C independent of n, and has support in $(\Omega_{2n}\setminus\Omega_n) \cup (B_{2n}\setminus B_n)$. By integrating by parts the identity $\int_{\mathbb{R}^N} (\lambda v - a\Delta v)v|v|^{p-2}\zeta_n^2 = 0$ (see [10, Section 3] if $1 < p < 2$), we obtain

$$0 = \lambda\int_{\mathbb{R}^N} |v|^p\zeta_n^2 + (p-1)\int_{\mathbb{R}^N} a|\nabla v|^2|v|^{p-2}\zeta_n^2$$
$$+ 2\int_{\mathbb{R}^N} a\zeta_n|v|^{p-2}v\nabla v\cdot\nabla\zeta_n + \int_{\mathbb{R}^N} \zeta_n^2|v|^{p-2}v\nabla a\cdot\nabla v.$$

By Hölder's inequality and observing that $a|\nabla\zeta_n|^2 \le C$ if $x \in E_n = (B_{2n}\backslash B_n) \cup (\Omega_{2n}\backslash\Omega_n)$ and $a\nabla\zeta_n = 0$ otherwise, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N} a\zeta_n|v|^{p-2}v\nabla v\cdot\nabla\zeta_n\right| &\le \left(\int_{\mathbb{R}^N} a\zeta_n^2|\nabla v|^2|v|^{p-2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N} a|v|^p|\nabla\zeta_n|^2\right)^{\frac{1}{2}} \\ &\le C_1\left(\int_{\mathbb{R}^N} a\zeta_n^2|\nabla v|^2|v|^{p-2}\right)^{\frac{1}{2}}\left(\int_{E_n}|v|^p\right)^{\frac{1}{2}} \\ &\le \varepsilon\int_{\mathbb{R}^N} a\zeta_n^2|\nabla v|^2|v|^{p-2} + \frac{C_2}{\varepsilon}\int_{E_n}|v|^p\end{aligned}$$

for every $\varepsilon > 0$ and some positive constants C_1, C_2. Since $|\nabla a| \le ca^{\frac{1}{2}}$ we also obtain in a similar way

$$\left|\int_{\mathbb{R}^N}\zeta_n^2|v|^{p-2}v\nabla a_1\cdot\nabla v\right| \le \varepsilon\int_{\mathbb{R}^N} a\zeta_n^2|\nabla v|^2|v|^{p-2} + \frac{C_3}{\varepsilon}\int_{\mathbb{R}^N}|v|^p$$

for every $\varepsilon > 0$ and some positive constant C_3. Combining the last inequalities we obtain

$$\left(\lambda - \frac{C_3}{\varepsilon}\right)\int_{\mathbb{R}^N}|v|^p\zeta_n^2 + (p-1-3\varepsilon)\int_{\mathbb{R}^N} a|\nabla v|^2|v|^{p-2}\zeta_n^2 - \frac{2C_2}{\varepsilon}\int_{E_n}|v|^p \le 0.$$

Finally, choosing $3\varepsilon < p-1$ and letting n to infinity, we obtain

$$\left(\lambda - \frac{C_3}{p-1}\right)\int_{\mathbb{R}^N}|v|^p \le 0$$

which implies $v \equiv 0$, if λ is large enough. □

As an immediate application of Theorem 3.3 and Proposition 3.5 we obtain the following result.

COROLLARY 3.6. *The operator $L = |x|^2\Delta$ with domain $D_p = D_{p,max}(L)$ generates an analytic semigroup in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The semigroup is positive and consistent with p.*

4. THE DEFINITION OF THE OPERATORS IN L^p

Let $L = |x|^\alpha\Delta$ in $\mathbb{R}^N$, $L_1 = |x|^\alpha\Delta$ in the ball B_R, with Dirichlet boundary conditions and $L_2 = |x|^\alpha\Delta$ in the exterior domain B_R^c, again with Dirichlet boundary conditions. In this section we define the domain of L, L_1, L_2 and we recall that $\Omega_R = B_R\backslash\{0\}$.

4.1. The domain of L_1

We define the maximal domain of L_1 as follows.

Definition 4.1.

$$D_{p,max}(L_1) = \{u \in L^p(B_R) \cap W^{2,p}(B_R \backslash B_\varepsilon)\ \forall \varepsilon > 0 : u(x) = 0 \text{ if } |x| = R,\\ |x|^\alpha \Delta u \in L^p(B_R)\}.$$

Observe that the Dirichlet boundary condition $u(x) = 0$ for $|x| = R$ makes sense, since u has second derivatives in L^p in a neighborhood of the boundary of B_R. By elliptic regularity L_1 is closed on its maximal domain. If $\alpha \geq 2$ the function $a(x) = |x|^\alpha$ satisfies the inequality $|\nabla a^{1/2}| \leq C$ in the ball B_R even though not globally in $\mathbb{R}^N$ when $\alpha > 2$. In analogy with Section 3 we define the domain of L_1 as follows.

Definition 4.2. If $\alpha \geq 2$ we set

$$D_p(L_1) = \{u \in L^p(B_R) \cap W^{2,p}(B_R \backslash B_\varepsilon)\ \forall \varepsilon > 0 : u(x) = 0 \text{ if } |x| = R,\\ |x|^{\alpha/2} \nabla u, |x|^\alpha D^2 u \in L^p(B_R)\}.$$

Observe that the Dirichlet boundary condition $u(x) = 0$ for $|x| = R$ makes sense, since u has second derivatives in L^p in a neighborhood of the boundary of B_R. Observe also that the "fundamental solution" of the Laplacian $u(x) = |x|^{2-N}$ (near the origin) belongs to $D_p(L_1)$, $\alpha \geq 2$, if and only if $p < N/(N-2)$.

Proposition 4.3. *If $\alpha \geq 2$, then $D_{p,max}(L_1) = D_p(L_1)$ and the operator L_1 is closed on its domain.*

Proof. Clearly $D_p(L_1) \subset D_{p,max}(L_1)$. To prove the opposite inclusion, let $u \in D_{p,max}(L_1)$ and η be a cut-off function such that $\eta(x) = 1$ if $|x| \leq R/2$ and $\eta(x) = 0$ if $|x| \geq 3R/4$. Finally, consider $L = a\Delta$ where $a(x) = |x|^\alpha$ if $|x| \leq R$ and $a(x) = R^\alpha$ if $|x| \geq R$. The operator L satisfies the assumption of the previous section and therefore, by Proposition 3.5, $D_p(L) = D_{p,max}(L)$. We write $u = \eta u + (1-\eta)u$ and observe that $(1-\eta)u \in D_p(L_1)$ since it vanishes in a neighborhood of the origin. Finally, $\eta u \in D_{p,max}(L) = D_p(L)$ and hence $|x|^{\alpha/2}\nabla u$, $|x|^\alpha D^2 u \in L^p(B_R)$, that is $\eta u \in D_p(L_1)$. The closedness of L_1 now follows since it is closed on its maximal domain. □

We consider next the case $\alpha < 0$.

Definition 4.4. If $\alpha < 0$ we set

$$D_p(L_1) = \{u \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R) : |x|^\alpha \Delta u \in L^p(B_R)\}.$$

Proposition 4.5. *If $\alpha < 0$, the operator $(L_1, D_p(L_1))$ is closed and invertible with compact resolvent. Its spectrum is independent of $1 < p < \infty$.*

Proof. If $(u_n) \subset D_p(L_1)$ converges to u in $L^p(B_R)$ and $|x|^\alpha \Delta u_n \to v$ in $L^p(B_R)$, then Δu_n converges in $L^p(B_R)$, since $\alpha < 0$. Since $u_n \subset W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$, by elliptic regularity u_n converges to u in $W^{2,p}(B_R)$. It follows that $v = |x|^\alpha \Delta u$, hence

$u \in D_p(L_1)$ and $L_1 u = v$. This shows the closedness. To show the invertibility, we observe that the equation $L_1 u = f$, $u = 0$ on ∂B_R, is equivalent to $\Delta u = f|x|^{-\alpha}$, $u = 0$ on ∂B_R which has a unique solution $u \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$. Such a u belongs to $D_p(L_1)$ and solves $L_1 u = f$. Since L_1 is clearly injective on its domain, this shows that it is invertible. The compactness of the resolvent follows from the compactness of the embedding of $W^{2,p}(B_R)$ into $L^p(B_R)$ and proves that the spectrum consists of eigenvalues which are independent of $1 < p < \infty$, see [1]. □

In order to deal with the case $0 < \alpha < 2$ we need some considerations which hold in the more general case $0 < \alpha < N$. If $f \in L^p(B_R)$ then $f|x|^{-\alpha} \in L^q(B_R)$ for some $q > 1$ if $p > N/(N-\alpha)$. In this case, Hölder's inequality yields any $1 < q < Np/(N + \alpha p)$.

Definition 4.6. If $0 < \alpha < 2$ and $p > N/(N - \alpha)$ we set

$$D_p(L_1) = \left\{ u \in W^{2,q}(B_R) \cap W_0^{1,q}(B_R) \text{ for every } q < \frac{Np}{N + \alpha p} : |x|^{\alpha} \Delta u \in L^p(B_R) \right\}.$$

Proposition 4.7. *If $0 < \alpha < 2$, $p > N/(N - \alpha)$, the operator $(L_1, D_p(L_1))$ is closed and invertible with compact resolvent. Its spectrum is independent of $1 < p < \infty$.*

Proof. If $(u_n) \subset D_p(L_1)$ converges to u in $L^p(B_R)$ and $|x|^{\alpha}\Delta u_n \to v$ in $L^p(B_R)$, then Δu_n converges in $L^q(B_R)$, for every $1 < q < Np/(N + \alpha p)$, by Hölder's inequality. Since $u_n \subset W^{2,q}(B_R) \cap W_0^{1,q}(B_R)$, by elliptic regularity u_n converges to u in $W^{2,q}(B_R)$. It follows that $v = |x|^{\alpha}\Delta u$, hence $u \in D_p(L_1)$ and $L_1 u = v$. This shows the closedness. To show the invertibility, we observe that the equation $L_1 u = f$, $u = 0$ on ∂B_R is equivalent to $\Delta u = f|x|^{-\alpha}$, $u = 0$ on ∂B_R, which has a unique solution $u \in W^{2,q}(B_R) \cap W_0^{1,q}(B_R)$. If $\frac{Np}{N+\alpha p} \le \frac{N}{2}$, then, by Sobolev embedding, $u \in L^s(B_R)$ where $1/s = 1/q - 2/N < 1/p$, if q is chosen sufficiently close to $Np/(N + \alpha p)$, since $\alpha < 2$. Otherwise we can choose $q > \frac{N}{2}$ and, by Sobolev embeddings again, $u \in L^{\infty}(B_R)$ and so $u \in L^p(B_R)$. Such a u belongs to $D_p(L_1)$ and solves $L_1 u = f$. Since L_1 is clearly injective on its domain, this shows that it is invertible. The compactness of the resolvent follows from the compactness of the embedding of $W^{2,q}(B_R)$ into $L^p(B_R)$ and the independence of the spectrum on $p > N/(N - \alpha)$ follows from [1]. □

Next we investigate the validity of the equality $D_{p,max}(L_1) = D_p(L_1)$ which we have already proved in the case $\alpha \ge 2$ in Proposition 4.3.

