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Mathematical Physics — *Social-economical mobility and redistribution policy*, by I. Borsi, G. Busoni, A. Fasano and M. Primicerio, communicated on 14 February 2014.

Abstract. — We present and discuss an integro-differential equation that models economic dynamics in a closed population. An "economic inequality index" is introduced and some examples are provided showing how changes in social mobility can produce increasing (or decreasing) social equity.

Key words: Social dynamics, social mobility, structured population, integro-differential equations.

Mathematics Subject Classification: 45G10, 92D25, 91D10.

1. Introduction

The aim of this paper is to present and discuss a mathematical model for the economic dynamics of a given closed population.

Let us start by assuming that, at a given time $t \geq 0$, each member of the population is characterized by a "social index" x. This index could correspond to the yearly salary, to the amount of income tax paid, etc. We assume that x ranges in a bounded interval and, upon a trivial normalization, we will take $x \in [0,1]$.

In this spirit, we can define a function $n(x,t)$, such that, for any fixed time t and any pair x_1, x_2, $0 \leq x_1 < x_2 \leq 1$, the quantity

$$\int_{x_1}^{x_2} n(x,t)\,dx, \tag{1.1}$$

represents the number of individuals of the given society whose social index (at time t) belongs to the interval (x_1, x_2). Sometimes, we will refer to $n(x,t)$ as to the "social density" at time t.

When $n(x,t)$ $(x \in [0,1], t \geq 0)$ for a given society is known, we say that the social dynamics of the society is known. It is clear that the social dynamics is influenced by three main factors: (i) immigration/emigration, (ii) age structure and birth/death rate, (iii) economic mobility (i.e. change of social index). In this article we neglect (i) just to keep the discussion simpler and shorter (for a discussion of these factors see e.g. [6]). Taking (ii) into account, on the contrary, would imply non-trivial modifications, including discussion on hereditary issues and so on, that are well beyond our present goals. We may just mention that one could partially include these aspects by assuming that the "individuals" that we consider in our model of society are rather households (as the latter appear in the usual economic statistics [3]).

Thus, we will confine our analysis to aspect (iii). In order to model economic mobility (see [5], [1] and the literature cited therein) we characterize it by a function $\gamma(x, y)$, $(x, y) \in [0, 1]^2$, representing the rate at which individuals having social index x pass to social index y. More precisely $\gamma(x, y)$ is such that, for any $0 \le t_1 < t_2$, the quantity

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} n(x, t) \int_0^1 \gamma(x, y)\, dy\, dx\, dt, \tag{1.2}$$

gives the number of individuals that, in the time interval (t_1, t_2) leave the social class identified by a social index belonging to the interval (x_1, x_2). Conversely, the number of "newcomers" in the same class in the same time interval is given by

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_0^1 n(z, t)\gamma(z, x)\, dz\, dx\, dt, \tag{1.3}$$

Of course, one could also consider the case of a time-dependent social mobility.

It is immediately seen that the balance for the social density $n(x, t)$ is expressed by the integrodifferential equation

$$\frac{\partial n}{\partial t}(x, t) = -n(x, t) \int_0^1 \gamma(x, y)\, dy + \int_0^1 \gamma(z, x) n(z, t)\, dz. \tag{1.4}$$

It is clear that an alternative discrete formulation can be obtained by partitioning the interval $(0, 1)$ into m parts and defining the numbers $n_1(t), \dots, n_m(t)$ of individuals in each of the social classes corresponding to the m sub-intervals. Then, a $(m \times m)$ matrix $\mathbb{A}$ can be introduced whose elements $a_{i,k} \ge 0$ represent the rate at which individuals pass from class i to class k. The balance is now expressed by the following dynamical system

$$\dot{n}_j(t) = -\sum_{k \ne j} n_j a_{jk} + \sum_{k \ne j} n_k a_{kj}. \tag{1.5}$$

A particular case of (1.5), i.e. $a_{jk} = 0$ if $|j - k| > 1$, was considered in [12] and [13].

In this paper we will first discuss (Section 2) the basic properties of the model and we give some different formulations and some examples. Then, we focus our attention (Section 3) on stationary solutions, that correspond to an equilibrium distribution of the population in the social classes. In Section 4 we include an index of social inequality associated with any $n(x, t)$ and we study the time variation of the social index x_p such that the first p-quantile of the total population has social index in the interval $(0, x_p)$.

Of course, the problem of economic redistribution is a key topic in political economy (see [2], [14]) since the economic mobility can be influenced by social policy ([7], [9], [15]). In this spirit, Section 5 is devoted to the problem of finding the kernel of social mobility $\gamma(x, y)$ (within a set of functions with a prescribed structure) that induces a desired stationary distribution. We will provide examples of non-existence and of non-uniqueness of solution for this problem.

In each section, examples and numerical simulations are displayed. Indeed the paper is more oriented to provide evidence of the reasonable outcomes of the model and of its potential usefulness, than to go deeper into its general mathematical structure.

