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Partial Differential Equations — *On the first curve of the Fučík spectrum for elliptic operators*, by Riccardo Molle and Donato Passaseo, communicated on 14 February 2014.

Abstract. — In this Note we present a new variational characterization of the first nontrivial curve of the Fučík spectrum for elliptic operators with Dirichlet or Neumann boundary conditions. Moreover, we describe the asymptotic behaviour and some properties of this curve and of the corresponding eigenfunctions. In particular, this new characterization allows us to compare the first curve of the Fučík spectrum with the infinitely many curves we obtained in previous works (see [8, 9]): for example, we show that these curves are all asymptotic to the same lines as the first curve, but they are all distinct from such a curve.

Key words: Elliptic operators, Fučík spectrum, first curve, asymptotic behaviours, variational methods.

Mathematics Subject Classification: 35J20, 35J60, 35J66.

1. Introduction

The Fučík spectrum plays an important role in the study of some elliptic problems with linear growth. Let us consider, for example, the Dirichlet problem

$$\Delta u + g(x,u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded connected domain of $\mathbb{R}^N$ and g is a Carathéodory function in $\Omega \times \mathbb{R}$ such that

$$\lim_{t\to-\infty} \frac{g(x,t)}{t} = \alpha, \quad \lim_{t\to+\infty} \frac{g(x,t)}{t} = \beta \quad \forall x \in \Omega,$$

with $\alpha, \beta \in \mathbb{R}$. Existence and multiplicity of solutions of problems of this type are strictly related to the position of the pair (α,β) with respect to the Fučík spectrum Σ which is defined as the set of all the pairs $(\alpha,\beta) \in \mathbb{R}^2$ such that the Dirichlet problem

$$\Delta u - \alpha u^- + \beta u^+ = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

has nontrivial solutions (i.e. $u \in H_0^1(\Omega)$, $u \not\equiv 0$). In analogous way one can define the Fučík spectrum $\tilde{\Sigma}$ when the Dirichlet boundary condition is replaced by the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

After the pioneering papers [2, 1] on these problems, the important role of the Fučík spectrum has been pointed out in [5, 3]. Then, several works have been devoted to describe the structure of Σ and $\tilde{\Sigma}$ (see, for instance, [4, 11, 12] and the references therein). The really interesting case is $N > 1$ since, in the case $N = 1$, Σ and $\tilde{\Sigma}$ are completely known and may be obtained by direct computation.

Let us denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$; it is clear that Σ includes the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, contains all the pairs (λ_i, λ_i) $\forall i \in \mathbb{N}$ (that are the only pairs (α, β) of Σ such that $\alpha = \beta$) and is symmetric with respect to the line $\{(\alpha, \beta) \in \mathbb{R}^2 : \alpha = \beta\}$; moreover, if $\alpha \neq \lambda_1$, $\beta \neq \lambda_1$ and $(\alpha, \beta) \in \Sigma$, then $\alpha > \lambda_1$, $\beta > \lambda_1$ and the Fučík eigenfunctions corresponding to (α, β) are sign changing functions (analogous properties hold for $\tilde{\Sigma}$).

In [3] it is shown that the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ are isolated in Σ. Many results (see [6, 7, 11, 12, 4] and the references therein) concern the curves in Σ emanating from each pair (λ_i, λ_i) (local existence and multiplicity, variational characterization, local and global properties, etc....). Combining these results, one can infer in particular that Σ contains a first nontrivial curve, which passes through (λ_2, λ_2) and extends to infinity. In [4] the authors prove directly the existence of such a first curve, show that it is asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, give a variational characterization and deduce that all the corresponding eigenfunctions have exactly two nodal domains (extending the well known Courant nodal domains theorem).

Recently (see [8, 9]) we have proved that, if $N > 1$, the Fučík spectrum Σ contains infinitely many curves asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ (notice that, on the contrary, if $N = 1$ Σ has only two curves asymptotic to $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$).

A similar result holds also in the case of Neumann boundary condition. In this case the first eigenvalue of $-\Delta$ is zero and, if $N > 1$, the Fučík spectrum $\tilde{\Sigma}$ includes infinitely many curves asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ (notice that, on the contrary, if $N = 1$ $\tilde{\Sigma}$ does not contain any curve asymptotic to $\{0\} \times \mathbb{R}$ or to $\mathbb{R} \times \{0\}$).

More precisely, the following theorem holds.