Proposition 4.8. *Let $u \in L^p(B_R)$ with $p \ge N/(N-2)$ and suppose that $\Delta u = 0$ in $\Omega_R = B_R \setminus \{0\}$. Then u is harmonic in B_R.*

Proof. By the mean value property

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy$$

for every $0 \neq x \in B_{\frac{R}{2}}$ and $r = |x|/2$. Hölder inequality (with $p = N/(N-2)$) yields

$$|u(x)| \leq C|x|^{2-N}\left(\int_{B(x,r)} |u(y)|^{N/(N-2)}dy\right)^{1-2/N}$$

and hence $u(x)|x|^{N-2} \to 0$ as $x \to 0$. By elementary properties of harmonic functions, u can be extended as an harmonic function in B_R. □

Observe that the limitation $p \geq N/(N-2)$ is necessary to exclude the fundamental solution $|x|^{2-N}$.

PROPOSITION 4.9. *If $\alpha < 0$, $p \geq N/(N-2)$, then $D_{p,max}(L_1) = D_p(L_1)$.*

PROOF. Let $u \in D_{p,max}(L_1)$. From $Lu = f$ we infer $\Delta u = f|x|^{-\alpha}$ in Ω_R. The function $g = f|x|^{-\alpha}$ belongs to $L^p(B_R)$, since $\alpha \leq 0$. Let $v \in W^{2,p}(B_R) \cap W_0^{1,p}(B_R)$ be such that $\Delta v = g$. Then $\Delta(u - v) = 0$ in Ω_R and, by Proposition 4.8, $u - v$ is harmonic in B_R. Therefore $u \in W^{2,p}(B_R)$ and belongs to $D_p(L_1)$ as defined in Definition 4.4. Since the converse inclusion is obvious, the proof is complete. □

The argument above can be generalized when $0 < \alpha < N$, as follows.

PROPOSITION 4.10. *Let $0 < \alpha < N$ and $p > N/(N-\alpha)$, $p \geq N/(N-2)$. Then all functions $u \in D_{p,max}(L_1)$ belong to $W^{2,q}(B_R)$ for every $1 \leq q < Np/(N+\alpha p)$. In particular $D_{p,max}(L_1) = D_p(L_1)$ when $0 < \alpha < 2$ and $p \geq N/(N-2)$.*

PROOF. Let $u \in D_{p,max}(L_1)$. As before $\Delta u = g = f|x|^{-\alpha}$ in Ω_R. By Hölder inequality $g \in L^q(B_R)$ for every $1 \leq q < Np/(N+\alpha p)$ (observe that $Np/(N+\alpha p) > 1$ since $p > N/(N-\alpha)$). Let $q > 1$ as in the statement and $v \in W^{2,q}(B_R) \cap W_0^{1,q}(B_R)$ be such that $\Delta v = g$. Then $\Delta(u - v) = 0$ in Ω_R. By Sobolev embedding $v \in L^{q^*}(B_R)$ with $1/q^* = 1/q - 2/N$ if $q < N/2$ and q^* any number if $q \geq N/2$. In each case $q^* \geq N/(N-2)$ and therefore Proposition 4.8 applies and $u - v$ is harmonic in B_R. Therefore $u \in W^{2,q}(B_R)$. If $0 < \alpha < 2$ and $p \geq N/(N-2)$, then $p > N/(N-\alpha)$ and therefore u belongs to $D_p(L_1)$, see Definition 4.6. Since the converse inclusion is obvious, the proof is complete. □

Summing up, we have proved in particular the following result.

COROLLARY 4.11. *The equality $D_p(L_1) = D_{p,max}(L_1)$ holds if $\alpha \geq 2$ and when $\alpha < 2$, $p \geq N/(N-2)$.*

Of course, when $0 < \alpha < 2$, we always assume that $p > N/(N-\alpha)$, otherwise the operator L_1 is not defined. We note the following easy consequence of Propositions 4.9, 4.10.

PROPOSITION 4.12. *If $\alpha \leq 0$ and $p > N/2$, $p \geq N/(N-2)$ or $0 < \alpha < 2$ and $p > N(2-\alpha)$, $p \geq N/(N-2)$, then all functions in $D_{p,max}(L_1)$ can be continuously extended to the origin.*

PROOF. We only note that for $0 < \alpha < 2$ the exponent $Np/(N + \alpha p)$ given by Lemma 4.10 is greater than $N/2$ when $p > N/(2 - \alpha)$. □

In particular, if $\alpha < 2$, then

$$D_{max}(L_1) = \{u \in C_0(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega) \text{ for all } p < \infty : Lu \in C_0(\Omega)\}.$$

EXAMPLE 4.13. *Let $\alpha \geq 2$, $u(x) = \sin \ln|x|$ near the origin. Then $u \in D_{max}(L_1)$ but u is not continuous at the origin.*

Let us show that in all cases smooth functions are a core for L_1.

PROPOSITION 4.14. *The space $C^\infty(\bar{B}_R)$ is a core for $(L_1, D_p(L_1))$.*

PROOF. If $\alpha \geq 2$ this follows arguing as in Lemma 3.1 or can be deduced from it, as in Proposition 4.3. Let then $\alpha < 2$ (and $p > N/(N - \alpha)$ when $0 < \alpha < 2$), $u \in D_p(L_1)$ and set $f = L_1 u \in L^p(B_R)$. We consider a sequence (f_n) of C^∞ functions with compact support contained in Ω_R (hence vanishing near 0) such that $f_n \to f$ in $L^p(B_R)$. Since $(L_1, D_p(L_1))$ is invertible, the functions $u_n = L_1^{-1} f_n$ are well-defined and converge to u in the graph norm of L_1. By elliptic regularity, since $\Delta u_n = |x|^{-\alpha} f_n$, the u_n are C^∞ functions in $\bar{B}_R \backslash B_\varepsilon$ for every $\varepsilon > 0$. Moreover, if $f_n \equiv 0$ in B_ε, then $\Delta u_n = 0$ in $B_\varepsilon \backslash \{0\}$. Since $u_n \in W^{2,q}(B_R)$, where $q = p$ if $\alpha < 0$ and $q > 1$ is any number less than $Np/(N + \alpha p)$ if $0 < \alpha < 2$, it follows that $\Delta u = 0$ in B_ε, hence $u_n \in C^\infty(\bar{B}_R)$. □

Next, we show the consistency of the resolvents with respect to p and their positivity for $\lambda > 0$.

PROPOSITION 4.15. *Let $\lambda \in \rho(L_1, D_p(L_1)) \cap \rho(L_1, D_{max}(L_1))$. Then the resolvents of L_1 in $L^p(B_R)$ and in $C_0(\Omega_R)$ coincide on $C_0(\Omega_R)$. In particular the resolvents in $L^p(B_R)$ and $L^q(B_R)$ coincide and are positive if λ is positive.*

PROOF. If $\alpha \geq 2$, then $D_{max}(L_1) \subset D_{p,max}(L_1) = D_p(L_1)$ and hence the solution $u \in D_{max}(L_1)$ of the equation $\lambda u - L_1 u = f \in C_0(\Omega_R)$ is also the unique solution in $D_p(L_1)$. This shows the consistency for $\alpha \geq 2$. If $\alpha < 2$, the above argument works only for $p \geq N/(N - 2)$, since then $D_p(L_1) = D_{p,max}(L_1)$ and we modify it as follows. Let $f \in C_0(\Omega_R)$ and $u \in D_p(L_1)$ solve $\lambda u - L_1 u = f$. By Definition 4.4, $u \in W^{2,p}(B_R)$ and vanishes at the boundary. If $p > N/2$, then $u \in C_0(B_R)$ and hence $L_1 u \in C_0(\Omega_R)$, that is $u \in D_{max}(L_1)$ and we are done. If $p < N/2$, by Sobolev embedding $u \in L^{p_1}(B_R)$ where $1/p_1 = 1/p - 2/N$, hence by elliptic regularity $u \in W^{2,p_1}(B_R)$. By iterating the procedure until $u \in W^{2,p_k}(B_R)$ with $p_k > N/2$, we conclude as before. The consistency of the resolvents in L^p, L^q follows by density, as well as the positivity of the resolvent in $L^p(B_R)$ for positive λ in the resovent set, since the resolvent of $(L_1, D_{max}(L_1))$ is positive. □

Finally, we show that when $0 < \alpha < 2$ and $p \leq N/(N - \alpha)$ the equation $\lambda u - |x|^\alpha \Delta u = f$ for positive λ has no positive solutions in $D_{p,max}(L_1)$ for certain positive $f \in L^p(B_R)$.

LEMMA 4.16. *Let* $0 < \alpha < 2$, $1 < p \le N/(N-\alpha)$ *and* u *solve the ordinary differential equation*

$$-r^{\alpha}\left(u'' + \frac{N-1}{r}u'\right) = g \tag{7}$$

in $]0,1]$, *with* $g \in L^p((0,1); r^{N-1}\,dr)$. *Then for* $\varepsilon > 0$ *sufficiently small*

$$\lim_{r\to 0} |r|^{N-\alpha-\varepsilon} u(r) = 0.$$

PROOF. We get

$$u'(r) = r^{1-N}\int_r^1 g(s)s^{N-1-\alpha}\,ds + cr^{1-N}$$

and Hölder's inequality with respect to the measure $r^{N-1}\,dr$ (the norms are taken in L^p with respect to the measure $r^{N-1}\,dr$) implies

$$\begin{aligned} |u'(r)| &\le \|g\|_p r^{1-N+N/p'-\alpha} + cr^{1-N} && \text{if } p < N/(N-\alpha)\\ |u'(r)| &\le \|g\|_p r^{1-N}|\log r|^{1/p'} + cr^{1-N} && \text{if } p = N/(N-\alpha) \end{aligned}$$

hence

$$\begin{aligned} |u(r)| &\le Cr^{2-N+N/p'-\alpha} && \text{if } p < N/(N-\alpha)\\ |u'(r)| &\le Cr^{2-N}|\log r|^{1/p'} && \text{if } p = N/(N-\alpha) \end{aligned}$$

near $r = 0$. These estimates easily imply the result, since $\alpha < 2$. □

LEMMA 4.17. *Let* $0 < \alpha < 2$, $1 < p \le N/(N-\alpha)$. *If* $f(r) = r^{\alpha-N}$ *for* $p < N/(N-\alpha)$ *or* $f(r) = r^{\alpha-N}|\log r\chi_{]]0,1/2[}|^{-1}$ *if* $p = N/(N-\alpha)$, *then for* $\lambda \ge 0$ *the ordinary differential equation*

$$\lambda u - r^{\alpha}\left(u'' + \frac{N-1}{r}u'\right) = f$$

has no positive solution $u \in L^p((0,1); r^{N-1}\,dr)$.