2. Different forms of the social dynamic equation

We will study the following problem

Problem (P). Find $n(x,t) \in L^1(0,1)$ $\forall t > 0$, continuously differentiable w.r.t. t, for $t \geq 0$, satisfying the following differential equation,

$$\frac{\partial n}{\partial t}(x,t) = -n(x,t)\int_0^1 \gamma(x,y)\,dy + \int_0^1 \gamma(z,x)n(z,t)\,dz. \tag{2.1}$$

and such that

$$n(x,0) = n_0(x), \quad x \in (0,1), \tag{2.2}$$

where

$$\gamma \in L^\infty(0,1)^2, \quad \gamma \geq 0 \ a.e. \tag{2.3}$$

$$n_0 \in L^1(0,1), \quad n_0 \geq 0 \ a.e. \tag{2.4}$$

□

Of course, problem (P) can be seen as an evolution problem in $L^1(0,1)$

$$\frac{dn}{dt} = Fn, \quad n(0) = n_0 \in L^1(0,1), \tag{2.5}$$

and the norm of the operator F is bounded by

$$\|F\| \leq 2\|\gamma\|_{L^\infty}.$$

As a particular case of the results of [4] we have

Theorem 2.1. *Problem (P), i.e. problem (2.5), has one unique solution. Moreover*

(i) *$n(x,t) \geq 0$ a.e. in $(0,1)$, $\forall t \geq 0$.*
(ii) *If γ and n_0 are continuously differentiable in $(0,1)^2$ and in $(0,1)$, respectively, till the order k, then $n \in C^k$ w.r.t. x, $n \in C^\infty$ w.r.t. t and $\frac{\partial^i n}{\partial x^i} \in C^\infty$ w.r.t. t $(i = 1, \ldots, k)$.*
(iii) *If $n_0 \in L^\infty(0,1)$, $n(x,t)$ is analytic w.r.t. t a.e. in x.* □

Moreover, the following conservation property is immediately proved.

PROPOSITION 2.1. *For any* $t > 0$

$$\|n(x,t)\|_{L^1(0,1)} = \|n_0(x)\|_{L^1(0,1)} \equiv N, \tag{2.6}$$

i.e. the total population is constant. □

Let us define

$$g(x) = \int_0^1 \gamma(x,y)\,dy, \tag{2.7}$$

and note that (2.1)–(2.2) can be re-written in the form (because of Theorem 2.1)

$$\begin{aligned} n(x,t) &= n_0(x)\exp[-g(x)t] \\ &\quad + \int_0^t \exp[-g(x)(t-s)]\Big(\int_0^1 \gamma(z,x)n(z,s)\,dz\Big)\,ds. \end{aligned} \tag{2.8}$$

It is sometimes useful to define the "total mobility rate" to class y,

$$\Gamma(y,t) = \int_0^1 \gamma(x,y)n(x,t)\,dx,$$

and to rewrite (2.8) as an integral equation for Γ,

$$\Gamma(y,t) = G(y,t) + \int_0^t\int_0^1 K(x,y,t-s)\Gamma(x,s)\,dx\,ds, \tag{2.9}$$

where

$$\begin{aligned} G(y,t) &= \int_0^1 \gamma(x,y)n_0(x)\exp[-g(x)t]\,dx, \\ K(x,y,t) &= \gamma(x,y)\exp[-g(x)t]. \end{aligned}$$

EXAMPLE 2.1. *If* $\gamma = c$, *then*

$$n(x,t) = N + (n_0(x) - N)e^{-ct}, \quad t \geq 0, \tag{2.10}$$

(*an immediate consequence of* (2.8)). □

EXAMPLE 2.2. *If* $\gamma(x,y) = q(y)$, *then, if* $q \neq 0$,

$$n(x,t) = \frac{N}{a}q(x)[1 - e^{-at}] + n_0(x)e^{-at}, \quad t \geq 0, \tag{2.11}$$

where

$$a = \int_0^1 q(y)\,dy.$$

Obviously, when $q \to 0$ *a.e., i.e.* $a \to 0$*, the solution* $n = n_0(x)$ *is easily retrieved from* (2.11). □

EXAMPLE 2.3. *If* $\gamma(x, y) = p(x)$*, then, if* $p \neq 0$*,*

$$n(x, t) = v(x, t)/p(x) \tag{2.12}$$

where $v(x, t)$ *solves*

$$v_t + p(x)v = p(x)\int_0^1 v(y, t)\,dy, \quad v(x, 0) = p(x)n_0(x),$$

i.e.

$$v(x, t) = v(x, 0)\exp(-p(x)t) + p(x)\int_0^t \exp(-p(x)(t - s))\int_0^1 v(y, s)\,dy\,ds. \quad \square$$

