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Partial Differential Equations — *A note on the monotonicity formula of Caffarelli-Jerison-Kenig*, by Bozhidar Velichkov, communicated on 14 February 2014.

Abstract. — The aim of this note is to prove the monotonicity formula of Caffarelli-Jerison-Kenig for functions, which are not necessarily continuous. We also give a detailed proof of the multiphase version of the monotonicity formula in any dimension.

Key words: Two-phase, multiphase, monotonicity formula, free boundary.

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1. Introduction

The Alt-Caffarelli-Friedman monotonicity formula is one of the most powerful tools in the study of the regularity of multiphase optimization problems as, for example, optimal partition problems for functionals involving some partial differential equation, a prototype being the multiphase Alt-Caffarelli problem

$$\min\left\{\sum_{i=1}^{m}\int_{\Omega}|\nabla u_i|^2 - f_i u_i + Q^2\mathbb{1}_{\{u_i>0\}}\,dx : (u_1,\dots,u_m)\in\mathscr{A}(\Omega)\right\}, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$ is a given (Lipschitz) bounded open set, $Q:\Omega\to\mathbb{R}$ is a measurable function, $f_1,\dots,f_m\in L^\infty(\Omega)$ and the admissible set $\mathscr{A}(\Omega)$ is given by

$$\begin{aligned}\mathscr{A}(\Omega) := \{&(u_1,\dots,u_m)\in[H^1(\Omega)]^m : u_i\geq 0, u_i = c \text{ on } \partial\Omega,\\ &u_iu_j = 0 \text{ a.e. on } \Omega, \forall i\neq j\},\end{aligned} \tag{1.2}$$

where $c\geq 0$ is a given constant.

Remark 1.1. • If $Q=0$, then we have a classical optimal partition problem as the ones studied in [8], [10], [11], [12] and [15].

- If $c=1$, $m=1$, $f_1=0$ and $0<a\leq Q^2\leq b<+\infty$, then (1.1) reduces to the problem considered in [1].
- If $m=1$, $Q\equiv 1$, $f_1=f$ and $f_2=-f$, then the solution of (1.1) is given by

$$u_1^* = u_+^* := \sup\{u^*,0\},\quad u_2^* = u_-^* := \sup\{-u^*,0\},$$

where $u^* \in H_0^1(\Omega)$ is a solution of the following problem, considered in [4],

$$\min\left\{\int_\Omega |\nabla u|^2 - fu\,dx + |\{u \neq 0\}| : u \in H_0^1(\Omega)\right\}.$$

- If, $Q \equiv 1$ and $f_1 = \cdots = f_m = f$, then (1.1) reduces to a problem considered in [6] and [3].

One of the main tools in the study of the Lipschitz continuity of the solutions $(u_1^*, \dots, u_m^*)$ of the multiphase problem (1.1) is the monotonicity formula, which relates the behaviour of the different phases u_i^* in the points on the common boundary $\partial\{u_i^* > 0\} \cap \partial\{u_j^* > 0\}$, the main purpose being to provide a bound for the gradients $|\nabla u_i^*|$ and $|\nabla u_j^*|$ in these points. The following estimate was proved in [7], as a generalization of the monotonicity formula from [2], and was widely used (for example in [4] and also [5]) in the study of free-boundary problems.

Theorem 1.2 (Caffarelli-Jerison-Kenig). *Let $B_1 \subset \mathbb{R}^d$ be the unit ball in R^d and let $u_1, u_2 \in H^1(B_1)$ be non-negative and continuous functions such that*

$$\Delta u_i + 1 \geq 0, \quad \text{for } i = 1, 2, \qquad \text{and} \qquad u_1 u_2 = 0 \quad \text{on } B_1.$$

Then there is a dimensional constant C_d such that for each $r \in (0,1)$ we have

$$\prod_{i=1}^{2}\left(\frac{1}{r^2}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right) \leq C_d\left(1 + \sum_{i=1}^{2}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\right)^2. \tag{1.3}$$

The aim of this paper is to show that the continuity assumption in Theorem 1.2 can be dropped (Theorem 3.1) and to provide the reader with a detailed proof of the multiphase version (Theorem 4.1 and Corollary 4.2) of Theorem 1.2, which was proved in [6]. We note that the proof of Theorem 3.1 follows precisely the one of Theorem 1.2 given in [7]. We report the estimates, in which the continuity assumption was used, in Section 2 and we adapt them, essentially by approximation, to the non-continuous case.

A strong initial motivation was provided by the multiphase version of the Alt-Caffarelli-Friedman monotonicity formula, proved in [10] in the special case of sub-harmonic functions u_i in $\mathbb{R}^2$ and also for non-linear eigenfunctions, for which the same technique can be applied. The argument from [10] allows to avoid the continuity assumption and applies also in the presence of more phases. As a conclusion of the Introduction section, we give the proof of this result, which has the advantage of avoiding the technicalities present in the general case, emphasising the stronger decay of the gradients in the multiphase case and showing that the continuous assumption is unnecessary.

Theorem 1.3 (Alt-Caffarelli-Friedman; Conti-Terracini-Verzini). *Consider the unit ball $B_1 \subset \mathbb{R}^2$ and let $u_1, \dots, u_m \in H^1(B_1)$ be m non-negative subharmonic* *functions such that* $\int_{\mathbb{R}^2} u_i u_j \, dx = 0$, *for every choice of different indices* $i, j \in \{1, \dots, m\}$. *Then the function*

$$\Phi(r) = \prod_{i=1}^{m} \Big(\frac{1}{r^m} \int_{B_r} |\nabla u_i|^2 \, dx \Big) \tag{1.4}$$

is non-decreasing on $[0, 1]$. *In particular,*

$$\prod_{i=1}^{m} \Big(\frac{1}{r^m} \int_{B_r} |\nabla u_i|^2 \, dx \Big) \leq \Big(\int_{B_1} |\nabla u_1|^2 \, dx + \cdots + \int_{B_1} |\nabla u_m|^2 \, dx \Big)^m. \tag{1.5}$$