PROOF. Assume that $u \in L^p((0,1); r^{N-1}\,dr)$ solves the above equation in $]0,1]$ and let $g = f - \lambda u$. Then u solves (7) and Lemma 4.16 yields $u(r)/f(r) \to 0$ as $r \to 0$, hence $\lambda u(r) \le (1/2)f(r)$ for small r. Since

$$u'(r)r^{N-1} = \int_r^1 (f(s) - \lambda u(s))s^{N-1-\alpha}\,ds + c,$$

it follows that $u'(r)r^{N-1} \ge 1$ for $r \le r_0$ hence $u(r_0) - u(r) \ge \int_r^{r_0} s^{1-N}\,ds$ implies that $u(r) \to -\infty$ as $r \to 0$. □

PROPOSITION 4.18. *If $0 < \alpha < 2$ and $p \le N/(N-\alpha)$ and $f \in L^p(B_1)$ is as in Lemma 4.17, then the equation $\lambda u - |x|^\alpha \Delta u = f$ for positive λ has no positive solutions in $D_{p,max}(L_1)$. In particular (L_1, D) cannot be the generator of a positive semigroup in $L^p(B_1)$ for any $D \subset D_{p,max}(L_1)$.*

PROOF. Assume that for a certain $\lambda \ge 0$ there exists $u \in D_{p,max}(L_1)$, $u \ge 0$, solving $\lambda u - |x|^\alpha \Delta u = f$. Since u is radial, the function

$$v(r) = \int_{S^{N-1}} u(r\omega)\, d\sigma(\omega),$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$ and $d\sigma$ its surface measure, still belongs to $D_{p,max}(L_1)$ and solves $\lambda v - |x|^\alpha \Delta v = f$ or

$$\lambda v - r^\alpha \Big(v'' + \frac{N-1}{r} v' \Big) = f.$$

By Lemma 4.17, v cannot be positive, hence neither u. □

4.2. The domains of L_2 and of L

The maximal domain of L_2 is defined in the usual way.

DEFINITION 4.19.

$$D_{p,max}(L_2) = \{u \in L^p(B_R^c) \cap W^{2,p}(B_R^c \cap B_r)\ \forall r > 0 : u(x) = 0 \text{ if } |x| = R,$$
$$|x|^\alpha \Delta u \in L^p(B_R^c)\}.$$

Observe that the Dirichlet boundary condition $u(x) = 0$ for $|x| = R$ makes sense, since u has second derivatives in L^p in a neighborhood of the boundary of B_R^c. By local elliptic regularity, L_2 is closed on its maximal domain. However, when $\alpha \le 2$ the function $a(x) = |x|^\alpha$ satisfies the inequality $|\nabla a^{1/2}| \le C$ in the exterior domain B_R^c. In analogy with Section 3 we can also define the domain $D_p(L_2)$ as follows.

DEFINITION 4.20. If $\alpha \le 2$ we set

$$D_p(L_2) = \{u \in L^p(B_R^c) \cap W^{2,p}(B_R^c \cap B_r)\ \forall r > 0 : u(x) = 0 \text{ if } |x| = R,$$
$$|x|^{\alpha/2} \nabla u, |x|^\alpha D^2 u \in L^p(B_R^c)\}.$$

The proof of the next proposition is similar to that of Proposition 4.3.

PROPOSITION 4.21. *If $\alpha \le 2$, then $D_{p,max}(L_2) = D_p(L_2)$ and the operator L_2 is closed on its domain.*

Therefore the domain of L_2 is always the maximal one and coincides with $D_p(L_2)$ when $\alpha \le 2$. Next we show the consistency of the resolvents with respect to p and their positivity for $\lambda > 0$.

PROPOSITION 4.22. *Let $0 < \lambda \in \rho(L_2, D_{p,max}(L_2)) \cap \rho(L_2, D_{max}(L_2))$. Then the resolvents of L_2 in $L^p(B_R^c)$ and in $C_0(B_R^c)$ coincide on $L^p(B_R^c) \cap C_0(B_R^c)$. In particular the resolvents in $L^p(B_R^c)$ and $L_q(B_R^c)$ coincide and are positive if λ is positive.*

PROOF. Assume first that $\alpha > 0$ and take $f \in C_c^\infty(B_R^c)$, $u \in D_{p,max}(L_2)$ such that $\lambda u - L_2 u = f$. Then $\Delta u \in L^p(B_R^c)$ and, since u vanishes at the boundary, by elliptic regularity, $u \in W^{2,p}(B_R^c)$. If $p > N/2$, then $u \in C_0(B_R^c)$, hence $u \in D_{max}(L_2)$ and the consistency of the resolvents follows by the density of $C_c^\infty(B_R^c)$ both in $C_0(B_R^c)$ and $L^p(B_R^c)$. If $p \le N/2$ we use Sobolev embedding as in the proof of Proposition 4.15 to conclude the proof.

Let now $\alpha < 0$. If $\rho > R$, $f \in C_c^\infty(B_R^c)$, $\lambda > 0$, we solve the Dirichlet problem $\lambda u - |x|^\alpha \Delta u = f$ in $C_\rho = B_\rho \backslash B_R$ with Dirichlet boundary conditions if $|x| = R$ or $|x| = \rho$. The solution u_ρ belongs to $W^{2,p}(C_\rho) \cap W_0^{1,p}(C_\rho)$ for every $p < \infty$ and satisfies $\lambda \|u_\rho\|_\infty \le \|f\|_\infty$. To get L^p estimates independent of ρ, we multiply the equation by $u|u|^{p-2}$ and integrate by parts. Since the boundary terms vanish we get

$$
\begin{aligned}
\lambda \int_{C_\rho} |u|^p &+ (p-1) \int_{C_\rho} |x|^\alpha |\nabla u|^2 |u|^{p-2} \\
&\le |\alpha| \int_{C_\rho} |x|^{\alpha-1} |\nabla u|\, |u|^{p-1} + \int_{C_\rho} |f|\, |u|^{p-1} \\
&\le C \int_{C_\rho} |x|^{\alpha/2} |\nabla u|\, |u|^{p-1} + \int_{C_\rho} |f|\, |u|^{p-1} \\
&\le C \Big(\int_{C\rho} |x|^\alpha |\nabla u|^2 |u|^{p-2} \Big)^{1/2} \Big(\int_{C_\rho} |u|^p \Big)^{1/2} + \|f\|_p \|u\|_p^{p-1}.
\end{aligned}
$$

From this we easily deduce the existence of $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ the estimate $(\lambda - \lambda_0)\|u_\rho\|_p \le \|f\|_p$ holds. A weak compactness argument based on local $W^{2,p}$ estimates now produces a function u satisfying $\lambda u - |x|^\alpha \Delta u = f$ with $u \in L^p(B_R^c) \cap C_0(B_R^c)$, hence the coherence of the resolvents. The coherence of the resolvents in L^p, L^q, as well as their positivity for $\lambda > 0$, now follows immediately. □

We shall construct a resolvent for L by gluing together the resolvents of L_1 and L_2. Accordingly, the domain of L will be defined in terms of the domains of L_1 and L_2, as in the following construction. We fix a radius $R > 0$ and we consider the operator L_1 in the ball B_{2R} and the operator L_2 in the exterior domain B_R^c, with the domains defined according to this section and depending on α.

DEFINITION 4.23.

$$
\begin{aligned}
D_p(L) = \{u \in L^p(\mathbb{R}^N) \cap W_{loc}^{2,p}(\Omega) : u = u_1 + u_2, u_1 \in D_p(L_1), u_2 \in D_{p,max}(L_2), \\
u_1, u_2 \text{ with compact support contained in } B_{2R}, B_R^c \text{ respectively}\}.
\end{aligned}
$$

Remark 4.24. *It is easily seen that the definition of* $D_p(L)$ *is independent of the choice of the radius R. Moreover,* $u \in D_p(L)$ *if and only if* $u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega)$, $\eta u \in D_p(L_1)$, $(1-\zeta)u \in D_{p,max}(L_2)$ *for fixed cut-off functions* η, ζ *with support in* B_{2R} *and equal to* 1 *near the origin.*

Also for the operator L we can define the maximal domain.

Definition 4.25.

$$D_{p,max}(L) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega) : |x|^\alpha \Delta u \in L^p(\mathbb{R}^N)\}.$$

The equality $D_{p,max}(L) = D_p(L)$ holds if and only if the same equality for L_1 holds, hence

Corollary 4.26. *The equality* $D_p(L) = D_{p,max}(L)$ *holds if* $\alpha \geq 2$ *and when* $\alpha < 2$, $p \geq N/(N-2)$.

5. Gluing the resolvents

We shall construct a resolvent for L by gluing together the resolvents of L_1 and L_2. In some other cases we also deduce results for L_1 or L_2 from properties of L in the whole space. This section is devoted to explain these methods. First let us fix some notations.

For $0 \leq \theta < \pi$, $\rho > 0$, we denote by $\Sigma_{\theta,\rho}$ the closed set

$$\Sigma_{\theta,\rho} = \{\lambda \in \mathbb{C} : |\lambda| \geq \rho, |\mathrm{Arg}\,\lambda| \leq \theta\}.$$

Even though more general operators can be considered, we confine to the case $A = a\Delta$ in $L^p(V)$ where V is an open set containing the annulus $C_R = B_{2R} \backslash B_R$. The function a is assumed to be locally Hölder continuous and strictly positive in any compact set contained in $V \cap \Omega$, hence having possible singularities only at 0, ∞. We shall apply these results to $a(x) = |x|^\alpha$.

Definition 5.1. Let (A, D) be the operator $a\Delta$ with $D \subset D_{p,max}(A)$ on $L^p(V)$, $1 < p < \infty$. We say that (A, D) satisfies $P(\theta, \rho, C, \gamma)$, where C, $\rho > 0$, $\gamma \geq 0$ and $0 \leq \theta < \pi$ if $\Sigma_{\theta,\rho} \subset \rho(A)$ and for every $\lambda \in \Sigma_{\theta,\rho}$ the following estimate holds

$$\|(\lambda - A)^{-1}\| \leq \frac{C}{|\lambda|^\gamma}. \tag{8}$$

Definition 5.2. We say that (A, D) satisfies $P(\theta, \rho, R, C, \gamma, \delta)$, $R > 0$, $\delta \in \mathbb{R}$ if it satisfies $P(\theta, \rho, C, \gamma)$, $a(x) \leq C$ for $x \in C(R)$ and moreover

$$\|(\lambda - A)^{-1}\|_{L^p(V) \to W^{1,p}(C(R))} \leq \frac{C}{|\lambda|^\delta}, \tag{9}$$

where the last norm is understood as the operator norm from $L^p(V)$ to $W^{1,p}(C(R))$.