THEOREM 1.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$. Then, there exists a nondecreasing sequence $(\beta_k)_k$ of positive numbers such that $\forall k \in \mathbb{N}$ and $\forall \beta > \beta_k$ there exists $\alpha_{k,\beta} > \lambda_1$ and $\tilde{\alpha}_{k,\beta} > 0$ such that $(\alpha_{k,\beta}, \beta) \in \Sigma$ and $(\tilde{\alpha}_{k,\beta}, \beta) \in \tilde{\Sigma}$; moreover, $\forall k \in \mathbb{N}$, $\alpha_{k,\beta}$ and $\tilde{\alpha}_{k,\beta}$ depend continuously on β in $]\beta_k, +\infty[$, $\alpha_{k,\beta} < \alpha_{k+1,\beta}$, $\tilde{\alpha}_{k,\beta} < \tilde{\alpha}_{k+1,\beta}$ $\forall \beta > \beta_{k+1}$ and $\lim_{\beta \to +\infty} \alpha_{k,\beta} = \lambda_1$, $\lim_{\beta \to +\infty} \tilde{\alpha}_{k,\beta} = 0$.*

In this Note we present a new variational characterization of the first nontrivial curve of the Fučík spectrum in the cases of Dirichlet and Neumann boundary conditions; this characterization, in particular, allows us to show that all the curves given by Theorem 1.1 (even for $k = 1$) are distinct from such a curve. This results, announced in this Note, will appear in [10], presented and proved in a more completed and detailed way.

For the sake of simplicity, here we present our results in the case of the Laplace operator; however, they may be easily extended in order to cover the case of more general elliptic operators in divergence form.

2. The main results

Theorems 2.1 and 2.2 give a variational characterization of the first nontrivial curve of the Fučík spectrum in the cases, respectively, of Dirichlet and Neumann boundary conditions.

Theorem 2.1 (see [10] for the proof). *Let Ω be a smooth bounded connected domain of $\mathbb{R}^N$ with $N \geq 2$. For all $\beta > \lambda_1$, let us set*

$$\alpha_\beta = \inf\left\{ \int_\Omega |Du^-|^2\,dx : u \in H_0^1(\Omega), \right. \quad \left. \|u^+\|_{L^2(\Omega)} = \|u^-\|_{L^2(\Omega)} = 1, \int_\Omega |Du^+|^2\,dx = \beta \right\}. \tag{2.1}$$

Then $\alpha_\beta > \lambda_1$, $(\alpha_\beta, \beta) \in \Sigma$ $\forall \beta > \lambda_1$ and $\alpha_\beta \leq \alpha$ for every $\alpha > \lambda_1$ such that $(\alpha, \beta) \in \Sigma$. Moreover, α_β is continuous and decreasing with respect to β in $]\lambda_1, +\infty[$, the infimum in (2.1) *is achieved $\forall \beta > \lambda_1$ and an eigenfunction corresponding to the pair (α_β, β) is given by $u_\beta = -\bar{u}_\beta^- + \mu_\beta \bar{u}_\beta^+$, where $\bar{u}_\beta$ is a minimizing function for* (2.1) *and μ_β is a suitable positive constant.*

As $\beta \to +\infty$, $\alpha_\beta \to \lambda_1$ and $u_\beta \to -e_1$ in $H_0^1(\Omega)$, where e_1 is the positive function in $H_0^1(\Omega)$ such that $\Delta e_1 + \lambda_1 e_1 = 0$ in Ω and $\|e_1\|_{L^2(\Omega)} = 1$; as $\beta \to \lambda_1$, $\alpha_\beta \to +\infty$ and $\|u_\beta^+\|_{L^2(\Omega)}^{-1} u_\beta \to e_1$ in $H_0^1(\Omega)$.

Theorem 2.2 (see [10] for the proof). *Let Ω be a smooth bounded connected domain of $\mathbb{R}^N$ with $N \geq 2$. For all $\beta > 0$, let us set*

$$\tilde{\alpha}_\beta = \inf\left\{ \int_\Omega |Du^-|^2\,dx : u \in H^1(\Omega), \right. \quad \left. \|u^+\|_{L^2(\Omega)} = \|u^-\|_{L^2(\Omega)} = 1, \int_\Omega |Du^+|^2\,dx = \beta \right\}. \tag{2.2}$$

Then $\tilde{\alpha}_\beta > 0$, $(\tilde{\alpha}_\beta, \beta) \in \tilde{\Sigma}$ $\forall \beta > 0$ and $\tilde{\alpha}_\beta \leq \alpha$ for every $\alpha > 0$ such that $(\alpha, \beta) \in \tilde{\Sigma}$. Moreover, $\tilde{\alpha}_\beta$ is continuous and decreasing with respect to β in $]0, +\infty[$, the infimum in (2.2) *is achieved $\forall \beta > 0$ and an eigenfunction corresponding to the pair $(\tilde{\alpha}_\beta, \beta)$ is given by $\tilde{u}_\beta = -\hat{u}_\beta^- + \tilde{\mu}_\beta \hat{u}_\beta^+$, where $\hat{u}_\beta$ is a minimizing function for* (2.2) *and $\tilde{\mu}_\beta$ is a suitable positive constant.*

As $\beta \to +\infty$, $\tilde{\alpha}_\beta \to 0$ and $\tilde{u}_\beta \to -|\Omega|^{-\frac{1}{2}}$ in $H^1(\Omega)$ while, as $\beta \to 0$, $\tilde{\alpha}_\beta \to +\infty$ and $\|\tilde{u}_\beta^+\|_{L^2(\Omega)}^{-1} \tilde{u}_\beta \to |\Omega|^{-\frac{1}{2}}$ in $H^1(\Omega)$.