PROOF. The function Φ is of bounded variation and calculating its derivative we get

$$\frac{\Phi'(r)}{\Phi(r)} \geq -\frac{m^2}{r} + \sum_{i=1}^{m} \frac{\int_{\partial B_r} |\nabla u_i|^2 \, d\mathcal{H}^1}{\int_{B_r} |\nabla u_i|^2 \, dx}. \tag{1.6}$$

We now prove that the right-hand side is positive for every $r \in (0, 1)$ such that $u_i \in H^1(\partial B_r)$, for every $i = 1, \dots, m$, and $\int_{\partial B_r} u_i u_j \, d\mathcal{H}^1 = 0$, for every $i \neq j \in \{1, \dots, m\}$. We use the sub-harmonicity of u_i to calculate

$$\int_{B_r} |\nabla u_i|^2 \, dx \leq \int_{\partial B_r} u_i \frac{\partial u_i}{\partial n} d\mathcal{H}^1 \leq \Big(\int_{\partial B_r} u_i^2 \, d\mathcal{H}^1 \Big)^{\frac{1}{2}} \Big(\int_{\partial B_r} |\nabla_n u_i|^2 \, d\mathcal{H}^1 \Big)^{\frac{1}{2}}, \tag{1.7}$$

and decomposing the gradient ∇u_i in the tangent and normal parts $\nabla_\tau u_i$ and $\nabla_n u_i$, we have

$$\begin{aligned} \int_{\partial B_r} |\nabla u_i|^2 \, d\mathcal{H}^1 &= \int_{\partial B_r} |\nabla_n u_i|^2 \, d\mathcal{H}^1 + \int_{\partial B_r} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1 \\ &\geq 2 \Big(\int_{\partial B_r} |\nabla_n u_i|^2 \, d\mathcal{H}^1 \Big)^{\frac{1}{2}} \Big(\int_{\partial B_r} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1 \Big)^{\frac{1}{2}}. \end{aligned} \tag{1.8}$$

Putting together (1.7) and (1.8), we obtain

$$\frac{\int_{\partial B_r} |\nabla u_i|^2 \, d\mathcal{H}^1}{\int_{B_r} |\nabla u_i|^2 \, dx} \geq 2 \Big(\frac{\int_{\partial B_r} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1}{\int_{\partial B_r} u_i^2 \, d\mathcal{H}^1} \Big)^{\frac{1}{2}} \geq 2\sqrt{\lambda_1(\partial B_r \cap \Omega_i)}, \tag{1.9}$$

where we use the notation $\Omega_i := \{u_i > 0\}$ and for an $\mathcal{H}^1$-measurable set $\omega \subset \partial B_r$ we define

$$\lambda_1(\omega) := \min\left\{ \frac{\int_{\partial B_r} |\nabla_\tau v|^2 \, d\mathcal{H}^1}{\int_{\partial B_r} v^2 \, d\mathcal{H}^1} : v \in H^1(\partial B_r),\ \mathcal{H}^1(\{v \neq 0\} \setminus \omega) = 0 \right\}.$$

By a standard symmetrization argument, we have $\lambda_1(\omega) \geq \left(\frac{\pi}{\mathcal{H}^1(\omega)}\right)^2$ and so, by (1.6) and the mean arithmetic-mean harmonic inequality, we obtain the estimate

$$\frac{\Phi'(r)}{\Phi(r)} \geq -\frac{m^2}{r} + \sum_{i=1}^{m} \frac{2\pi}{\mathcal{H}^1(\partial B_r \cap \Omega_i)} \geq 0,$$

which concludes the proof. □

2. Preliminary results on the monotonicity factors

In this section we consider non-negative functions $u \in H^1(B_2)$ such that

$$\Delta u + 1 \geq 0 \quad \text{weakly in } [H_0^1(B_2)]',$$

and we study the energy functional

$$A_u(r) := \int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}} \, dx,$$

for $r \in (0, 1)$, which is precisely the quantity that appears in (3.2) and (4.1). We start with a lemma, which was first proved in [7, Remark 1.5].

Lemma 2.1. *Suppose that $u \in H^1(B_2)$ is a non-negative Sobolev function such that $\Delta u + 1 \geq 0$ on $B_2 \subset \mathbb{R}^d$. Then, there is a dimensional constant C_d such that*

$$\int_{B_1} \frac{|\nabla u|^2}{|x|^{d-2}} \, dx \leq C_d \left(1 + \int_{B_2 \setminus B_1} u^2 \, dx\right). \tag{2.1}$$

Proof. Let $u_\varepsilon = \phi_\varepsilon * u$, where $\phi_\varepsilon \in C_c^\infty(B_\varepsilon)$ is a standard molifier. Then $u_\varepsilon \to u$ strongly in $H^1(B_2)$, $u_\varepsilon \in C^\infty(B_2)$ and $\Delta u_\varepsilon + 1 \geq 0$ on $B_{2-\varepsilon}$. We will prove (2.1) for u_ε. We note that a brief computation gives the inequality

$$\Delta(u_\varepsilon^2) = 2|\nabla u_\varepsilon|^2 + 2u_\varepsilon \Delta u_\varepsilon \geq 2|\nabla u_\varepsilon|^2 - 2u_\varepsilon \quad \text{in } [H_0^1(B_{2-\varepsilon})]'. \tag{2.2}$$