Clearly, A generates an analytic semigroup if and only if $P(\theta, \rho, C, \gamma)$ holds for some $\theta > \pi/2$, $\gamma = 1$. Assume that (A_1, D_1), (A_2, D_2), where $A_1 = a_1(x)\Delta$, $A_2 = a_2(x)\Delta$, are given in $L^p(B_{2R})$, $L^p(B_R^c)$ respectively. We also assume that if $u_i \in D_i$ and η_i are C^∞ function with compact support in B_{2R}, B_R^c, respectively, such that $\eta_1 = 0$ in a neighborhood of the origin and $\eta_2 = 1$ in a neighborhood of infinity, then $\eta_i u_j \eta_i \in D_i$, for $i, j = 1, 2$. These conditions are clearly verified for L_1, L_2, by the definition of the corresponding domains given in Section 4. If $a(x) = a_2(x)$ in C_R we set $A = a(x)\Delta$ where $a(x) = a_1(x)$ for $|x| \le R$ and $a(x) = a_2(x)$ for $|x| \ge R$. The domain of A, say D, is defined as follows.

Definition 5.3.

$$
\begin{aligned}
D = \{u \in L^p(\mathbb{R}^N) : u = u_1 + u_2, u_1 \in D_1, u_2 \in D_2,\\
u_1, u_2 \text{ with compact support contained in } B_{2R}, B_R^c \text{ respectively}\}.
\end{aligned}
$$

Remark 5.4. *Observe that if $u \in D$, then $\eta_i u \in D_i$. In fact, writing $u = u_1 + u_2$ with u_i as in Definition 5.3, then $\eta_i u_j \in D_i$.*

Proposition 5.5. *Under the above assumptions, suppose that (A_1, D_1), (A_2, D_2) satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in $L^p(B_{2R})$ and in $L_p(B_R^c)$ respectively, with $\delta > 0$. If there exists $\lambda_0 > 0$ such that $(\lambda - A, D)$ is injective for $\lambda > \lambda_0$, then (A, D) satisfies $P(\theta, \rho_1, C_1, \gamma)$ in $L^p(\mathbb{R}^N)$, where ρ_1, C_1 depend only on p, θ, ρ, R, C, γ. If $\gamma > \frac{1}{2}$ then (A, D) satisfies $P(\theta, \rho_1, C_1, \gamma)$ in $L^p(\mathbb{R}^N)$ without the extra injectivity assumption on L. Finally, if $P(\theta, \rho_1, R, C, \gamma, \delta)$ are satisfied both in L^p, L^q and the resolvents of A_1, A_2 are coherent in L^p, L^q, then the resolvents of A are coherent in L^p, L^q.*

Proof. Let $0 \le \eta_1, \eta_2 \le 1$ be positive C^∞-functions supported in B_{2R} and $\mathbb{R}^N \backslash B_R$, respectively, such that $\eta_1^2 + \eta_2^2 = 1$. For $\lambda \in \Sigma_{\theta,\rho}$ $f \in L^p(\mathbb{R}^N)$, set $R_i(\lambda)f = \eta_i(\lambda - A_i)^{-1}(\eta_i f) \subset D_i \cap D$ for $i = 1, 2$. Observing that $A\eta_i = A_i\eta_i$, $\eta_i A = \eta_i A_i$ it follows that

$$
\begin{aligned}
(\lambda - A)R_i(\lambda)f &= (\lambda - A)\eta_i(\lambda - A_i)^{-1}(\eta_i f)\\
&= \eta_i(\lambda - A_i)(\lambda - A_i)^{-1}(\eta_i f) + [\eta_i, A](\lambda - A_i)^{-1}(\eta_i f)\\
&= \eta_i^2 f + [\eta_i, A](\lambda - A_i)^{-1}(\eta_i f)
\end{aligned}
$$

where

$$
[\eta_i, A_i]g = \eta_i a\Delta g - a\Delta(\eta_i g) = -2a\nabla\eta_i\nabla g - a(\Delta\eta_i)g
$$

is a first order operator supported on C_R. Therefore $(\lambda - A)R_i(\lambda)f = \eta_i^2 f + S_i(\lambda)f$ where $S_i(\lambda)f = -2a\nabla\eta_i\nabla(\lambda - L_i)^{-1}(\eta_i f) - a(\Delta\eta_i)(\lambda - L_i)^{-1}(\eta_i f)$. By (9), it follows that

$$
\|S_i(\lambda)\|_{L^p(\mathbb{R}^N)} \le \frac{c_1}{|\lambda|^\delta}
$$

for $\lambda \in \Sigma_{\theta,\rho}$ and with c_1 depending only on C, R. Then $(\lambda - A)R(\lambda)f = f + S(\lambda)f$ where

$$R(\lambda) = \sum_{i=1}^{2} R_i(\lambda), \quad S(\lambda) = \sum_{i=1}^{2} S_i(\lambda).$$

Choosing $|\lambda| > \rho_1$ large enough, we find $\|S(\lambda)\|_{L^p(\mathbb{R}^N)} \leq \frac{1}{2}$ and we deduce that the operator $I + S(\lambda)$ is invertible in $L^p(\mathbb{R}^N)$. Setting $V(\lambda) = (I + S(\lambda))^{-1}$ we have

$$(\lambda - A)R(\lambda))V(\lambda)f = f$$

and hence the operator $R(\lambda)V(\lambda)$, which maps $L^p(\mathbb{R}^N)$ into D, is a right inverse of $\lambda - A$ and, by (8), satisfies

$$\|R(\lambda)V(\lambda)\| \leq \frac{2C}{|\lambda|^\gamma} \tag{10}$$

for $\lambda \in \Sigma_{\theta,\rho_1}$. Clearly $R(\lambda)V(\lambda)$ coincides with $(\lambda - A)^{-1}$ whenever this last is injective. If $\lambda - A$ is injective for $\lambda > \lambda_0$, then $]\lambda_0, \infty[\subset \rho(A)$ and the a-priori estimates (10) show that the norm of the resolvent cannot explode in the set $\Sigma_{\theta,\bar{\rho}}$, hence this set is contained in $\rho(A)$ where the resolvent operator coincide with $R(\lambda)V(\lambda)$ and satisfies (10).

If $\gamma > \frac{1}{2}$ we have to prove the injectivity of $\lambda - A$ for $|\lambda|$ large enough. Let $u \in D$, $\lambda \in \Sigma_{\theta,\rho}$. Then $\eta_i u \in D_i$, see Remark 5.4, $\eta_i Lu = \eta_i L_i u$ and

$$\begin{aligned} R(\lambda)(\lambda - L)u &= \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}\eta_i(\lambda - A)u = \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}\eta_i(\lambda - A_i)u \\ &= \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}(\lambda - A_i)\eta_i u + \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}[A, \eta_i]u. \end{aligned}$$

Suppose that $(\lambda - A)u = 0$. Then $u = -\sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u$. It follows that

$$\|Au\|_p \leq \sum_{i=1}^{2} \|A\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p.$$

Since $\nabla\eta_i$, $\Delta\eta_i$ have support contained in C_R and by the definition of $[A_i, \eta_i]u$, we have

$$\begin{aligned} \|A\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p &\leq \|(\lambda - A_i)\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p \\ &\quad + \|\lambda\eta_i(\lambda - A_i)^{-1}[A, \eta_i]u\|_p \leq \|\eta_i(\lambda - A_i)(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p \\ &\quad + \|[A_i, \eta_i](\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p + \|\lambda\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p \\ &\leq \|\eta_i[A_i, \eta_i]u\|_p + \|[A_i, \eta_i](\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p + \|\lambda\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p \\ &\leq C\left[\|u\|_{W^{1,p}(C(R))} + \frac{1}{|\lambda|^\delta}\|u\|_{W^{1,p}(C(R))} + |\lambda|^{1-\gamma}\|u\|_{W^{1,p}(C(R))}\right]. \end{aligned}$$

By the interpolative estimates [6, Theorem 7.28] there exists $C > 0$ such that for every $\varepsilon > 0$

$$\|u\|_{W^{1,p}(C_R)} \le \varepsilon \|u\|_{W^{2,p}(C_R)} + \frac{C}{\varepsilon}\|u\|_{L_p(C_R)}.$$

Using the interior estimates for elliptic operators (note that a is positive far from the origin) as in [6, Theorem 9.11] we deduce the existence of a constant $C > 0$ such that for every $\varepsilon > 0$

$$\|u\|_{W^{1,p}(C(R))} \le C\left[\varepsilon\|Au\|_{L^p(B_{2R+1}\backslash B_{\frac{R}{2}})} + \frac{1}{\varepsilon}\|u\|_{L^p(B_{2R+1}\backslash B_{\frac{R}{2}})}\right].$$

It follows that for every $\varepsilon, \varepsilon_1 > 0$ and some C independent of ε, ε_1

$$\|Au\|_p \le C\left[\varepsilon\|Au\|_p + \frac{1}{\varepsilon}\|u\|_p + \varepsilon_1|\lambda|^{1-\gamma}\|Au\|_p + \frac{1}{\varepsilon_1}|\lambda|^{1-\gamma}\|u\|_p\right],$$

where all norms are taken over $\mathbb{R}^N$. By choosing $\varepsilon_1 = \varepsilon|\lambda|^{\gamma-1}$, it follows that

$$\|Au\|_p \le C\left[\varepsilon\|Au\|_p + \frac{1}{\varepsilon}\|u\|_p + \frac{1}{\varepsilon}|\lambda|^{2-2\gamma}\|u\|_p\right].$$

By choosing ε small enough, $\|Au\|_p \le C|\lambda|^{2-2\gamma}\|u\|_p$. Since $Au = \lambda u$ and $\gamma > \frac{1}{2}$, $u = 0$ for $|\lambda| > \bar{\rho}$, $\bar{\rho}$ large enough, and $\lambda - A$ is injective. Finally, if the hypotheses hold in L^p, L^q and the resolvents of A_1, A_2 are coherent in L^p, L^q (in B_{2R}, B_R^c, respectively, we have seen in the proof that the resolvent of A is the operator $R(\lambda)V(\lambda)$ which is coherent in $L^p(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ by construction. □

The above proof can be adapted to deduce results both in exterior and interior domains from the whole space. We consider (A, D) in $L^p(\mathbb{R}^N)$, where $A = a\Delta$ with $D \subset D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega) : Au \in L^p(\mathbb{R}^N)\}$. Next we introduce $A_1 = a\Delta$ in B_{2R} with Dirichlet boundary conditions if $|x| = 2R$. More precisely we define its domain D_1 as

$$D_1 = \{u \in L^p(B_{2R} \cap W^{2,p}(B_{2R}\backslash B_\varepsilon)\ \forall\varepsilon > 0 : u_{|\partial B_{2R}} = 0, \quad \eta u \in D\ \forall\eta \in C_c^\infty(B_{2R}), \eta \equiv 1 \text{ near } 0\}. \tag{11}$$

Similarly, we consider A_2 in B_R^c, where $A_2 = a\Delta$ and

$$D_2 = \{u \in L^p(B_R^c \cap W^{2,p}(B_\rho\backslash B_R)\ \forall\rho > R : u_{|\partial B_R = 0}, \quad \eta u \in D\ \forall\eta \in C^\infty(B_R^c), \eta \equiv 1 \text{ near } \infty, \eta \equiv 0 \text{ near } \partial B_R\}. \tag{12}$$

Proposition 5.6. *Let $(A, D) = (a\Delta, D)$ satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in $L^p(\mathbb{R}^N)$. Let (A_2, D_2) in B_{R^c} as defined in* (12). *If there exists $\lambda_0 > 0$ such that $\lambda - A_2$ is injective for $\lambda > \lambda_0$, then A_2 satisfies $P(\theta, \rho_2, C_2, \gamma)$ in L^p, where ρ_2, C_2 depend only on p, θ, ρ, R, C, γ. If $\gamma > \frac{1}{2}$ then A_2 satisfies $P(\theta, \rho_1, C_1, \gamma)$ in L^p without the extra injectivity* *assumption on A_2. Finally, if $P(\theta,\rho_1,R,C,\gamma,\delta)$ is satisfied both in L^p, L^q and the resolvent of A are coherent in L^p, L^q, then the resolvents of A_2 are coherent in L^p, L^q.*

Proof. The proof is similar to that of Proposition 5.5 and we only outline the main steps. Let $0 \le \eta_1,\eta_2 \le 1$ be positive C^∞-functions supported in B_{2R} and B_R^c, respectively, such that $\eta_1^2+\eta_2^2=1$. Let A_R be the operator $a\Delta$ in the annulus C_R with Dirichlet boundary conditions, that is with domain

$$D_p(A_R)=\{u \in W^{2,p}(C_R)\cap W_0^{1,p}(C_R)\}.$$

Since $a>0$ in C_R, A_R is uniformly elliptic and generates an analytic semigroup in C_R, see [9]. In particular, A_R satisfies $P(\theta,\rho,R,C,\gamma,\delta)$.