Finally, for all $\beta > 0$, there exist $\tilde{x}_\beta \in \Omega$ and $\rho_\beta > 0$ such that $\lim_{\beta \to +\infty} \rho_\beta = 0$ and $\operatorname{supp}(\tilde{u}_\beta^+) \subset B(\tilde{x}_\beta, \rho_\beta)$ $\forall \beta > 0$; if (up to a subsequence) $\lim_{\beta \to +\infty} \tilde{x}_\beta = \tilde{x}$ for a *suitable* $\tilde{x}$, *then* $\tilde{x} \in \partial\Omega$ *and, if we set* $\tilde{H} = \{x \in \mathbb{R}^N : (x \cdot \tilde{v}) < 0\}$ *where* $\tilde{v}$ *denotes the outward normal to* $\partial\Omega$ *in* $\tilde{x}$, *then we have*

$$\lim_{\beta\to+\infty} \Big(\sup_{\Omega} \tilde{u}_\beta\Big)^{-1} \tilde{u}_\beta\Big(\frac{x}{\sqrt{\beta}} + \tilde{x}_\beta\Big) = U(x) \quad \forall x \in \tilde{H},$$

where U *is the radial solution of the problem* $\Delta U + U^+ = 0$ *in* $\mathbb{R}^N$, $U(0) = 1$. *Similar properties hold for* $-\|\tilde{u}_\beta^+\|^{-1}_{L^2(\Omega)}\tilde{u}_\beta$ *as* $\beta \to 0$.

All the properties of the first nontrivial curve of the Fučík spectrum and of the corresponding eigenfunctions, in the cases of Dirichlet and Neumann boundary conditions, may be deduced from the variational characterization given by Theorems 2.1 and 2.2. In particular, one can deduce that $\alpha_{\lambda_2} = \lambda_2$ (namely, the first curve of Σ passes through (λ_2, λ_2)) and, analogously, that $\tilde{\alpha}_{\tilde{\lambda}_2} = \tilde{\lambda}_2$, where $\tilde{\lambda}_2$ denotes the second eigenvalue of the Laplace operator $-\Delta$ with the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. In next proposition we use this variational characterization to show that the curves given by Theorem 1.1, contained in Σ [respectively, in $\tilde{\Sigma}$], are all distinct from the first nontrivial curve of Σ [respectively, of $\tilde{\Sigma}$].

PROPOSITION 2.3. *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^N$ *with* $N \geq 2$. *Then, there exists* $\bar{\beta} > 0$ *such that* $\alpha_\beta < \alpha_{k,\beta}$ *and* $\tilde{\alpha}_\beta < \tilde{\alpha}_{k,\beta}$ $\forall k \in \mathbb{N}$, $\forall \beta > \max\{\bar{\beta}, \beta_k\}$.

Sketch of the proof. For simplicity, here we consider only the case $N \geq 3$ (the case $N = 2$, which requires more refined estimates, is considered in [10]).

In order to prove that $\alpha_\beta < \alpha_{k,\beta}$, for β large enough, notice that (as we proved in [9])

$$\lim_{\beta\to+\infty} \beta^{\frac{N-2}{2}}(\alpha_{k,\beta} - \lambda_1) = k\,\mathrm{cap}(\bar{r}_1)\Big(\max_{\Omega} e_1\Big)^2 \quad \forall k \in \mathbb{N}, \tag{2.3}$$

where $\bar{r}_1$ is the radius of the balls in $\mathbb{R}^N$ for which the first eigenvalue of $-\Delta$ in H^1_0 is equal to 1 and $\mathrm{cap}(\bar{r}_1)$ denotes the capacity of these balls. Then, choose $\bar{x} \in \Omega$ such that $e_1(\bar{x}) < \max_\Omega e_1$ and (for β large enough so that $B(\bar{x}, \bar{r}_1/\sqrt{\beta}) \subset \Omega$) set

$$\alpha_\beta(\bar{x}) = \inf\Big\{\int_\Omega |Du|^2\,dx : u \in H^1_0(\Omega), u \leq 0 \text{ in } \Omega,$$
$$u = 0 \text{ in } B\Big(\bar{x}, \frac{\bar{r}_1}{\sqrt{\beta}}\Big), \|u\|_{L^2(\Omega)} = 1\Big\}. \tag{2.4}$$