We now choose a positive and radially decreasing function $\phi \in C_c^\infty(B_{3/2})$ such that $\phi = 1$ on B_1. By (2.2) we get

$$
\begin{aligned}
(2.3)\quad 2\int_{B_{3/2}} \frac{\phi(x)|\nabla u_\varepsilon|^2}{|x|^{d-2}}\,dx &\le \int_{B_{3/2}} \phi(x)\frac{2u_\varepsilon + \Delta(u_\varepsilon^2)}{|x|^{d-2}}\,dx \\
&= \int_{B_{3/2}} 2\frac{\phi(x)u_\varepsilon}{|x|^{d-2}} + u_\varepsilon^2\Delta\Big(\frac{\phi(x)}{|x|^{d-2}}\Big)\,dx \\
&= \int_{B_{3/2}} \bigg\{2\frac{\phi(x)u_\varepsilon}{|x|^{d-2}} + u_\varepsilon^2\frac{\Delta\phi(x)}{|x|^{d-2}} \\
&\qquad + u_\varepsilon^2\nabla\phi(x)\cdot\nabla(|x|^{2-d})\bigg\}\,dx - C_d u_\varepsilon^2(0) \\
&\le 2\int_{B_{3/2}} \frac{\phi(x)u_\varepsilon}{|x|^{d-2}}\,dx + C_d\int_{B_2\setminus B_1} u_\varepsilon^2\,dx.
\end{aligned}
$$

Thus, in order to obtain (2.1), it is sufficient to estimate the norm $\|u_\varepsilon\|_{L^\infty(B_1)}$ with the r.h.s. of (2.1). To do that, we first note that since $\Delta(u_\varepsilon(x) + |x|^2/2d) \ge 0$, we have

$$
(2.4)\qquad \max_{x\in B_1}\{u_\varepsilon(x) + |x|^2/2d\} \le C_d + C_d \fint_{\partial B_r} u_\varepsilon\, d\mathcal{H}^{d-1}, \quad \forall r \in (3/2, 2-\varepsilon),
$$

and, after integration in r and the Cauchy-Schwartz inequality, we get

$$
(2.5)\qquad \|u_\varepsilon\|_{L^\infty(B_1)} \le C_d + C_d\Big(\int_{B_2\setminus B_1} u_\varepsilon^2\,dx\Big)^{1/2},
$$

which, together with (2.3), gives (2.1). □

Remark 2.2. For a non-negative function $u \in H^1(B_r)$, satisfying

$$
\Delta u + 1 \ge 0 \quad \text{in } [H_0^1(B_r)]',
$$

we denote with $A_u(r)$ the quantity

$$
(2.6)\qquad A_u(r) := \int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx < +\infty.
$$

- The function $r \mapsto A_u(r)$ is bounded and increasing in r.
- We note that $A_u(r)$ is invariant with respect to the rescaling $u_r(x) := u(rx)$. Indeed, for any $0 < r \le 1$ we have

$$
\Delta u_r + 1 \ge 0 \quad \text{and} \quad A_{u_r}(1) = A_u(r).
$$

The next result is implicitly contained in [7, Lemma 2.8] and it is the point in which the continuity of u_i was used. The inequality (2.7) is the analogue of the estimate (1.9), which is the main ingredient of the proof of Theorem 1.3.

Lemma 2.3. *Let $u \in H^1(B_2)$ be a non-negative function such that $\Delta u + 1 \geq 0$ on B_2. Then for Lebesgue almost every $r \in (0,1)$ we have the estimate*

$$\frac{1}{r^4}\int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx \leq C_d\Big(1 + \frac{r^{-2}}{\sqrt{\lambda(u,r)}}\Big(\fint_{\partial B_r} |\nabla u|^2\,d\mathcal{H}^{d-1}\Big)^{\frac{1}{2}}\Big) + \frac{d\omega_d r^{-3}}{2\alpha(u,r)}\fint_{\partial B_r} |\nabla u|^2\,d\mathcal{H}^{d-1}, \tag{2.7}$$

where

$$\lambda(u,r) := \min\left\{\frac{\int_{\partial B_r} |\nabla v|^2\,d\mathcal{H}^{d-1}}{\int_{\partial B_r} v^2\,d\mathcal{H}^{d-1}} : v \in H^1(\partial B_r),\ \text{such that}\ \mathcal{H}^{d-1}(\{v \neq 0\} \cap \{u = 0\}) = 0\right\}, \tag{2.8}$$

and $\alpha(u,r) \in \mathbb{R}^+$ is the characteristic constant of $\{u > 0\} \cap \partial B_r$, i.e. the non-negative solution of the equation

$$\alpha(u,r)\Big(\alpha(u,r) + \frac{d-2}{r}\Big) = \lambda(u,r). \tag{2.9}$$

Proof. We start by determining the subset of the interval $(0,1)$ for which we will prove that (2.7) holds. Let $u_\varepsilon := u * \phi_\varepsilon$, where ϕ_ε is a standard molifier. Then we have that:

(i) for almost every $r \in (0,1)$ the restriction of u to ∂B_r is Sobolev. i.e. $u_{|\partial B_r} \in H^1(\partial B_r)$;
(ii) for almost every $r \in (0,1)$ the sequence of restrictions $(\nabla u_\varepsilon)_{|\partial B_r}$ converges strongly in $L^2(\partial B_r;\mathbb{R}^d)$ to $(\nabla u)_{|\partial B_r}$.

We now consider $r \in (0,1)$ such that both (i) and (ii) hold. Using the scaling $u_r(x) := r^{-2}u(rx)$, we have that

$$\fint_{\partial B_r} |\nabla u|^2\,d\mathcal{H}^{d-1} = r^2\fint_{\partial B_1} |\nabla u_r|^2\,d\mathcal{H}^{d-1},$$

$$\frac{1}{r^4}\int_{B_r} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx = \int_{B_1} \frac{|\nabla u_r|^2}{|x|^{d-2}}\,dx,$$

$$\alpha(u_r,1) = r\alpha(u,r) \quad \text{and} \quad \lambda(u_r,1) = r^2\lambda(u,r).$$

Substituting in (2.7), we can suppose that $r = 1$ and set $\alpha := \alpha(u, 1)$ and $\lambda := \lambda(u, 1)$.

If $\mathscr{H}^{d-1}(\{u = 0\} \cap \partial B_1) = 0$, then $\lambda = 0$. Now if $\int_{\partial B_1} |\nabla u|^2 \, d\mathscr{H}^{d-1} > 0$, then the inequality (2.7) is trivial. If on the other hand, $\int_{\partial B_1} |\nabla u|^2 \, d\mathscr{H}^{d-1} = 0$, then u is a constant on ∂B_1 and so, we may suppose that $u = 0$ on $\mathbb{R}^d \setminus B_1$, which again gives (2.7), by choosing C_d large enough. Thus, it remains to prove the Lemma in the case $\mathscr{H}^{d-1}(\{u = 0\} \cap \partial B_1) > 0$.