For $\lambda \in \Sigma_{\theta,\rho}$ $f \in L^p(B_R^c)$ (extended to zero outside B_R^c) we set

$$R(\lambda)f=\eta_1(\lambda-A_R)^{-1}(\eta_1 f)+\eta_2(\lambda-A)^{-1}(\eta_2 f)\in D_2.$$

Then we argue as in Proposition 5.5. □

Proposition 5.7. *Let $(A,D)=(a\Delta,D)$ satisfy $P(\theta,\rho,R,C,\gamma,\delta)$ in $L^p(\mathbb{R}^N)$. Let (A_1,D_1) in B_{2R} as defined in (11). If there exists $\lambda_0>0$ such that $\lambda-A_1$ is injective for $\lambda>\lambda_0$, then A_1 satisfies $P(\theta,\rho_1,C_1,\gamma)$ in L^p, where ρ_1, C_1 depend only on p, θ, ρ, R, C, γ. If $\gamma>\frac{1}{2}$ then A_1 satisfies $P(\theta,\rho_1,C_1,\gamma)$ in L^p without the extra injectivity assumption on L_1. Finally, if $P(\theta,\rho_1,R,C,\gamma,\delta)$ is satisfied both in L^p, L^q and the resolvents of A are coherent in L^p, L^q, then the resolvents of A_1 are coherent in L^p, L^q.*

Proof. Keeping the notation of the proof of Proposition 5.6, for $f \in L_p(B_{2R})$ (extended to zero outside B_R) we set

$$R(\lambda)f=\eta_1(\lambda-A)^{-1}(\eta_1 f)+\eta_2(\lambda-A_R)^{-1}(\eta_2 f)\in D_1$$

and we argue as in Proposition 5.5. □

6. Generation results for L_2

We indicate by $(T_2(t))_{t\ge 0}$ the semigroup generated by $(L_2, D_{max}(L_2))$ in $C_0(B_R^c)$, see Section 2.

Proposition 6.1. *Let $\alpha \le 2$. Then the operator $(L_2, D_p(L_2))$ generates coherent positive analytic semigroups $(T_{2,p}(t))_{t\ge 0}$ in $L^p(B_R^c)$ for $1<p<\infty$. If $f \in C_0(B_R^c) \cap L^p(B_R^c)$, then $T_{2,p}(t)f=T_2(t)f$.*

Proof. We extend the operator to the whole $\mathbb{R}^N$ by setting

$$\tilde a(x)=\begin{cases} R^\alpha & |x|\le R\\ |x|^\alpha & |x|\ge R\end{cases}$$

and $\tilde{L} = \tilde{a}\Delta$. Since $\alpha \le 2$, the operator $(\tilde{L}, D_p(\tilde{L}))$ belongs to the class studied in Subsection 3 and then generates coherent analytic semigroups in $L^p(\mathbb{R}^N)$, by Theorem 3.3. The equality $D_{p,max}(\tilde{L}) = D_p(\tilde{L})$ follows from Proposition 3.5. Finally, by Proposition 5.6, $(L_2, D_p(L_2))$ generates an analytic semigroup in $L^p(B_R^c)$ for $1 < p < \infty$. Coherence and positivity of the semigroups follows from consistency and positivity of the resolvents proved in Proposition 4.22. □

Next we consider the case $\alpha > 2$.

Proposition 6.2. *Let $\alpha > 2$. Then $(L_2, D_{p,max}(L_2))$ generates coherent positive analytic semigroup $(T_{2,p}(t))_{t \ge 0}$ in $L^p(B_R^c)$ for $N/(N-2) < p < \infty$. If $f \in C_0(B_R^c) \cap L^p(B_R^c)$, then $T_{2,p}(t)f = T_2(t)f$. Finally, if $p > N/(N-\alpha)$,*

$$D_{p,max}(L_2) = \{u \in W^{2,p}(B_R^c) : (1+|x|^{\alpha-2})u, (1+|x|^{\alpha-1})\nabla u, \\ (1+|x|^{\alpha})D^2 u \in L^p(B_R^c)\}.$$

Proof. As before we extend the operator to the whole $\mathbb{R}^N$ by setting $\tilde{L} = \tilde{a}\Delta$. By [14, Theorem 5.5], $(\tilde{L}, D_{p,max}(\tilde{L}))$ generates coherent analytic semigroup in $L^p(\mathbb{R}^N)$ for $N/(N-2) < p < \infty$ and the domain characterization follows from [14, Theorem 9.8]. Finally, by Proposition 5.6, $(L_2, D_{p,max}(L_2))$ generates an analytic semigroup in $L^p(B_R^c)$ for $1 < p < \infty$. Also in this case, coherence and positivity of the semigroups are consequence of Proposition 4.22. □

7. Generation results for L_1

We indicate by $(T_1(t))_{t \ge 0}$ the semigroup generated by $(L_1, D_{max}(L_1))$ in $C_0(\Omega_R)$, see Section 2. When $\alpha \ge 2$ the operator L_1 belongs to the class studied in Subsection 3.

Proposition 7.1. *Let $\alpha \ge 2$ and $1 < p < \infty$. Then the operator $(L_1, D_p(L_1))$ generates coherent positive analytic semigroups $(T_{1,p}(t))_{t \ge 0}$ in $L^p(B_R)$. If $f \in C_0(\Omega_R)$ then $T_{1,p}(t)f = T_1(t)f$.*

Proof. We extend the operator to the whole $\mathbb{R}^N$ by setting

$$\tilde{a}(x) = \begin{cases} |x|^{\alpha} & |x| \le R \\ R^{\alpha} & |x| \ge R \end{cases}$$

and $\tilde{L} = \tilde{a}\Delta$. Since $\alpha \ge 2$ the operator $(\tilde{L}, D_p(\tilde{L}))$ belongs to the class studied in Subsection 3 and then generates coherent analytic semigroups in $L^p(\mathbb{R}^N)$, by Theorem 3.3. The equality $D_{p,max}(\tilde{L}) = D_p(\tilde{L})$ follows from Proposition 3.5. By Proposition 5.7, $(L_1, D_p(L_1))$ generates an analytic semigroup in $L^p(B_R)$ for $1 < p < \infty$. Coherence and positivity of the semigroups follow from the consistency and positivity of the resolvents proved in Proposition 4.15. □

The case $\alpha < 2$ is more involved and we proceed in several steps.

Proposition 7.2. *Let* $\alpha < 0$, $1 < p < \infty$ *or* $0 < \alpha < 2$ *and* $p > \frac{N}{N-2}$. *If* $\operatorname{Re}\lambda \geq 0$, *then the operator* $\lambda - L_1$ *is invertible on* $D_p(L_1)$ *and*

$$\|(\lambda - L_1)^{-1}\|_p \leq \|(-L_1)^{-1}\|_p.$$

Moreover $(\lambda - L_1)^{-1} \geq 0$ *for* $\lambda \geq 0$.

Proof. First consider positive λ. Let ρ be the resolvent set of $(L_1, D_p(L_1))$ and observe that Propositions 4.5, 4.7 and 4.15 show that $0 \in \rho$ and if $0 \leq \lambda \in \rho$, then $(\lambda - L_1)^{-1} \geq 0$. By the resolvent equation $(\lambda - L)^{-1} \leq (-L)^{-1}$ and therefore $\|(\lambda - L_1)^{-1}\|_p \leq \|(-L_1)^{-1}\|_p$. Let $E = [0, \infty[\cap \rho$. Then E is non empty and open in $[0, \infty[$, since ρ is open, and closed since the operator norm of $(\lambda - L_1)^{-1}$ is bounded in E. Then $E = [0, \infty[$. If $\operatorname{Re}\lambda > 0$, $f \in C_0(\Omega_R)$,

$$|(\lambda - L)^{-1} f| \leq \int_0^\infty e^{-\operatorname{Re}\lambda t} T(t)|f|\, dt = (\operatorname{Re}\lambda - L)^{-1}|f|.$$

By the coherence of the resolvents (Proposition 4.15), we deduce that

$$\|(\lambda - L_1)^{-1}\|_p \leq \|(\operatorname{Re}\lambda - L_1)^{-1}\|_p \leq \|(-L_1)^{-1}\|_p$$

whenever $\lambda \in \rho$. Repeating the argument used in $[0, \infty[$ one concludes the proof. □

The next step consists in proving that, for large p, the operator L_1 generates an analytic semigroup in $L^p(B_R)$. We apply the Kelvin transform in order to deduce results for L_1 from those of L_2.

Proposition 7.3. *Let* $\alpha < 2$ *and* $\frac{2N}{N-2} \leq p < \infty$. *Then the operator*

$$(L_1, D_p(L_1)) = (L_1, D_{p,max}(L_1))$$

generates coherent positive analytic semigroups $(T_{1,p}(t))_{t \geq 0}$ *in* $L^p(B_R)$. *If* $f \in C_0(\Omega_R)$ *then* $T_{1,p}(t)f = T_1(t)f$.

Proof. We may assume that $R = 1$ and write B for B_R. The equality $D_p(L_1) = D_{p,max}(L_1)$ follows from Corollary 4.11 since $p \geq 2N/(N-2)$. To indicate the dependence of L_1 on α we write L_1^α. Similarly for L_2.