Since there exists $u \in H^1_0(\Omega)$ such that $u \geq 0$ in Ω, $u = 0$ in $\Omega\backslash B(\bar{x}, \bar{r}_1/\sqrt{\beta})$, $\|u\|_{L^2(\Omega)} = 1$ and $\int_\Omega |Du|^2\,dx = \beta$, we have $\alpha_\beta \leq \alpha_\beta(\bar{x})$. As $\beta \to +\infty$ (after rescaling) we obtain

$$\lim_{\beta\to+\infty} \beta^{\frac{N-2}{2}}[\alpha_\beta(\bar{x}) - \lambda_1] = \mathrm{cap}(\bar{r}_1)[e_1(\bar{x})]^2 < \mathrm{cap}(\bar{r}_1)\Big(\max_\Omega e_1\Big)^2. \tag{2.5}$$

It follows that there exists $\bar{\beta} \geq \beta_1$ such that $\alpha_\beta \leq \alpha_\beta(\bar{x}) < \alpha_{1,\beta}$ $\forall \beta > \bar{\beta}$ and, as a consequence, $\alpha_\beta < \alpha_{k,\beta}$ $\forall k \in \mathbb{N}$, $\forall \beta > \max\{\bar{\beta}, \beta_k\}$.

In analogous way one can prove that $\tilde{\alpha}_\beta < \tilde{\alpha}_{k,\beta}$ $\forall k \in \mathbb{N}$, for $\beta > 0$ large enough, because $\lim_{\beta\to+\infty} \beta^{\frac{N-2}{2}} \tilde{\alpha}_{k,\beta} = k\,\mathrm{cap}(\bar{r}_1)|\Omega|^{-1}$ $\forall k \in \mathbb{N}$ while $\lim_{\beta\to+\infty} \beta^{\frac{N-2}{2}} \tilde{\alpha}_\beta = (1/2)\,\mathrm{cap}(\bar{r}_1)|\Omega|^{-1}$ (see [10] for more details).

REMARK 2.4. The eigenfunctions $u_{k,\beta} \in H^1_0(\Omega)$ and $\tilde{u}_{k,\beta} \in H^1(\Omega)$, corresponding respectively to the pairs $(\alpha_{k,\beta}, \beta) \in \Sigma$ and $(\tilde{\alpha}_{k,\beta}, \beta) \in \tilde{\Sigma}$, present k interior bumps; moreover, for $u_{k,\beta}$ and for $\tilde{u}_{k,\beta}$, the asymptotic profile of the rescaled bumps is described by the function U introduced in Theorem 2.2 (see [8]).

On the contrary, the eigenfunctions $u_\beta \in H^1_0(\Omega)$ and $\tilde{u}_\beta \in H^1(\Omega)$, corresponding respectively to the pairs $(\alpha_\beta, \beta) \in \Sigma$ and $(\tilde{\alpha}_\beta, \beta) \in \tilde{\Sigma}$, have only one bump which, for β large enough, is localized near the boundary of Ω. In Theorem 2.2 we have described this property for $\tilde{u}_\beta$; a similar property holds also for u_β, but the asymptotic profile of the rescaled bump of u_β is not described by the function U; in fact, as $\beta \to +\infty$, the functions u^+_β concentrate near suitable points x_β such that $\lim_{\beta\to+\infty} \mathrm{dist}(x_\beta, \partial\Omega) = 0$; if (up to a subsequence) $\lim_{\beta\to+\infty} x_\beta = \hat{x}$, for a suitable $\hat{x} \in \partial\Omega$, and if we set $\hat{H} := \{x \in \mathbb{R}^N : (x \cdot \hat{v}) < 0\}$ where $\hat{v}$ denotes the outward normal to $\partial\Omega$ in $\hat{x}$, then $\lim_{\beta\to+\infty} (\sup_\Omega u_\beta)^{-1} u_\beta(x/\sqrt{\beta} + x_\beta) = \hat{U}(x)$ $\forall x \in \hat{H}$, where $\hat{U}$ is a function in $H^1_0(\hat{H})$, which solves a suitable limit equation in $\hat{H}$.

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Riccardo Molle
Dipartimento di Matematica
Universita' di Roma "Tor Vergata"
Via della Ricerca Scientica no 1 - 00133 Roma
molle@mat.uniroma2.it

Donato Passaseo
Dipartimento di Matematica "E. De Giorgi"
Universita' di Lecce
P.O. Box 193 - 73100 Lecce, Italia