EXAMPLE 2.4. *If* $\gamma(x, y) = p(x)q(y)$, $q \neq 0$*, then the function*

$$P(t) = \int_0^1 p(z)n(z, t)\,dz,$$

solves the Volterra integral equation

$$P(t) = \int_0^1 p(z)n_0(z)\exp[-g(z)t]\,dz + \int_0^t P(s)Z(t - s)\,ds, \tag{2.13}$$

where

$$Z(t) = \int_0^1 p(x)q(x)\exp[-g(x)t]\,dx,$$

(*a simple consequence of* (2.9)). *Moreover, once* $P(t)$ *has been determined, the function*

$$Q(t) = \int_0^1 q(z)n(z, t)\,dz$$

satisfies

$$Q(t) = H(t) + \int_0^t K(t - s)P(s)\,ds,$$

where

$$H(t) = \int_0^1 q(x) n_0(x) e^{-p(x)t\int_0^1 q(y)\,dy}, \quad K(t) = \int_0^1 q(x) e^{-p(x)t\int_0^1 q(y)\,dy}. \qquad \square$$

3. Stationary solutions

To simplify notation, in the sequel we will deal with the special case in which $n_0(x) \in C[0,1]$ and $\gamma(x,y) \in C([0,1]^2)$, but it is clear that all the results can be translated into the L^∞ framework, just by substituting notations like $max_{[0,1]}$ by $essup_{(0,1)}$ etc.

From now on we tacitly exclude the trivial case $\gamma \equiv 0$, unless explicitly stated.

Recalling definition (2.7) and introducing also

$$h(x) = \int_0^1 \gamma(y,x)\,dy, \tag{3.1}$$

we prove the following

Theorem 3.1. *Assume $g(x) > 0$ in $[0,1]$. Then Problem (P) admits a constant stationary solution*

$$n_\infty(x) = N, \quad x \in [0,1], \tag{3.2}$$

if and only if

$$g(x) = h(x), \quad x \in [0,1]. \tag{3.3}$$

Proof. It is obvious that (3.2) implies (3.3). To prove the converse, from (2.1) we get

$$n_\infty(x) = \frac{\int_0^1 \gamma(z,x) n_\infty(z)\,dz}{g(x)}. \tag{3.4}$$

Let

$$\bar{\bar{n}} = \max_{[0,1]} n_\infty(x) \tag{3.5}$$

and assume that there exists an $\hat{x}$ such that $n_\infty(\hat{x}) < \bar{\bar{n}}$. Then from (3.4) we get

$$\bar{\bar{n}} < \bar{\bar{n}} \max \frac{h(x)}{g(x)}, \tag{3.6}$$

thus concluding the proof by contradiction. $\square$

Corollary 3.1. *If $\gamma(x,y)$ has the form*

$$\gamma(x,y) = p(x)q(y), \quad p(x) > 0,$$

then problem (P) *has the stationary solution*

$$n_\infty(x) = N\frac{q(x)}{p(x)}\Big(\int_0^1 \frac{q(z)}{p(z)}dz\Big)^{-1}.$$

In particular the stationary solution is constant if and only if

$$q(x) = Ap(x)$$

for any positive constant A. □

REMARK 3.1. *Of course, multiplying* γ *by any positive constant just changes the time scale of the evolution but does not affect the asymptotic solution.* □

Let us define

$$\bar{g} = \min_{x\in[0,1]} g(x), \quad \bar{h} = \min_{x\in[0,1]} h(x), \quad \bar{\gamma}(x) = \min_{z\in[0,1]} \gamma(z,x), \tag{3.7}$$

and

$$\bar{\bar{g}} = \max_{x\in[0,1]} g(x), \quad \bar{\bar{h}} = \max_{x\in[0,1]} h(x), \quad \bar{\bar{\gamma}}(x) = \max_{z\in[0,1]} \gamma(z,x). \tag{3.8}$$

For any x such that $g(x) > 0$, we get from (2.8), (3.7), (3.8),

$$\begin{aligned} n_0(x)\exp(-g(x)t) + N\bar{\gamma}(x)\frac{1-\exp(-g(x)t)}{g(x)} &\le n(x,t) \\ \le n_0(x)\exp(-g(x)t) + N\bar{\bar{\gamma}}(x)&\frac{1-\exp(-g(x)t)}{g(x)}, \end{aligned}$$

so that, for the asymptotic value we have

$$\frac{N\bar{\gamma}(x)}{\bar{\bar{g}}} \le n_\infty(x) \le \frac{N\bar{\bar{\gamma}}(x)}{\bar{g}}.$$

Let us now drop the assumption $g(x) > 0$ in $[0,1]$, and consider the sets

$$A_g = \{x \in (0,1) : g(x) = 0\}, \quad A_h = \{x \in (0,1) : h(x) = 0\}.$$

Of course, if $A_g = (0,1)$ (implying also $A_h = (0,1)$) then

$$n(x,t) = n_0(x), \quad \forall t \ge 0.$$

In general we have

$$\begin{aligned} n_0(x) \le n(x,t) \le n_0(x) + N\bar{\bar{\gamma}}(x)t, &\quad x \in A_g, \\ n_0(x) \ge n(x,t) \ge n_0(x)e^{-\bar{\bar{g}}(x)t}, &\quad x \in A_h, \end{aligned}$$

(more precisely we could put instead of $\bar{\bar{\gamma}}$ and $\bar{\bar{g}}$, $\max_{z\in(0,1),x\in A_g}\gamma(z,x)$ and $\max_{x\in A_h} g(x)$, respectively). Of course if $x \in A_h \cap A_g$, it is $n(x,t) = n_0(x)$. Concerning asymptotic solution we find that if $x \in A_h$ then $n_\infty(x) = 0$, $\forall x$ such that $\bar{\gamma}(x) > 0$ (i.e. where the arrival rate is zero and the leaving rate is not, the only possible stationary solution is $n_\infty(x) = 0$).