First we consider the case $p = p_0 := \frac{2N}{N-2}$. For $u \in L^{p_0}(B)$ we define its Kelvin transform Tu on B^c by

$$(Tu)(x) = |x|^{2-N} u\Big(\frac{x}{|x|^2}\Big).$$

If $y = x/|x|^2$, then $dy = |x|^{-2N}$, $T : L^{p_0}(B) \to L^{p_0}(B^c)$ is an invertible isometry and its inverse T^{-1} has the same expression. Setting $v = Tu$ it follows that (see [4, Theorem 2.70])

$$\Delta u(x) = |x|^{-N-2}\Delta v\Big(\frac{x}{|x|^2}\Big). \tag{13}$$

It follows that, if $v \in L^{p_0}(B^c)$,

$$(TL_1^\alpha T^{-1}v) = |x|^{4-\alpha}\Delta v = L_2^{4-\alpha}v.$$

This identity implies that $T^{-1}D_{p_0,max}(L_2^{4-\alpha}) = D_{p_0,max}(L_1^\alpha)$ and therefore

$$TL_1^\alpha T^{-1} = L_2^{4-\alpha}$$

as operators. By Proposition 6.2, the operator $(L_2^{4-\alpha}, D_{p_0,max}(L_2^{4-\alpha}))$ generates a positive analytic semigroup in $L^{p_0}(B^c)$ and therefore $(L_1, D_{p_0,max}(L_1^\alpha))$ generates a positive analytic semigroup $(T_{1,p_0}(t))_{t\geq 0}$ in $L^{p_0}(B)$.

Since $T_{1,p_0}(t)f = T_1(t)f$ for $f \in C_0(\Omega_R)$ by Proposition 4.15 (and then for $f \in L^\infty(\Omega_R)$ by the integral representation through a kernel (see [12, Theorem 4.4])), by Stein's interpolation Theorem [9, Chapter 5] it follows that $(T_1(t))_{t\geq 0}$ extends to an analytic semigroup $(T_{1,p}(t))_{t\geq 0}$ in $L^p(B_R)$ for every $\frac{2N}{N-2} \leq p < \infty$. Let (A, D_p) its generator in $L^p(B_R)$. From the description of the domain in continuous function space (see [12, Section 5, pag. 184]), it follows that $D_{max}(L_1) \subset D_p$ and $Au = L_1u$ for every $u \in D_{max}(L_1)$. Since $C_0(\Omega_R)$ is dense in $L^p(B_R)$ then $D_{max}(L_1) = (\lambda - L_1)^{-1}(C_0(\Omega_R))$ is dense in D_p with respect to the graph norm and hence $D_p \subset D_{p,max}(L_1)$, since $D_{max}(L_1) \subset D_{p,max}(L_1)$ and this last is closed. The equality $D_p = D_{p,max}(L_1)$ now follows since L_1 is injective on $D_{p,max}(L_1)$. □

Theorem 7.4. *Let $\alpha < 2$ and $\frac{N}{N-\alpha} < p < \infty$. Then the operator $(L_1, D_p(L_1))$ generates coherent positive analytic semigroups $(T_{1,p}(t))_{t\geq 0}$ in $L^p(B_R)$. If $f \in C_0(\Omega_R)$ then $T_{1,p}(t)f = T_1(t)f$.*

Proof. By the provious result we may assume that $p \leq 2N/(N-2)$.

Let $N/(N-\alpha) < q < \frac{2N}{N-2} = p_0 < \infty$ and let $\operatorname{Re}\lambda > 0$. By Proposition 7.3, $(L_1, D_{p_0}(L_1))$ satisfies $P(\theta_1, \rho, C, 1)$ (see (5.1)) in $L^{p_0}(B_R)$ for some $\theta_1 > \pi/2$, ρ, $C_1 > 0$. Since all resolvents are consistent, see Proposition 4.15, we use the Riesz Thorin Theorem to interpolate between the resolvent estimates given by Proposition 7.3, 7.2 and deduce that for every $0 \leq \tau < \frac{1}{2}$ and

$$\frac{1}{p_1} = \frac{1-\tau}{p_0} + \frac{\tau}{q},$$

there are constants $\rho_1 = \rho \vee 1$, $C_1 > 0$ such that for every $\lambda \in \mathbb{C}$ with $|\lambda| > \rho_1$ and $\operatorname{Re}\lambda > 0$

$$\|(\lambda - L_1)^{-1}\|_{p_1} \leq C_1|\lambda|^{\tau-1}.$$

From $\lambda u - L_1 u = f$ and the bound on u we deduce that $\| |x|^\alpha \Delta u\|_{p_1} \le C(1+|\lambda|^\tau)\|f\|_{p_1} \le 2C|\lambda|^\tau \|f\|_{p_1}$ if $|\lambda| > \rho_1$. By the estimates ([6, Theorem 9.13]) and the interpolative estimates [6, Theorem 7.28], the gradient estimate in the annulus $C = B_R \backslash B_{\frac{R}{2}}$

$$\|\nabla(\lambda - L_1)^{-1}\|_{L^{p_1}(C)} \le C_1 |\lambda|^{\tau - \frac{1}{2}} \tag{14}$$

follows as in the proof of Proposition 5.5. Now we use a scaling argument to prove that the resolvent set of L_1 contains a sector of angle $\theta > \pi/2$ and that the analyticity estimate holds. Since scaling is allowed in the whole space, first we use the results of Section 5 to show property $P(\theta, \rho, C, 1-\tau)$, see (5.2), for the operator L with the same p_1 as above.

By Proposition 6.1, the operator $(L_2, D_{p_1,max}(L_2))$ generates an analytic semigroup in $L^{p_1}(B^c_{\frac{R}{2}})$ and therefore it satisfies

$$P\Big(\theta_2, \rho_2, C, \frac{R}{2}, \frac{1}{2}, 1\Big)$$

for some $\theta_2 > \frac{\pi}{2}$, ρ_2, $C > 0$. By Proposition 5.5, since $\tau < 1/2$, $(L, D_{p_1}(L))$ satisfies $P(\theta, \rho, C, 1-\tau)$ for some $\theta > \frac{\pi}{2}$, ρ, $C > 0$. In particular the resolvent tends to zero when $|\mathrm{Im}\,\lambda| \to \infty$ and the resolvent set ρ intersects the left half-plane. For $s > 0$ let $I_s : L^{p_1} \to L^{p_1}$ defined by $I_s u(x) = u(sx)$. Clearly I_s is invertible with inverse $I_{s^{-1}}$ and $\|I_s u\|_{p_1} = s^{-N/p}\|u\|_{p_1}$. Since $L = s^{2-\alpha} I_s L I_s^{-1}$, $I_s D_{p_1}(L) = D_{p_1}(L)$, then the resolvent set is a cone and contains a closed sector of angle $\theta > \pi/2$, since it intersects the left half-plane. If $\lambda \in \mathbb{C}$, $\lambda = r\omega$ with $|\omega| = 1$, $|\mathrm{Arg}\,\omega| \le \theta$, then the equality

$$\lambda - L = I_s r\Big(\omega - \frac{s^{2-\alpha}}{r}\Big) I_s^{-1}$$

yields the decay

$$\|(\lambda - L)^{-1}\|_{p_1} \le \frac{C}{|\lambda|}\|(\omega - L)^{-1}\|_{p_1} \le \frac{C}{|\lambda|},$$

provided that $s = r^{\frac{1}{2-\alpha}}$. As before we deduce the gradient estimate (14) with $\tau = 0$ with L instead of L_1 and, by Proposition 5.7, we deduce that $(L_1, D_{p_1}(L_1))$ generates an analytic semigroup in $L^{p_1}(B_R)$.

The above procedure does not allow to reach any $p > N/(N-\alpha)$ in one step, since $\tau < 1/2$. However, it can be iterated starting from p_1 instead of p_0. For a fixed $N/(N-\alpha) < p < 2N/(N-2)$ we fix $N/(N-\alpha) < q < p$ and set $p_0 = 2N/(N-2)$ and

$$\frac{1}{p_{n+1}} = \frac{(1-\tau)}{p_n} + \frac{\tau}{q}.$$

We apply repeatedly the above computations obtaining sequences $\theta_n > \pi/2$, ρ_n, $C_n > 0$ such that $(L_1, D_{p_n}(L_1))$ satisfies $P(\theta_n, \rho_n, C_n, 1)$. Since p_n converges to q we can find m such that $p_m < p$ and then $(L_1, D_{p_m}(L_1))$ is sectorial in $L^{p_m}(B_R)$. Since $(L_1, D_{p_0}(L_1))$ is also sectorial in $L^{p_0}(B_R)$ and all resolvents are coherent, by interpolation $(L_1, D_p(L_1))$ is sectorial in $L^p(B_R)$. □

8. Generation results for L

We denote by $(T(t))_{t\geq 0}$ the semigroup generated by $(L, D_{max}(L))$ in $C_0(\Omega)$.

Theorem 8.1. *Let $\alpha = 2$ and $1 < p < \infty$, $\alpha < 2$ and $\frac{N}{N-\alpha} < p < \infty$ or $\alpha > 2$ and $\frac{N}{N-2} < p < \infty$. Then the operator $(L, D_p(L))$ generate coherent positive analytic semigroups $(T_p(t))_{t\geq 0}$ in $L^p(\mathbb{R}^N)$. If $f \in L^p(\mathbb{R}^N) \cap C_0(\Omega)$ then $T_p(t)f = T(t)f$.*

Proof. The case $\alpha = 2$ has been already treated in Corollary 3.6. In the other cases, analyticity and consistency follow by Proposition 5.5 after observing that, since L_1 and L_2 generate analytic and consistent semigroups in $L^p(B_{2R})$ and $L^p(B_R^c)$ by Propositions 6.1, 6.2, 7.1 and 7.4, respectively, then they satisfy $P\big(\theta, \rho, C, R, 1, \frac{1}{2}\big)$ for suitable $\theta > \pi/2$, ρ, C.

The positivity of $T_p(t)$ follows from the equality $T_p(t)f = T(t)f$ for $f \in L^p(\mathbb{R}^N) \cap C_0(\Omega)$ or, equivalently, from the fact that the resolvents of $(L, D_p(L))$ and $(L, D_{max}(L))$ coincide for positive λ on $L^p(\mathbb{R}^N) \cap C_0(\Omega)$. To show this we notice that by Propositions 4.15 and 4.22 the resolvents of L_1 and L_2 in L^p and in C_0 are coherent and that the resolvent of $(L, D_p(L))$ is constructed by gluing together the resolvents of L_1, L_2 as in Proposition 5.5. Therefore it is sufficient to show that also the resolvent of $(L, D_{max}(L))$ in $C_0(\Omega)$ can be obtained by the resolvents of L_1, L_2 in $C_0(\Omega_{2R})$, $C_0(B_R^c)$ with the same procedure which is recalled below.