We first note that since $\mathscr{H}^{d-1}(\{u = 0\} \cap \partial B_1) > 0$, the eigenvalue λ is strictly positive. Using the restriction of u on ∂B_1 as a test function in (2.8) we get

$$\lambda \int_{\partial B_1} u^2 \, d\mathscr{H}^{d-1} \leq \int_{\partial B_1} |\nabla_\tau u|^2 \, d\mathscr{H}^{d-1},$$

where ∇_τ is the tangential gradient on ∂B_1. In particular, we have

$$\lambda \int_{\partial B_1} u^2 \, d\mathscr{H}^{d-1} \leq \int_{\partial B_1} |\nabla_\tau u|^2 \, d\mathscr{H}^{d-1} \leq \int_{\partial B_1} |\nabla u|^2 \, d\mathscr{H}^{d-1} =: B_u(1). \tag{2.10}$$

For every $\varepsilon > 0$, using the inequality

$$\Delta(u_\varepsilon^2) = 2u_\varepsilon \Delta u_\varepsilon + 2|\nabla u_\varepsilon|^2 \geq -2u_\varepsilon + 2|\nabla u_\varepsilon|^2,$$

and the fact that $\Delta(u_\varepsilon + |x|^2/2d) \geq 0$, we have

$$\begin{aligned} 2\int_{B_1} \frac{|\nabla u_\varepsilon|^2}{|x|^{d-2}} dx &\leq \int_{B_1} \frac{2u_\varepsilon + \Delta(u_\varepsilon^2)}{|x|^{d-2}} dx \\ &\leq C_d + C_d \Big(\int_{\partial B_1} u_\varepsilon^2 \, d\mathscr{H}^{d-1} \Big)^{1/2} + \int_{B_1} \frac{\Delta(u_\varepsilon^2)}{|x|^{d-2}} dx. \end{aligned} \tag{2.11}$$

We now estimate the last term on the right-hand side.

$$\begin{aligned} \int_{B_1} \frac{\Delta(u_\varepsilon^2)}{|x|^{d-2}} dx &= \int_{B_1} \Delta(|x|^{2-d}) u_\varepsilon^2 \, dx \\ &\quad + \int_{\partial B_1} \left[\frac{\partial(u_\varepsilon^2)}{\partial n} |x|^{2-d} - \frac{\partial(|x|^{2-d})}{\partial n} u_\varepsilon^2 \right] d\mathscr{H}^{d-1} \\ &\leq -d(d-2)\omega_d u_\varepsilon^2(0) + \int_{\partial B_1} 2u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\mathscr{H}^{d-1} \\ &\quad + (d-2) \int_{\partial B_1} u_\varepsilon^2 \, d\mathscr{H}^{d-1} \\ &\leq \int_{\partial B_1} 2u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} d\mathscr{H}^{d-1} + (d-2) \int_{\partial B_1} u_\varepsilon^2 \, d\mathscr{H}^{d-1}, \end{aligned} \tag{2.12}$$

where we used that $-\Delta(|x|^{2-d}) = d(d-2)\omega_d\delta_0$ (see for example [13, Section 2.2.1]). Since (ii) holds, we may pass to the limit in (2.11) and (2.12), as $\varepsilon \to 0$. Using (2.10) we obtain the inequality

$$\begin{aligned}
2\int_{B_1} \frac{|\nabla u|^2}{|x|^{d-2}}\,dx &\le C_d + C_d\Big(\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1}\Big)^{1/2} + (d-2)\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1} \\
&\quad + 2\Big(\int_{\partial B_1} u^2\,d\mathcal{H}^{d-1}\Big)^{\frac{1}{2}}\Big(\int_{\partial B_1}\Big|\frac{\partial u}{\partial n}\Big|^2\,d\mathcal{H}^{d-1}\Big)^{\frac{1}{2}} \\
&\le C_d + C_d\sqrt{\frac{B_u(1)}{\lambda}} + \frac{1}{\alpha}\int_{\partial B_1}\Big|\frac{\partial u}{\partial n}\Big|^2\,d\mathcal{H}^{d-1} \\
&\quad + \frac{\alpha + (d-2)}{\lambda}\int_{\partial B_1}\Big|\frac{\partial u}{\partial \tau}\Big|^2\,d\mathcal{H}^{d-1} \\
&= C_d + C_d\sqrt{\frac{B_u(1)}{\lambda}} + \frac{B_u(1)}{\alpha},
\end{aligned}$$

where the last equality is due to the definition of α from (2.9). □

3. The two-phase monotonicity formula

In this section we prove the Caffarelli-Jerison-Kenig monotonicity formula for Sobolev functions. We follow precisely the proof given in [7], since the only estimates, where the continuity of u_i was used are now isolated in Lemma 2.1 and Lemma 2.3.

Theorem 3.1 (Two-phase monotonicity formula). *Let $B_1 \subset \mathbb{R}^d$ be the unit ball in $\mathbb{R}^d$ and $u_1, u_2 \in H^1(B_1)$ be two non-negative Sobolev functions such that*

$$\Delta u_i + 1 \ge 0, \quad \text{for } i = 1,2, \qquad \text{and} \qquad u_1u_2 = 0 \quad \text{a.e. in } B_1. \tag{3.1}$$

Then there is a dimensional constant C_d such that for each $r \in (0,1)$ we have

$$\prod_{i=1}^{2}\Big(\frac{1}{r^2}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\Big) \le C_d\Big(1 + \sum_{i=1}^{2}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx\Big)^2. \tag{3.2}$$

For the sake of simplicity of the notation, for $i = 1,2$ and u_1, u_2 as in Theorem 3.1, we set

$$A_i(r) := A_{u_i}(r) = \int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx. \tag{3.3}$$

In the next Lemma we estimate the derivative (with respect to r) of the quantity that appears in the left-hand side of (3.2) from Theorem 3.1.