4. Describing the economic dynamics

We introduce two definitions that may be helpful in the description of the economic dynamics of the population:

$$W_{ab}(t) = \int_a^b xn(x,t)\,dx, \tag{4.1}$$

is the cumulative richness at time t of the social class having wealth index x between a and b, $(0 \le a < b \le 1)$.

$$\hat{W}_{ab}(t) = W_{ab}(t)/N_{ab}(t), \tag{4.2}$$

is the average (per capita) richness in the same social class. Here

$$N_{ab}(t) = \int_a^b n(x,t)\,dx,$$

and clearly

$$a \le \hat{W}_{ab} \le b. \tag{4.3}$$

In particular, taking $a = 0$, $b = 1$ we have the total and average wealth of the population, $W(t)$ and $\hat{W}(t)$, respectively.

We introduce an index of economic inequality

$$i(t) = \frac{1}{N\hat{W}(t)(1-\hat{W}(t))}\int_0^1 (x-\hat{W}(t))^2 n(x,t)\,dx. \tag{4.4}$$

An alternative definition of i is

$$i(t) = \frac{\hat{W}_2(t) - \hat{W}^2(t)}{\hat{W}(t)[1-\hat{W}(t)]}, \tag{4.5}$$

where $\hat{W}_2(t) = \dfrac{1}{N}\displaystyle\int_0^1 x^2 n(x,t)\,dx$. Since $\hat{W}_2 \le \hat{W}$, from (4.5) we have $0 \le i \le 1$.

Remark 4.1. *Of course* (4.4) *only makes sense if $\hat{W}(t) \neq 0, 1$. The cases $\hat{W} = 0$, $\hat{W} = 1$ arise just when, at some time $\bar{t}$, $n(x,\bar{t}) = 2N\delta(x)$, $n(x,\bar{t}) = 2N\delta(x-1)$, respectively, where δ is the Dirac distribution. In these cases we will define $i(\bar{t})$ as the* *limit for* $\varepsilon \to 0$ *of the expression* (4.4) *where* $n(x, \bar{t})$ *is replaced by* $n_\varepsilon(x) = 2N\delta(\varepsilon)$ *or* $n_\varepsilon(x) = 2N\delta(x - 1 + \varepsilon)$, *and* $\hat{W}_\varepsilon(\bar{t})$ *is defined accordingly.* □

It is well known, since the classical paper [11], that several indexes of economic inequality have been proposed, see e.g. the Gini's paper [8] and the references in [10]. We believe that (4.4) is the most natural definition of inequality in the spirit of our approach.

We note that for a population in which all the members have the same wealth (necessarily $\hat{W}$; from now on we do not indicate time dependence explicitly, to simplify notation) we would have $n(x) = N\delta(x - \hat{W})$. In this case $i = 0$.

On the contrary, for the same total wealth $N\hat{W}$ the most unequal distribution corresponds to the case in which $N\hat{W}$ individuals have wealth index 1 and $N(1 - \hat{W})$ have wealth index 0. This corresponds to

$$n(x) = 2(1 - \hat{W})\delta(x) + 2N\hat{W}\delta(x - 1), \tag{4.6}$$

(the factor 2 is introduced since $\int_0^1 \delta(x)\, dx = \int_0^1 \delta(x - 1)\, dx = 1/2$). Hence, in this case,

$$i = \frac{1}{\hat{W}(1 - \hat{W})}[(1 - \hat{W})\hat{W}^2 + \hat{W}(1 - \hat{W})^2] = 1. \tag{4.7}$$

To see, on some other examples, how index i depends on the distribution function $n(x)$, we consider the case

$$n(x) = N\alpha x^a (1 - x)^b, \tag{4.8}$$

where $a, b \in \mathbb{N}$ and where α is a normalization constant such that $\int_0^1 n(x)\, dx = N$. After some lengthy algebra we get

PROPOSITION 4.1. *If* $n(x)$ *is given by* (4.8), *then the inequality index* i *has the form*

$$i = \frac{1}{a + b + 3}. \tag{4.9}$$ □

To provide some numerical simulations, let us take the mobility function

$$\gamma(x, y) = \beta(x)\theta(|x - y|)H(x - y) + \alpha(x)\theta(|x - y|)H(y - x) \tag{4.10}$$

where

- H is the Heaviside function,
- $\alpha(x) = (1 - x)$,
- $\beta(x) = x$,
- $\theta(z) = e^{-z^2/2\sigma^2}$ (with $\sigma = 0.3$, $z = |x - y|$) is a function modulating the kernel with the distance between x and y.

In (4.10) α denotes the social promotion from class x to class y and β the social relegation. Both are assumed to depend (linearly) just on the wealth of class x. The mobility is weighted by a function θ that is decreasing with respect to $|x-y|$.