Let $\lambda > 0$ $0 \leq \eta_1, \eta_2 \leq 1$ positive C^∞-functions as in Proposition 5.5 and set $R_i(\lambda)f = \eta_i(\lambda - L_i)^{-1}(\eta_i f)$ for $i = 1, 2$. It follows that $(\lambda - L)R_i(\lambda)f = \eta_i^2 f + S_i(\lambda)f$ where $S_i(\lambda) = -2a\nabla\eta_i\nabla(\lambda - L_i)^{-1}\eta_i - a(\Delta\eta_i)(\lambda - L_i)^{-1}\eta_i$ is a first order operator supported in a compact set K of the the annulus C_R. Fix $p > N$. Fix $s > 0$ such that $K_1 = \{x : \text{dist}(x, K) \leq s\}$ is a compact subset of C_R. Combining the Morrey estimates

$$|\nabla u(x)| \leq C(r^{-N/p}\|\nabla u\|_{L^p(K_1)} + r^{1-N/p}\|D^2 u\|_{L^p(K_1)}$$

for $x \in K$ and the interpolative estimates for small ε

$$\|\nabla u\|_{L^p(K_1)} \leq \varepsilon\|D^2 u\|_{L^p(K_1)} + \frac{C}{\varepsilon}\|u\|_{L_p(K_1)},$$

we deduce that $\|\nabla u\|_{L^\infty(K)} \leq \varepsilon\|D^2 u\|_{L^p(K_1)} + \frac{C}{\varepsilon^\tau}\|u\|_{L_p(K_1)}$ for small ε and with $\tau = (N + p)/(p - N)$. Using the interior estimates for elliptic operators (see

[6, Theorem 9.11]) we deduce the existence of a constant $C > 0$ such that $\|D^2u\|_{L^p(K_1)} \le C[\|L_iu\|_{L^\infty(C_R)} + \|u\|_{L^\infty(C_R)}]$ and therefore for small ε

$$\|\nabla u\|_{L^\infty(K)} \le \varepsilon\|L_iu\|_{L^\infty(C_R)} + \frac{C}{\varepsilon^\tau}\|u\|_{L_\infty(C_R)}.$$

Applying this last inequality to $S_i(\lambda)f$ and taking into account that $\lambda\|(\lambda - L_i)^{-1}f\|_\infty \le \|f\|_\infty$ it follows that $\|S_i(\lambda)f\|_\infty \le C\lambda^{-1/(1+\tau)}$. We have $(\lambda - L)R(\lambda)f = f + S(\lambda)f$ where

$$R(\lambda) = \sum_{i=1}^{2} R_i(\lambda), \quad S(\lambda) = \sum_{i=1}^{2} S_i(\lambda).$$

Choosing λ large enough, we find $\|S(\lambda)\|_{L^\infty(\mathbb{R}^N)} \le \frac{1}{2}$ and we deduce that the operator $I + S(\lambda)$ is invertible in $C_0(\Omega)$. Setting $V(\lambda) = (I + S(\lambda))^{-1}$ we have $(\lambda - A)R(\lambda)V(\lambda)f = f$ and hence the operator $R(\lambda)V(\lambda)$ is a right inverse of $\lambda - L$. Clearly $R(\lambda)V(\lambda)$ coincides with $(\lambda - L)^{-1}$ since this last is injective by Proposition 2.3.

Since $R(\lambda)V(\lambda)$ is also the resolvent of $(L, D_p(L))$, see Proposition 5.5, the consistency of the resolvents is proved and the proof is complete. □

Standard perturbation arguments as in [14, Theorem 9.8] allow us to show that the operator $m(x)a(x)\Delta$, where a is as above, generates an analytic semigroup in $L^p(\mathbb{R}^N)$ if $a\Delta$ does it, whenever $m \in C(\mathbb{R}^N)$, $m(x) > 0$ for every $x \in \mathbb{R}^N$ and $\lim_{|x|\to\infty} m(x) = l > 0$.

9. Dissipativity

Theorem 9.1. *Let $N \ge 3$, $2 - N \le \alpha \le (p-1)(N-2)$. Then $(L, D_p(L))$ is a dissipative operator.*

Proof. For $f \in C_c^\infty(\mathbb{R}^N)$, $\rho > 0$, $\lambda > 0$, we consider the Dirichlet problem in $L^p(B_\rho \backslash B_{\frac{1}{\rho}})$

$$\begin{cases} \lambda u - Lu = f & \text{in } B_\rho \backslash B_{\frac{1}{\rho}}, \\ u = 0 & \text{on } \partial(B_\rho \backslash B_{\frac{1}{\rho}}). \end{cases} \tag{15}$$

According to Theorem 9.15 in [6], for $\lambda > 0$ there exists a unique solution u_ρ in $W^{2,p}(B_\rho \backslash B_{\frac{1}{\rho}}) \cap W_0^{1,p}(B_\rho \backslash B_{\frac{1}{\rho}})$. We set $u^\star = u_\rho|u_\rho|^{p-2}$, multiply Lu_ρ by $u^\star$ and integrate over $B_\rho \backslash B_{\frac{1}{\rho}}$. The integration by parts is straightforward when $p \ge 2$. For $1 < p < 2$, $|u_\rho|^{p-2}$ becomes singular near the zeros of u_ρ. It is possible to prove that the integration by parts is allowed also in this case (see [10]). Notice also that all boundary terms vanish since $u_\rho = 0$ at the boundary. We obtain

$$
\begin{aligned}
-\int_{B_\rho\setminus B_{\frac{1}{\rho}}} Lu_\rho u^\star &= (p-1)\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |x|^\alpha |u_\rho|^{p-2}|\nabla u_\rho|^2 \\
&\quad + \alpha \int_{B_\rho\setminus B_{\frac{1}{\rho}}} u_\rho \nabla u_\rho |u_\rho|^{p-2}|x|^{\alpha-1}\frac{x}{|x|} \\
&= (p-1)\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |x|^\alpha |u_\rho|^{p-2}|\nabla u_\rho|^2 \\
&\quad - \frac{\alpha(\alpha-2+N)}{p}\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |u_\rho|^p |x|^{\alpha-2}.
\end{aligned}
$$

Clearly, if $2-N\le\alpha\le 0$, $-\int_{B_\rho\setminus B_{\frac{1}{\rho}}} Lu_\rho u^\star \ge 0$. If $\alpha>0$, by Hardy's inequality (see for example [14, Proposition 9.10])

$$
\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |u_\rho|^p|x|^{\alpha-2} \le \frac{p^2}{(N+\alpha-2)^2}\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |x|^\alpha |u_\rho|^{p-2}|\nabla u_\rho|^2
$$

and hence

$$
-\int_{B_\rho\setminus B_{\frac{1}{\rho}}} Lu_\rho u^\star \ge \left(p-1-\frac{\alpha p}{N+\alpha-2}\right)\int_{B_\rho\setminus B_{\frac{1}{\rho}}} |u_\rho|^{p-2}|x|^\alpha|\nabla u_\rho|^2.
$$

Observe that $p-1-\dfrac{p\alpha}{N+\alpha-2}$ is positive for $\alpha\le(p-1)(N-2)$.

Summing up, L is dissipative in $B_\rho\setminus B_{\frac{1}{\rho}}$ for $2-N\le\alpha\le(p-1)(N-2)$ and therefore

$$
\lambda\|u_\rho\|_p \le \|f\|_{L^p}. \tag{16}
$$

Next we use weak compactness arguments to produce a function $u\in D_{p,max}(L)$ satisfying $\lambda u - Lu = f$. Let us fix a radius r and apply the interior L^p estimates ([6, Theorem 9.11]) together with (16) to the functions u_ρ with $\rho<r+1$

$$
\begin{aligned}
\|u_\rho\|_{W^{2,p}(B_\rho\setminus B_{\frac{1}{\rho}})} &\le C_1[\|\lambda u_\rho - Lu_\rho\|_{L^p(B_{r+1}\setminus B_{\frac{1}{r+1}})} + \|u_\rho\|_{L^p(B_{r+1}\setminus B_{\frac{1}{r+1}})}] \\
&\le C_2\|f\|_{L^p}.
\end{aligned}
$$

By weak compactness and a diagonal argument, we can find a sequence $(\rho_n)\to\infty$ such that the functions (u_{ρ_n}) converge weakly in $W^{2,p}_{loc}(\Omega)$ to a function u. Clearly u satisfies $\lambda u - Lu = f$ and, by (16), $\lambda\|u\|_{L^p}\le\|f\|_{L^p}$. In particular $u\in D_{p,max}(L)$ and is a solution of the equation $\lambda u - Lu = f$.

If $\alpha \geq 2$ or $\alpha < 2$ and $p \geq N/(N-2)$, then $u \in D_p(L) = D_{p,max}(L)$ and, by density, the estimate $\lambda \|R(\lambda, L)\|_p \leq 1$ follows for $\lambda > 0$. If $\alpha < 2$ and $N/(N-\alpha) < p < N/(N-2)$, we fix $q \geq N/(N-2)$ and use the consistency of the resolvents in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ proved in Theorem 8.1. For large λ, say $\lambda \geq \lambda_0$, both the resolvent of $(L, D_p(L))$ and $(L, D_q(L))$ exist and coincide on $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Therefore, since $u \in D_q(L)$, then $u \in D_p(L)$ and hence $\lambda \|R(\lambda, L)\|_p \leq 1$ holds for $\lambda \geq \lambda_0$. If $0 < \lambda < \lambda_0$ and $\lambda u - Lu = f$, then $\lambda_0 u - Lu = (\lambda_0 - \lambda)u + f$, hence $\lambda_0 \|u\|_p \leq (\lambda_0 - \lambda)\|u\|_p + \|f\|_p$ and $\lambda \|u\|_p \leq \|f\|_p$. From this a-priori estimate it follows that $]0, \infty[\subset \rho(L)$ and that $(L, D_p(L))$ is dissipative. □

Finally we show that the condition $2 - N \leq \alpha \leq (p-1)(N-2)$ is necessary for the dissipativity.

Proposition 9.2. *Suppose that the operator $(L, D_p(L))$ is dissipative. Then $2 - N \leq \alpha \leq (p-1)(N-2)$.*

Proof. Assume L dissipative. Then, for every $u \in D_p(L)$, $\int_{\mathbb{R}^N} |x|^\alpha u|u|^{p-2} \Delta u \leq 0$. If $u \in C_c^\infty(\Omega)$ (hence equal to 0 in a neighborhood of the origin), we may integrate by parts twice and, using the identity $\nabla |u|^p = pu|u|^{p-2}\nabla u$, we get

$$\alpha(N + \alpha - 2) \int_{\mathbb{R}^N} |u|^p |x|^{\alpha-2} \leq p(p-1) \int_{\mathbb{R}^N} |x|^\alpha |u|^{p-2} |\nabla u|^2.$$