Lemma 3.2. *Let u_1 and u_2 be as in Theorem 3.1. Then there is a dimensional constant $C_d > 0$ such that the following implication holds: if $A_1(1/4) \geq C_d$ and $A_2(1/4) \geq C_d$, then*

$$\frac{d}{dr}\left[\frac{A_1(r)A_2(r)}{r^4}\right] \geq -C_d\Big(\frac{1}{\sqrt{A_1(r)}}+\frac{1}{\sqrt{A_2(r)}}\Big)\frac{A_1(r)A_2(r)}{r^4},$$

for Lebesgue almost every $r \in [1/4, 1]$.

Proof. We set, for $i = 1, 2$ and $r > 0$,

$$B_i(r) = \int_{\partial B_r} |\nabla u_i|^2 \, d\mathscr{H}^{d-1}.$$

Since A_1 and A_2 are increasing functions, they are differentiable almost everywhere on $(0, +\infty)$. Moreover, $A_i'(r) = r^{2-d}B_i$, for $i = 1, 2$, in sense of distributions. Thus, the function

$$r \mapsto r^{-4}A_1(r)A_2(r),$$

differentiable a.e. and we have

$$\frac{d}{dr}\left[\frac{A_1(r)A_2(r)}{r^4}\right] = \Big(-\frac{4}{r}+\frac{r^{2-d}B_1(r)}{A_1(r)}+\frac{r^{2-d}B_2(r)}{A_2(r)}\Big)\frac{A_1(r)A_2(r)}{r^4}.$$

Thus, it is sufficient to prove, that for almost every $r \in [1/4, 1]$ we have

$$-\frac{4}{r}+\frac{r^{2-d}B_1(r)}{A_1(r)}+\frac{r^{2-d}B_2(r)}{A_2(r)} \geq -C_d\Big(\frac{1}{\sqrt{A_1(r)}}+\frac{1}{\sqrt{A_2(r)}}\Big). \tag{3.4}$$

Using the rescaling $u_{i,r}(x) := r^{-2}u_i(rx)$, we have the identities

$$\int_{\partial B_1} |\nabla u_{i,r}|^2 \, d\mathscr{H}^{d-1} = \frac{1}{r^{d+1}}\int_{\partial B_r} |\nabla u_i|^2 \, d\mathscr{H}^{d-1}, \tag{3.5}$$

$$\int_{B_1} \frac{|\nabla u_{i,r}|^2}{|x|^{d-2}}\,dx = \frac{1}{r^4}\int_{B_r} \frac{|\nabla u_i|^2}{|x|^{d-2}}\,dx,$$

and so, it is sufficient to prove (3.4) in the case $r = 1$. We consider two cases:

(A) Suppose that $B_1(1) \geq 4A_1(1)$ or $B_2(1) \geq 4A_2(1)$. In both cases we have

$$-4+\frac{B_1(1)}{A_1(1)}+\frac{B_2(1)}{A_2(1)} \geq 0,$$

which gives (3.4).

(B) Suppose that $B_1(1) \le 4A_1(1)$ and $B_2(1) \le 4A_2(1)$. By Lemma 2.3 we have

$$A_1(1) \le C_d + C_d\sqrt{\frac{B_1(1)}{\lambda_1}} + \frac{B_1(1)}{2\alpha_1} \le C_d + C_d\sqrt{\frac{A_1(1)}{\lambda_1}} + \frac{B_1(1)}{2\alpha_1}, \tag{3.6}$$

where we used the notation $\alpha_i := \alpha(u_i, 1)$ and $\lambda_i = \lambda(u_i, 1)$ where α and λ are as in Lemma 2.3. We now consider two sub-cases:

(B1) Suppose that $\alpha_1 \ge 4$ or $\alpha_2 \ge 4$. By (3.6), we get

$$A_1(1) \le 2C_d\sqrt{\frac{A_1(1)}{\lambda_1}} + \frac{B_1(1)}{\alpha_1}.$$

Now since $\sqrt{\lambda_1} \ge \alpha_1 \ge 4$ we obtain

$$4A_1(1) \le 2C_d\sqrt{A_1(1)} + B_1(1) = A_1(1)\Big(\frac{2C_d}{\sqrt{A_1(1)}} + \frac{B_1(1)}{A_1(1)}\Big),$$

which gives (3.4).

(B2) Suppose that $\alpha_1 \le 4$ and $\alpha_2 \le 4$. Then for both $i = 1, 2$, we have $C_d \le \sqrt{A_i/\lambda}$ and so, by (3.6)

$$2\alpha_i A_i(1) \le C_d\sqrt{A_i(1)} + B_i(1).$$

Thus (3.4) reduces to $\alpha_1 + \alpha_2 \ge 2$, which was proved in [14] (see also [9]). □

The following is the discretized version of Lemma 3.2 and also the main ingredient in the proof of Theorem 3.1.

LEMMA 3.3. *Let u_1 and u_2 be as in Theorem 3.1. Then there is a dimensional constant $C_d > 0$ such that the following implication holds: if for some $r \in (0, 1)$*

$$\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_1|^2}{|x|^{d-2}}dx \ge C_d \quad \text{and} \quad \frac{1}{r^4}\int_{B_r}\frac{|\nabla u_2|^2}{|x|^{d-2}}dx \ge C_d,$$

then we have the estimate

$$4^4 A_1(r/4)A_2(r/4) \le (1 + \delta_{12}(r))A_1(r)A_2(r), \tag{3.7}$$

where

$$\delta_{12}(r) := C_d\Big(\Big(\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_1|^2}{|x|^{d-2}}dx\Big)^{-1/2} + \Big(\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_2|^2}{|x|^{d-2}}dx\Big)^{-1/2}\Big). \tag{3.8}$$

PROOF. Using the rescaling $u_r(x) = r^{-2}u(rx)$, we can suppose that $r = 1$. We consider two cases:

(A) If $A_1(1) \geq 4^4 A_1(1/4)$ or $A_2(1) \geq 4^4 A_2(1/4)$, then

$$A_1(1)A_2(1) - 4^4 A_1(1/4)A_2(1/4) \geq A_1(1)(A_2(1) - 4^4 A_2(1/4)) \geq 0,$$

and so, we have the claim.