Let us take four different initial conditions (with the same initial population $N = 1/6$)

$$n_1(x) = \frac{1}{3s\sqrt{2\pi}} e^{-x^2/2s^2}, \quad s = 0.2, \tag{4.11}$$

$$n_2(x) = \frac{1}{6s\sqrt{2\pi}} e^{-(x-0.5)^2/2s^2}, \quad s = 0.2, \tag{4.12}$$

$$n_3(x) = \begin{cases} -a(x-x_0)/x_0, & 0 < x \le x_0, \\ b(x-(1-x_0))/x_0, & (1-x_0) \le x \le 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{4.13}$$

$$\textit{with } x_0 = 1/8,\ a = 2,\ b = (1/3 - a x_0)/x_0,$$

$$n_4(x) = \frac{1}{6}(1 + \sin(4\pi x)), \tag{4.14}$$

and compute $n(x,t)$. We use a finite difference scheme, with an explicit forward method in time. The integrals appearing in the equations are solved by the trapezoidal rule integration method. The computation shows that the equilibrium solution corresponding to the four initial conditions coincide (Fig. 1). On the other hand, the evolution of $i(t)$ and $W(t)$ are obviously different (Fig. 2, 3).

Coming back to definitions (4.1) and (4.2), we find

$$\dot{\hat{W}}_{ab} = \frac{1}{N_{ab}}[\dot{W}_{ab} - \hat{W}_{ab}\dot{N}_{ab}] = \frac{1}{N_{ab}} \int_a^b (x - \hat{W}_{ab}) n_t \, dx. \tag{4.15}$$

This means that in order to get an increase of the per capita wealth of the class considered it is sufficient that the number of the individuals with wealth index less then the average level of the class is increasing with time, and vice-versa.

Another way of visualizing the economic dynamics is to define the time dependent wealth index $x_p(t)$ such that the fraction p (the p-quantile) of the total population is the class $(0, x_p(t))$ at time t, i.e.[1]

$$\int_0^{x_p(t)} n(x,t)\, dx = pN. \tag{4.16}$$

Moreover, since

[1] Alternatively, one could fix $k \in (0,1]$ and define $p_k(t)$ such that $\int_0^k n(x,t)\,dx = p_k(t)N$.

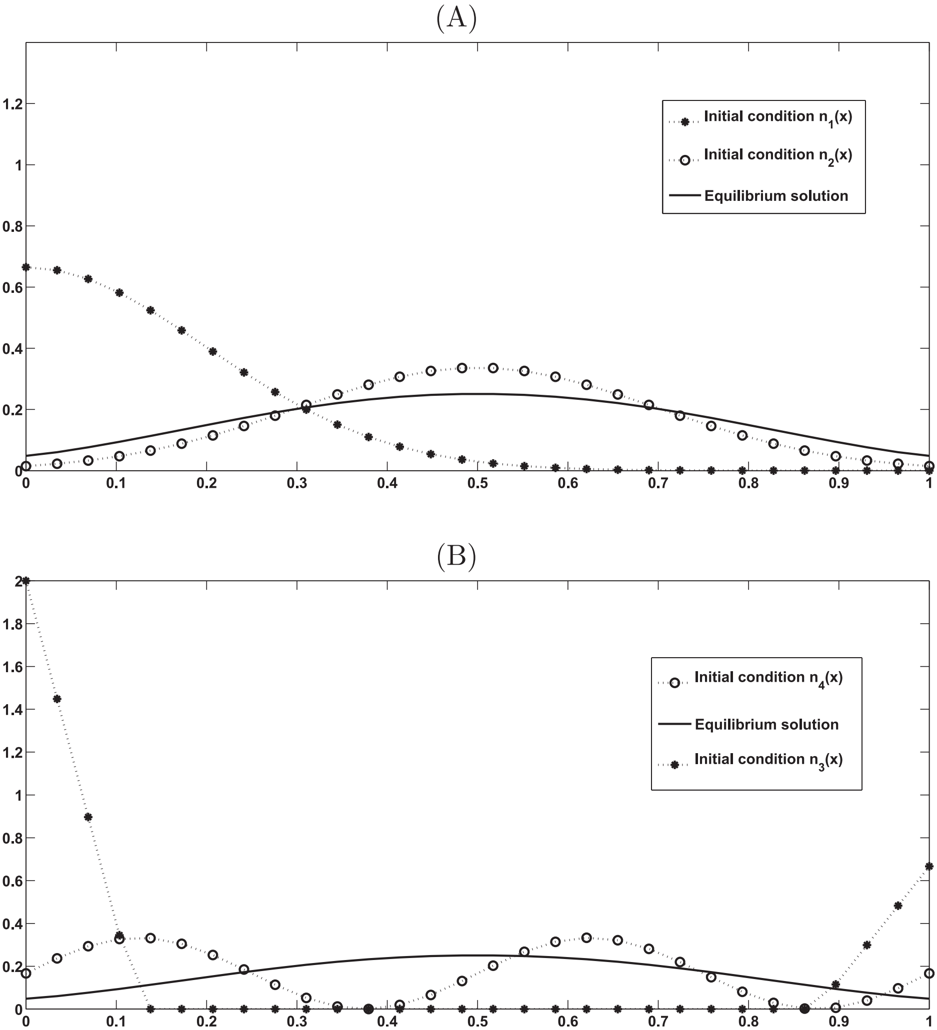

Figure 1: Reaching the equilibrium solution from different initial data. The equilibrium solution (solid line) is the same in all cases, which have been split in two figures only for the reader convenience. (A): Initial conditions (4.11) and (4.12). (B): Initial conditions (4.13) and (4.14).