It follows that $\alpha(N + \alpha - 2) \leq 0$, that is $2 - N \leq \alpha \leq 0$ or

$$\int_{\mathbb{R}^N} |u|^p |x|^{\alpha-2} \leq \frac{p(p-1)}{\alpha(N+\alpha-2)} \int_{\mathbb{R}^N} |x|^\alpha |u|^{p-2} |\nabla u|^2 \tag{17}$$

for every $u \in C_c^\infty(\Omega)$. Since $\left(\frac{p}{N+\alpha-2}\right)^2$ is the best constant in Hardy's inequality above (see [14, Proposition 9.10]), we obtain

$$\frac{p(p-1)}{\alpha(N+\alpha-2)} \geq \left(\frac{p}{N+\alpha-2}\right)^2,$$

which implies $2 - N \leq \alpha \leq (p-1)(N-2)$. □

10. Heat kernel estimates

As in [16], we can prove kernel estimates for L, L_1, L_2 by using the equivalence between weighted Nash inequalities and ultracontractivity ([2]). We give the details only for L, the other cases being similar. We introduce the Hilbert spaces $L^2_\mu = L^2_\mu(\mathbb{R}^N)$, where $d\mu(x) = |x|^{-\alpha}\, dx$, endowed with its canonical inner product and $H = \{u \in L^2_\mu : \nabla u \in L^2(\mathbb{R}^N)\}$ endowed with the inner product $(u,v)_H = \int_{\mathbb{R}^N} (u\bar{v}\, d\mu + \nabla u \cdot \nabla \bar{v}\, dx)$. Let $\mathscr{V}$ be the closure of $C_c^\infty(\Omega)$ in H, with respect to the norm of H and observe that Sobolev inequality $\|u\|^2_{2^*} \le C\|\nabla u\|_2^2$ holds in $\mathscr{V}$. We next introduce the form

$$a(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v}\, dx \tag{18}$$

for $u, v \in \mathscr{V}$ and the self-adjoint operator $\mathscr{L}$ defined by

$$D(\mathscr{L}) = \left\{ u \in L^2_\mu : \text{there exists } f \in L^2_\mu : a(u,v) = -\int_{\mathbb{R}^N} f\bar{v}\, d\mu \ \forall v \in \mathscr{V} \right\} \quad \mathscr{L}u = f.$$

Since $a(u,u) \ge 0$, the operator $\mathscr{L}$ generates an analytic semigroup of contractions $e^{t\mathscr{L}}$ in L^2_μ. An application of the Beurling-Deny criteria shows that the generated semigroup is positive and L^∞-contractive. For our purposes we need to show that the resolvent of $\mathscr{L}$ and of $(L, D_p(L))$ are coherent. This is done in the following proposition.

Proposition 10.1. *If* $\lambda > 0$ *and* $f \in L^p(\mathbb{R}^N) \cap L^2_\mu$, *then* $(\lambda - \mathscr{L})^{-1} f = (\lambda - L)^{-1} f$.

Proof. Let $0 \le f \in C_c(\Omega)$ and fix an annulus $C_n = B_n \setminus B_{\frac{1}{n}}$ in $\mathbb{R}^N$. Consider the problem

$$\begin{cases} \lambda u - Lu = f & x \in C_n, \\ u = 0 & x \in \partial C_n. \end{cases}$$

Since the operator L is uniformly elliptic in C_n, there exists a unique solution $u_n \in W^{2,2}(C_n) \cap W^{1,2}_0(C_n)$ of the previous problem satisfying

$$\lambda \int_{C_n} u\bar{v}\, d\mu + \int_{C_n} \nabla u_n \nabla \bar{v}\, dx = \int_{C_n} fv\, d\mu \tag{19}$$

for every $v \in W^{1,2}_0(C_n)$. Moreover, as in Section 2, se also [13, Theorem 3.4], u_n is positive, increasing and converges pointwise to a function $u \in D_{max}(L)$ which satisfies $\lambda u - Lu = f$. Setting $v = u_n$ in (19) we obtain $\lambda\|u_n\|_{L^2_\mu} + \lambda^{1/2}\|\nabla u_n\|_{L^2_\mu} \le \|f\|_{L^2_\mu}$. Letting n to infinity, by monotone convergence, we deduce that $u_n \to u$ in L^2_μ. Moreover, for a suitable subsequence (n_k), ∇u_{n_k} converges weakly, hence $u \in H$ and u belongs to the closure in H of $W^{1,2}$ functions with compact support, that is $u \in \mathscr{V}$. Letting $n \to \infty$ in (19) we obtain $a(u,v) = -(\lambda u - f, v)_{L^2_\mu}$, for every $v \in W^{1,2}$ having compact support and, by density, for every $v \in \mathscr{V}$, that is $u \in D(\mathscr{L})$ and $\lambda u - \mathscr{L}u = f$. This shows the coeherence of the resolvents of $(L, D_{max}(L)$ and $\mathscr{L}$ for $\lambda > 0$, hence of $\mathscr{L}$ and $(L, D_p(L))$, by Theorem 8.1. □

Since the operator L is symmetric with respect to the measure $d\mu(x) = (|x|^\alpha)^{-1}\,dx$ we shall reepresent the generated semigroup $T(t)$ through a kernel with respect to $d\mu$, namely

$$T(t)f(x) = \int_{\mathbb{R}^N} p_\mu(x,y,t)f(y)\,d\mu(y).$$

Clearly $p_\mu(x,y,t) = |y|^\alpha p(x,y,t)$, if p is the kernel with respect to the Lebesgue measure. Our goal consists in obtaining upper bounds for p_μ following the approach of [2], which now describe. A positive C^2 function V is a Lyapunov function for L if $LV \le cV$ for some positive c. This implies that $T(t)V \le V \exp ct$, see for example [12, Lemma 3.9]. It turns out that the estimate $p_\mu(x,y,t) \le Ct^{-N/2}e^{ct}V(x)V(y)$ is equivalent to the weighted Nash inequality

$$\left(\int_{\mathbb{R}^N} |u|^2\,d\mu\right)^{1+\frac{2}{N}} \le Ca(u,u)\left(\int_{\mathbb{R}^N} |u|V\,d\mu\right)^{\frac{4}{N}},$$

see [2, Theorem 2.5]. Let $V(x) = |x|^\beta$; since $LV = \beta(N+\beta-2)|x|^{\alpha+\beta-2}$ it is clear the inequality $LV \le cV$ will be satisfied in B and in B^c for different values of β. This explains why different choices of the parameters will be done in B and B^c.

Proposition 10.2. *Let $u \in \mathscr{V}$. Then*

$$\left(\int_{\mathbb{R}^N} |u|^2\,d\mu\right)^{1+\frac{2}{N}} \le Ca(u,u)\left(\int_{\mathbb{R}^N} |u|V\,d\mu\right)^{\frac{4}{N}}$$

where

$$V(x) = \begin{cases} 1 & x \in B, \\ |x|^{\alpha\frac{2-N}{4}} & x \in B^c \end{cases} \qquad \alpha < 0$$

$$V(x) = |x|^{\alpha\frac{2-N}{4}} \qquad 0 < \alpha \le 4$$

$$V(x) = \begin{cases} |x|^{\alpha\frac{2-N}{4}} & x \in B, \\ |x|^{2-N} & x \in B^c \end{cases} \qquad \alpha > 4.$$

Proof. Suppose $\alpha \le 4$ and let $u \in \mathscr{V}$. Then, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^2\,d\mu &= \int_{\mathbb{R}^N} |u|^{\frac{2N}{N+2}}|u|^{\frac{4}{N+2}}(|x|^\alpha)^{-1}\,dx \\ &\le \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N+2}\frac{N+2}{N-2}}\,dx\right)^{\frac{N-2}{N+2}}(|u|\,|x|^{-\alpha\frac{N+2}{4}}\,dx)^{\frac{4}{N+2}} \\ &= \left(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\right)^{\frac{1}{2^*}\frac{2N}{N+2}}\left(\int_{\mathbb{R}^N} |u|\,|x|^{\alpha\frac{2-N}{4}}\,d\mu\right)^{\frac{4}{N+2}},\end{aligned}$$

where $2^* = 2N/(N-2)$. By Sobolev embedding,

$$\int_{\mathbb{R}^N} |u|^2\, d\mu \le C\Big(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\Big)^{\frac{N}{N+2}} \Big(\int_{\mathbb{R}^N} |u|\, |x|^{\alpha\frac{2-N}{4}}\, d\mu\Big)^{\frac{4}{N+2}}$$
$$\le Ca(u,u)\Big(\int_{\mathbb{R}^N} |u|\, |x|^{\alpha\frac{2-N}{4}}\, d\mu\Big)^{\frac{4}{N}}.$$

Since $|x|^{\alpha\frac{2-N}{4}} \le V(x)$, the claim follows. If $\alpha \ge 4$ we split the integral on $\mathbb{R}^N$ as the sum of the integrals over B and B^c. By Hölder's inequality

$$\begin{aligned}\Big(\int_{\mathbb{R}^N} |u|^2\, d\mu\Big)^{1+\frac{2}{N}} &\le C\Big(\int_B |u|^{2^*}\, dx\Big)^{\frac{2}{2^*}} \Big(\int_B |u|\, |x|^{\alpha\frac{2-N}{4}}\, d\mu\Big)^{\frac{4}{N}} \\ &\quad + C\Big(\int_{B^c} |u|^{2^*}(1+|x|)^4 \frac{1}{1+|x|^\alpha}\, dx\Big)^{\frac{2}{2^*}} \\ &\quad \times \Big(\int_{B^c} |u|(1+|x|)^{2-N}\, d\mu\Big)^{\frac{4}{N}} \\ &\le C\Big(\int_B |u|^{2^*}\, dx\Big)^{\frac{2}{2^*}} \Big(\int_B |u|\, |x|^{\alpha\frac{2-N}{4}}\, d\mu\Big)^{\frac{4}{N}} \\ &\quad + C\Big(\int_{B^c} |u|^{2^*}\, dx\Big)^{\frac{2}{2^*}} \Big(\int_{B^c} |u|\, |x|^{2-N}\, d\mu\Big)^{\frac{4}{N}} \\ &\le C\Big(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\Big)^{\frac{2}{2^*}} \Big(\int_{\mathbb{R}^N} |u| V\, d\mu\Big)^{\frac{4}{N}}.\end{aligned}$$

By Sobolev embedding the proof follows also in this case. □

THEOREM 10.3. *Let V be as in Proposition* 10.2*. Then the kernel p_μ satisfies*

$$p_\mu(x,y,t) \le \frac{e^{ct}}{t^{\frac{N}{2}}} V(x)V(y)$$

for every $t > 0$, $x, y \in \mathbb{R}^N$ and for some $c \in \mathbb{R}$.

PROOF. Let W be a $C^2(\Omega)$ function such that

$$W(x) = \begin{cases} 1 & x \in B_1, \\ |x|^{\alpha\frac{2-N}{4}} & x \in \mathbb{R}^N \setminus B_2 \end{cases} \qquad \alpha < 0$$
$$W(x) = V(x) \quad 0 < \alpha \le 4$$
$$W(x) = \begin{cases} |x|^{\alpha\frac{2-N}{4}} & x \in B_1, \\ |x|^{2-N} & x \in \mathbb{R}^N \setminus B_2 \end{cases} \qquad \alpha > 4.$$

It easily follows that W is a Lyapunov function for L. Since $c_1 W \le V \le c_2 W$, for suitable $c_1, c_2 > 0$, the statement follows from Proposition 10.2 and [2, Theorem 2.5]. □

Since $p(x, y, t) = |y|^{-\alpha} p_\mu(x, y, t)$ it follows that $p(x, y, t) \le \frac{e^{ct}}{t^{\frac{N}{2}}} |y|^{-\alpha} V(x) V(y)$ for every $t > 0$, $x, y \in \mathbb{R}^N$. Heat kernel estimates for L_1, L_2 easily follows from above, since the semigroups generated by L_1, L_2 are pointwise dominated by that generated by L.

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