(B) Suppose that $A_1(1) \leq 4^4 A_1(1/4)$ or $A_2(1) \leq 4^4 A_2(1/4)$. Then $A_1(r) \geq C_d$ and $A_2(r) \geq C_d$, for every $r \in (1/4, 1)$ and so, we may apply Lemma 3.2

$$\begin{aligned} &A_1(1)A_2(1) - 4^4 A_1(1/4)A_2(1/4) \\ &\quad \geq -C_d \int_{1/4}^{1} \Big(\frac{1}{\sqrt{A_1(r)}} + \frac{1}{\sqrt{A_2(r)}} \Big) A_1(r)A_2(r)\, dr \\ &\quad \geq -C_d \frac{3}{4} \Big(\frac{1}{\sqrt{A_1(1/4)}} + \frac{1}{\sqrt{A_2(1/4)}} \Big) A_1(1)A_2(1) \\ &\quad \geq -C_d \frac{3}{4} \Big(\frac{16}{\sqrt{A_1(1)}} + \frac{16}{\sqrt{A_2(1)}} \Big) A_1(1)A_2(1), \end{aligned}$$

where in the second inequality we used the monotonicity of A_1 and A_2. □

The following lemma corresponds to [7, Lemma 2.9] and its proof implicitly contains [7, Lemma 2.1] and [7, Lemma 2.3]. We state it here as a single separate result since it is only used in the proof of the two-phase monotonicity formula (Theorem 3.1).

Lemma 3.4. *Let u_1 and u_2 be as in Theorem* 3.1. *Then there are dimensional constants $C_d > 0$ and $\varepsilon > 0$ such that the following implication holds: if $A_1(1) \geq C_d$, $A_2(1) \geq C_d$ and $4^4 A_1(1/4) \geq A_1(1)$, then $A_2(1/4) \leq (1 - \varepsilon)A_2(1)$.*

Proof. The idea of the proof is roughly speaking to show that if $A_1(1/4)$ is not too small with respect to $A_1(1)$, then there is a big portion of the set $\{u_1 > 0\}$ in the annulus $B_{1/2} \backslash B_{1/4}$. This of course implies that there is a small portion of $\{u_2 > 0\}$ in $B_{1/2} \backslash B_{1/4}$ and so $A_2(1/4)$ is much smaller than $A_2(1)$. We will prove the Lemma in two steps.

Step 1. *There are dimensional constants $C > 0$ and $\delta > 0$ such that if $A_1(1) \geq C$ and $4^4 A_1(1/4) \geq A_1(1)$, then $|\{u_1 > 0\} \cap B_{1/2} \backslash B_{1/4}| \geq \delta |B_{1/2} \backslash B_{1/4}|$.*

By Lemma 2.1 we have that

$$A_1(1/4) \leq C_d + C_d \int_{B_{1/2} \backslash B_{1/4}} u_1^2\, dx,$$

and by choosing $C > 0$ large enough we get

$$A_1(1/4) \leq C_d \int_{B_{1/2} \backslash B_{1/4}} u_1^2\, dx.$$

Now if $|\{u_1 > 0\} \cap B_{1/2} \backslash B_{1/4}| > 1/2 |B_{1/2} \backslash B_{1/4}|$, then there is nothing to prove. Otherwise, there is a dimensional constant C_d such that the Sobolev inequality holds

$$\Big(\int_{B_{1/2} \backslash B_{1/4}} u_1^{\frac{2d}{d-2}} \, dx \Big)^{\frac{d-2}{d}} \le C_d \int_{B_{1/2} \backslash B_{1/4}} |\nabla u_1|^2 \, dx \le C_d A_1(1).$$

By the Hölder inequality, we get

$$\begin{aligned} A_1(1/4) &\le C_d |\{u_1 > 0\} \cap B_{1/2} \backslash B_{1/4}|^{\frac{2}{d}} A_1(1) \\ &\le C_d |\{u_1 > 0\} \cap B_{1/2} \backslash B_{1/4}|^{\frac{2}{d}} 4^4 A_1(1/4), \end{aligned}$$

which gives the claim of *Step 1* since $A_1(1/4) > 0$. The proof in dimension 2 is analogous.

Step 2. *Let $\delta \in (0,1)$. Then there are constants $C > 0$ and $\varepsilon > 0$, depending on δ and the dimension, such that if $A_2(1) \ge C$ and $|\{u_2 > 0\} \cap B_{1/2} \backslash B_{1/4}| \le (1-\delta)|B_{1/2} \backslash B_{1/4}|$, then $A_2(1/4) \le (1-\varepsilon) A_2(1)$.*

Since $|\{u_2 = 0\} \cap B_{1/2} \backslash B_{1/4}| \ge \delta |B_{1/2} \backslash B_{1/4}|$, there is a constant $C_\delta > 0$ such that

$$\int_{B_{1/2} \backslash B_{1/4}} u_2^2 \, dx \le C_\delta \int_{B_{1/2} \backslash B_{1/4}} |\nabla u_2|^2 \, dx.$$