$$n(x_p(t), t)\dot{x}_p(t) = \int_0^{x_p(t)} n(x,t)g(x)\,dx - \int_0^{x_p(t)} dx \int_0^1 \gamma(y,x)n(y,t)\,dy$$

it is immediately seen that a sufficient condition for the positivity of $\dot{x}_p(t)$ is

$$p\bar{g} > \int_0^{x_p(t)} \bar{\bar{\gamma}}(x)\,dx. \tag{4.17}$$

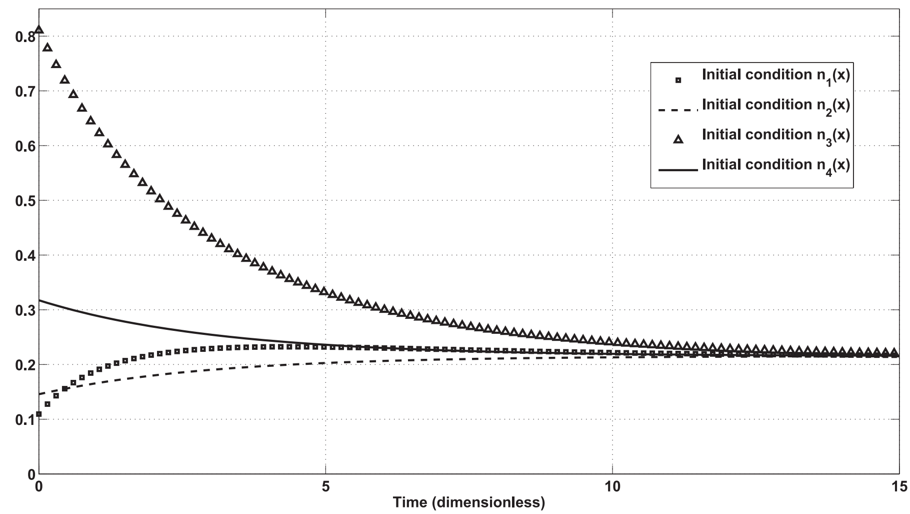

Figure 2: Inequality index computed using the mobility function (4.10) and initial conditions (4.11), (4.12), (4.13) and (4.14).

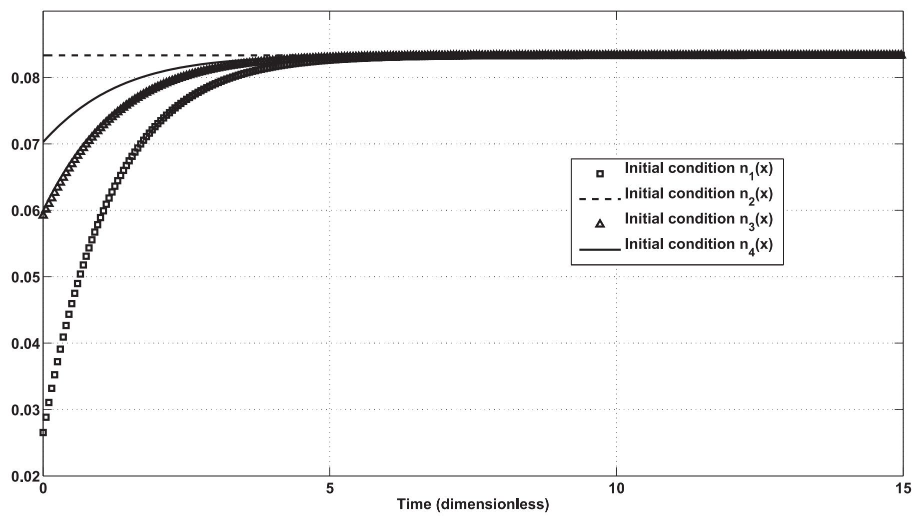

Figure 3: Total wealth $W(t)$ corresponding to initial conditions (4.11), (4.12), (4.13) and (4.14).

5. Inverse problems

It can be worth considering an inverse problem for the integro-differential equation (2.1) i.e. to look for a function $\tilde{\gamma}(x, y)$ such that (2.1) admits a given stationary solution $\tilde{n}(x)$. Of course, we do not expect that such problem is uniquely solvable (apart form the fact that $\tilde{\gamma}$ is in any case defined up to a multiplicative constant). This is obvious for the case $\tilde{n} = N$ (see Theorem 3.1) and is clearly shown in a general case

Proposition 5.1. *Given* $\tilde{n}(x) \geq 0$, *for any*

$$\tilde{\gamma}(x, y) = p(x)p(y)\tilde{n}(y), \tag{5.1}$$

where $p(x)$ *is an arbitrary positive function, equation* (2.1) *has the time-independent solution* $\tilde{n}(x)$. □

For a numerical check, let us take $n_3(x)$ (see (4.13)) to play the role of target equilibrium solution and prove that taking

$$\tilde{\gamma}(x, y) = p(x)p(y)n_3(y)$$

for any positive $p(x)$, problem (P) has the asymptotic solution $n_3(x)$, for any initial datum. Fig. 4 and 5 verify this fact, taking $n(x, 0) = n_4(x)$ as initial condition and taking two different functions $p(x)$.