We can suppose that

$$\int_{B_{1/4}} |\nabla u_2|^2 \, dx \ge \frac{1}{2} \int_{B_1} |\nabla u_2|^2 \, dx \ge \frac{C}{2},$$

since otherwise the claim holds with $\varepsilon = 1/2$. Applying Lemma 2.1 we obtain

$$\begin{aligned} \int_{B_{1/4}} |\nabla u_2|^2 \, dx &\le C_d + C_d \int_{B_{1/2} \backslash B_{1/4}} u_2^2 \, dx \\ &\le C_d + C_d C_\delta \Big(\int_{B_1} |\nabla u_2|^2 \, dx - \int_{B_{1/4}} |\nabla u_2|^2 \, dx \Big) \\ &\le \Big(C_d C_\delta + \frac{1}{2} \Big) \int_{B_1} |\nabla u_2|^2 \, dx - C_d C_\delta \int_{B_{1/4}} |\nabla u_2|^2 \, dx, \end{aligned} \tag{3.9}$$

where for the last inequality we chose $C > 0$ large enough. □

The proof of Theorem 3.1 continues exactly as in [7]. In what follows, for $i = 1, 2$, we adopt the notation

$$A_i^k := A_i(4^{-k}), \quad b_i^k := 4^{4k} A_i(4^{-k}) \quad \text{and} \quad \delta_k := \delta_{12}(4^{-k}),$$

where A_i was defined in (3.3) and δ_{12} in (3.8).

Proof of Theorem 3.1. Let $M > 0$ be a fixed constant, larger than the dimensional constants in Lemma 3.2, Lemma 3.3 and Lemma 3.4.

Suppose that $k \in \mathbb{N}$ is such that

$$4^{4k} A_1^k A_2^k \geq M(1 + A_1^0 + A_2^0)^2. \tag{3.10}$$

Then we have

$$b_1^k = 4^{4k} A_1^k \geq M \quad \text{and} \quad b_2^k = 4^{4k} A_2^k \geq M. \tag{3.11}$$

Thus, applying Lemma 3.3 we get that if $k \in \mathbb{N}$ does not satisfy (3.10), then

$$4^4 A_1^{k+1} A_2^{k+1} \leq (1 + \delta_k) A_1^k A_2^k. \tag{3.12}$$

We now denote with $S_1(M)$ the set

$$S_1(M) := \{k \in \mathbb{N} : 4^{4k} A_1^k A_2^k \leq M(1 + A_1^0 + A_2^0)^2\},$$

and with S_2 the set

$$S_2 := \{k \in \mathbb{N} : 4^4 A_1^{k+1} A_2^{k+1} \leq A_1^k A_2^k\}.$$

Let $L \in \mathbb{N}$ be such that $L \notin S_1(M)$ and let $l \in \{0, 1, \dots, L\}$ be the largest index such that $l \in S_1(M)$. Note that if $\{l+1, \dots, L-1\} \setminus S_2 = \emptyset$, then we have

$$4^{4L} A_1^L A_2^L \leq 4^{4(L-1)} A_1^{L-1} A_2^{L-1} \leq \cdots \leq 4^{4(l+1)} A_1^{l+1} A_2^{l+1} \leq 4^4 4^{4l} A_1^l A_2^l,$$

which gives that $L \in S_1(4^4 M)$.

Repeating the proof of [7, Theorem 1.3], we consider the decreasing sequence of indices

$$l + 1 \leq k_m < \cdots < k_2 < k_1 \leq L,$$

constructed as follows:

- k_1 is the largest index in the set $\{l+1, \dots, L\}$ such that $k_1 \notin S_2$;
- k_{j+1} is the largest integer in $\{l+1, \dots, k_j - 1\} \setminus S_2$ such that

$$b_1^{k_{j+1}+1} \leq (1 + \delta_{k_{j+1}}) b_1^{k_j} \quad \text{and} \quad b_2^{k_{j+1}+1} \leq (1 + \delta_{k_{j+1}}) b_2^{k_j}. \tag{3.13}$$

We now conclude the proof in four steps.

Step 1. $4^{4L} A_1^L A_2^L \leq 4^{4(k_1+1)} A_1^{k_1} A_2^{k_1}$.

Indeed, since $\{k_1 + 1, \dots, L\} \subset S_2$, we have

$$4^{4L} A_1^L A_2^L \leq 4^{4(L-1)} A_1^{L-1} A_2^{L-1} \leq \cdots \leq 4^{4(k_1+1)} A_1^{k_1+1} A_2^{k_1+1} \leq 4^4 4^{4k_1} A_1^{k_1} A_2^{k_1}.$$

Step 2. $4^{4k_m} A_1^{k_m} A_2^{k_m} \leq 4^4 M(1 + A_1^0 + A_2^0)^2$.

Let $\tilde{k} \in \{l+1, \dots, k_m - 1\}$ be the smallest integer such that $\tilde{k} \notin S_2$. If no such $\tilde{k}$ exists, then we have

$$4^{4k_m} A_1^{k_m} A_2^{k_m} \leq \dots \leq 4^{4(l+1)} A_1^{l+1} A_2^{l+1} \leq 4^4 4^{4l} A_1^l A_2^l \leq 4^4 M(1 + A_1^0 + A_2^0)^2.$$

Otherwise, since k_m is the last index in the sequence constructed above, we have that

$$b_1^{\tilde{k}+1} > (1+\delta_{\tilde{k}}) b_1^{k_m} \quad \text{or} \quad b_2^{\tilde{k}+1} > (1+\delta_{\tilde{k}}) b_2^{k_m}.$$

Assuming, without loss of generality that the first inequality holds, we get

$$\begin{aligned} 4^{4k_m} A_1^{k_m} A_2^{k_m} &\leq \frac{4^{4(\tilde{k}+1)} A_1^{\tilde{k}+1}}{1+\delta_{\tilde{k}}} A_2^{\tilde{k}+1} \leq 4^{4\tilde{k}} A_1^{\tilde{k}} A_2^{\tilde{k}} \leq \dots \leq 4^4 4^{4l} A_1^l A_2^l \\ &\leq 4^4 M(1 + A_1^0 + A_2^0)^2, \end{aligned}$$

where in the second inequality we used Lemma 3.3 and afterwards we used the fact that $\{l+1, \dots, \tilde{k}-1\} \subset S_2$.

Step 3. $4^{4k_j} A_1^{k_j} A_2^{k_j} \leq (1+\delta_{k_{j+1}}) 4^{4k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}}$.