From a practical point of view it makes sense to assume that the social mobility is the sum of a given function $\gamma_0(x, y) \geq 0$ and of a control function $\gamma_c(x, y)$

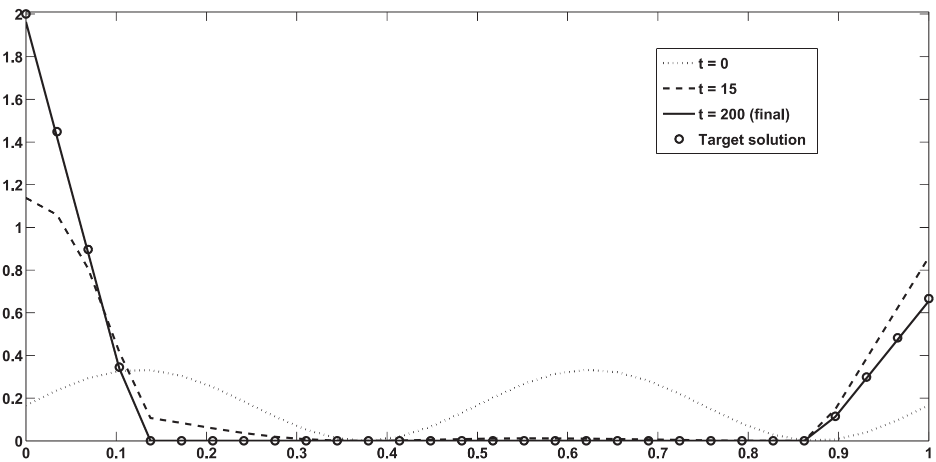

Figure 4: An example of result stated in Proposition 5.1, with $n(x, 0) = n_4(x)$ and $p(x) = x + 0.1$.

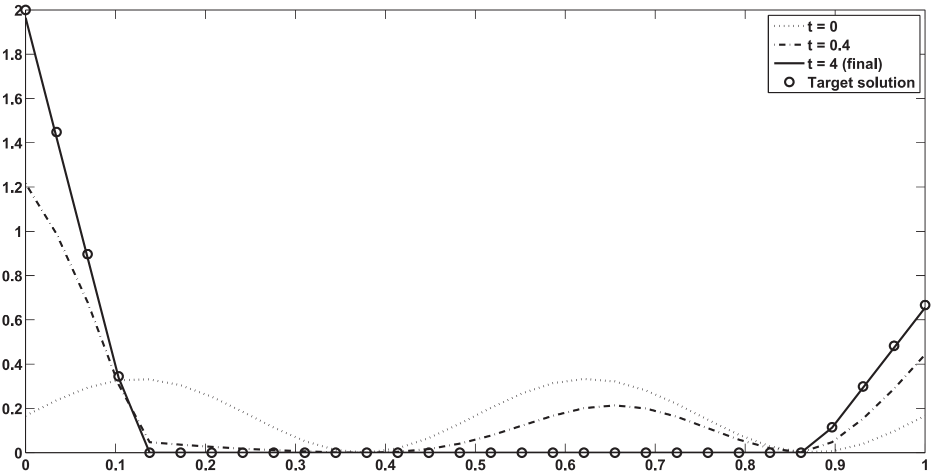

Figure 5: An example of result stated in Proposition 5.1, with $n(x,0) = n_4(x)$ and $p(x) = 1.5 + \sin(2\pi x)$.

that is superimposed by some "authority" to reach or to approach a desired stationary solution

$$\gamma(x, y) = \gamma_0(x, y) + \gamma_c(x, y). \tag{5.2}$$

Let us assume a target stationary solution

$$\tilde{n}(x) > 0, \tag{5.3}$$

and define

$$S(x) = \frac{1}{\tilde{n}(x)}\left[\tilde{n}(x)\int_0^1 \gamma_0(x, y)\,dy - \int_0^1 \gamma_0(z, x)\tilde{n}(z)\,dz\right], \tag{5.4}$$

so that if $\tilde{n}$ is a stationary solution of problem (P) it is

$$\int_0^1 \gamma_c(x, y)\,dy = -S(x) + \frac{1}{\tilde{n}(x)}\int_0^1 \gamma_c(z, x)\tilde{n}(z)\,dz. \tag{5.5}$$

We consider the example in which $\gamma_c(x, y)$ is supposed to be dependent on just one of the two variables.

Proposition 5.2. *If γ_c depends on the first variable, the function*

$$\gamma_c(x) = -S(x) + \frac{c}{\tilde{n}(x)}, \tag{5.6}$$

(*where* c *is any constant such that* $\gamma_c \geq 0$) *is such that* (2.1) *has the stationary solution* $\tilde{n}(x)$. (*Just note that* $\int_0^1 S(x)\tilde{n}(x)\,dx = 0$). □

To check this result on a special case let us choose

$$\tilde{n}(x) = x(1-x) + 0.1,$$

and take

$$\gamma_0(x, y) = p(x)p(y)(n_3(y) + 0.1),$$

where e.g. $p(x) = x + 0.1$.

According to Proposition 5.1 the equilibrium solution corresponding to γ_0 is $n_3(x) + 0.1$. But if we take

$$\gamma_c(x) = -S(x) + \frac{0.1}{\tilde{n}(x)},$$

we find (Fig. 6) that the equilibrium solution is $\tilde{n}(x)$ (in Fig. 6, $n_1(x)$ defined in (4.11) was taken as initial condition, to be specific).

Example 5.1. *If* γ_c *depends just on the second variable, then any positive function of the form*

$$\gamma_c(y) = \frac{\tilde{n}(y)}{N}(C + S(y)), \quad C = const. \tag{5.7}$$

is such that (2.1) *has the solution* $\tilde{n}(x)$. □

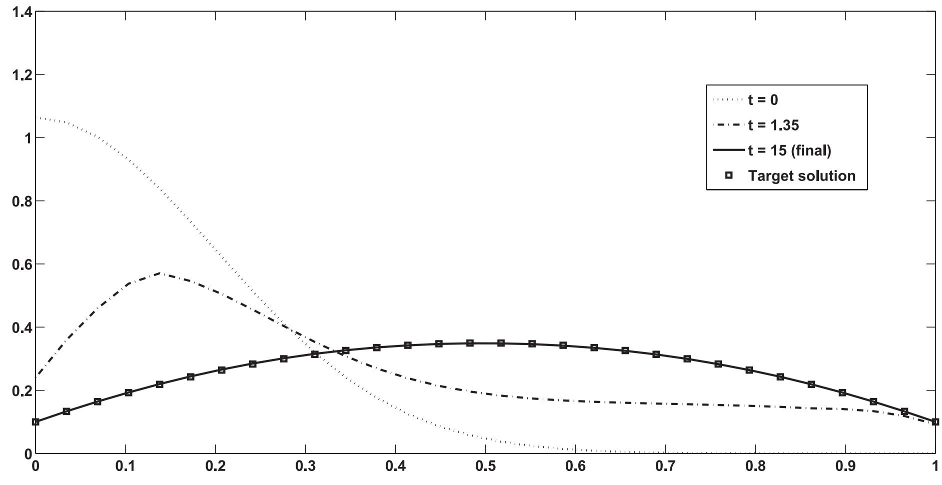

Figure 6: An example of result stated in Example 5.1.

Another particular case is provided by the following

EXAMPLE 5.2. *Look for some $\gamma_c(x, y)$ of the form*

$$\gamma_c(x, y) = \lambda(y + \psi(x)), \tag{5.8}$$

where λ is a positive constant and ψ is a function to be determined. In this case (5.5) *reads as*

$$\tilde{n}(x)\psi(x) = -\frac{\tilde{n}(x)}{2} - \frac{S(x)}{\lambda}\tilde{n}(x) + Nx + \int_0^1 \tilde{n}(z)\psi(z)\,dz \equiv Q(x) + \int_0^1 \tilde{n}(z)\psi(z)\,dz.$$

Noting that $\int_0^1 Q(x)\,dx = 0$, we find

$$\psi(x) = \frac{Q(x)}{\tilde{n}(x)} + K,$$

where K is any constant such that $\psi(x) \geq 0$. □

EXAMPLE 5.3. *Look for a $\gamma_c(x, y)$ of the form*

$$\gamma_c(x, y) = \beta(x)H(x - y) + \alpha(x)H(y - x),$$

corresponding to promotion $\alpha(x)$ and relegation $\beta(x)$ depending on the social index x. Now (5.5) *gives*

$$\tilde{n}(x)x\beta(x) + \tilde{n}(x)\alpha(x)(1 - x) = -\tilde{n}(x)S(x) + \int_0^x \alpha(z)\tilde{n}(z)\,dz + \int_x^1 \beta(z)\tilde{n}(z)\,dz.$$

If we suppose that $\tilde{n}(x)$ and $\alpha(x)$ are given and look for the unknown function $F(x) = \beta(x)\tilde{n}(x)$ we find

$$xF(x) = \hat{Q}(x) + \int_x^1 F(z)\,dz$$

where

$$\hat{Q}(x) = -\tilde{n}(x)[S(x) + \alpha(x)(1 - x)] + \int_0^x \alpha(z)\tilde{n}(z)\,dz.$$

It is easily seen that if $\hat{Q}(x) = \mathcal{O}(x)$ a nonsingular solution is given by

$$F(x) = \frac{\hat{Q}(x)}{x} + \frac{1}{x^2}\int_x^1 \hat{Q}(\xi)\xi\,d\xi.$$ □

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