We reason as in **Step 2** choosing $\tilde{k} \in \{k_{j+1}+1, \dots, k_j - 1\}$ to be the smallest integer such that $\tilde{k} \notin S_2$. If no such $\tilde{k}$ exists, then $\{k_{j+1}+1, \dots, k_j - 1\} \subset S_2$ and so we have

$$\begin{aligned} 4^{4k_j} A_1^{k_j} A_2^{k_j} \leq 4^{4(k_j-1)} A_1^{k_j-1} A_2^{k_j-1} &\leq \dots \leq 4^{4(k_{j+1}+1)} A_1^{k_{j+1}+1} A_2^{k_{j+1}+1} \\ &\leq (1+\delta_{k_{j+1}}) 4^{4k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}}, \end{aligned}$$

where the last inequality is due to Lemma 3.3. Suppose now that $\tilde{k}$ exists. Since k_j and k_{j+1} are consecutive indices, we have that

$$b_1^{\tilde{k}+1} > (1+\delta_{\tilde{k}}) b_1^{k_j} \quad \text{or} \quad b_2^{\tilde{k}+1} > (1+\delta_{\tilde{k}}) b_2^{k_j}.$$

As in **Step 2**, we assume that the first inequality holds. By Lemma 3.3 we have

$$\begin{aligned} 4^{4k_j} A_1^{k_j} A_2^{k_j} &\leq \frac{4^{4(\tilde{k}+1)} A_1^{\tilde{k}+1}}{1+\delta_{\tilde{k}}} A_2^{\tilde{k}+1} \leq 4^{4\tilde{k}} A_1^{\tilde{k}} A_2^{\tilde{k}} \leq \cdots \\ &\leq 4^{4(k_{j+1}+1)} A_1^{k_{j+1}+1} A_2^{k_{j+1}+1} \leq (1+\delta_{k_{j+1}}) 4^{4k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}}. \end{aligned}$$

which concludes the proof of **Step 3**.

Step 4. *Conclusion*. Combining the results of Steps 1, 2 and 3, we get

$$4^{4L} A_1^L A_2^L \leq 4^8 M(1 + A_1^0 + A_2^0)^2 \prod_{j=1}^{m} (1+\delta_{k_j}). \tag{3.14}$$

We now prove that the sequences $b_1^{k_j}$ and $b_2^{k_j}$ can both be estimated from above by a geometric progression. Indeed, since $k_j \notin S_2$, we have

$$A_1^{k_j} A_2^{k_j} \le 4^4 A_1^{k_j+1} A_2^{k_j+1} \le 4^4 A_1^{k_j+1} A_2^{k_j}.$$

Thus $A_1^{k_j} \le 4^4 A_1^{k_j+1}$ and analogously $A_2^{k_j} \le 4^4 A_2^{k_j+1}$. Applying Lemma 3.4 we get

$$A_1^{k_j+1} \le (1-\varepsilon) A_1^{k_j} \quad \text{and} \quad A_2^{k_j+1} \le (1-\varepsilon) A_2^{k_j}.$$

Using again the fact that $k_j \notin S_2$, we obtain

$$A_1^{k_j} A_2^{k_j} \le 4^4 A_1^{k_j+1} A_2^{k_j+1} \le 4^4 A_1^{k_j+1} (1-\varepsilon) A_2^{k_j},$$

and so

$$b_1^{k_j} \le (1-\varepsilon) b_1^{k_j+1} \quad \text{and} \quad b_2^{k_j} \le (1-\varepsilon) b_2^{k_j+1}, \quad \text{for every } j = 1, \dots, m. \tag{3.15}$$

By the construction of the sequence k_j, we have that for $i = 1, 2$

$$b_i^{k_j} \ge \frac{b_i^{k_{j+1}+1}}{1+\delta_{k_{j+1}}} \ge \frac{b_i^{k_{j+1}}}{(1+\delta_{k_{j+1}})(1-\varepsilon)} \ge \left(1 - \frac{\varepsilon}{2}\right)^{-1} b_i^{k_{j+1}},$$

where for the last inequality we choose M large enough such that $k \notin S_1(M)$ implies $\delta_k \le \varepsilon/2$, where ε is the dimensional constant from Lemma 3.4. Setting $\sigma = (1 - \varepsilon/2)^{1/2}$, we have that

$$b_i^{k_j} \ge \sigma^{-2} b_i^{k_{j+1}} \ge \cdots \ge \sigma^{2(j-m)} b_i^{k_m} \ge M \sigma^{2(j-m)},$$

which by the definition of δ_{k_j} gives $\delta_{k_j} \le \dfrac{C_d}{M} \sigma^{m-j} \le C_d \sigma^{m-j}$, for $M > 0$ large enough, and

$$\begin{aligned} 4^{4L} A_1^L A_2^L &\le \prod_{j=1}^m (1 + C_d \sigma^j) 4^8 M (1 + A_1^0 + A_2^0)^2 \\ &\le \exp\Big(\sum_{j=1}^m \log(1 + C_d \sigma^j) \Big) 4^8 M (1 + A_1^0 + A_2^0)^2 \\ &\le \exp\Big(C_d \sum_{j=1}^m \sigma^j \Big) 4^8 M (1 + A_1^0 + A_2^0)^2 \\ &\le \exp\Big(\frac{C_d}{1-\sigma} \Big) 4^8 M (1 + A_1^0 + A_2^0)^2, \end{aligned} \tag{3.16}$$

which concludes the proof. □

4. Multiphase monotonicity formula

This section is dedicated to the multiphase version of Theorem 3.1, proved in [6]. The proof follows the same idea as in [7]. The major technical difference with respect to the two-phase case consists in the fact that we only need Lemma 3.3 and its three-phase analogue Lemma 4.5, while the estimate from Lemma 3.4 is not necessary.

Theorem 4.1 (Three-phase monotonicity formula). *Let $B_1 \subset \mathbb{R}^d$ be the unit ball in $\mathbb{R}^d$ and let $u_i \in H^1(B_1)$, $i = 1, 2, 3$, be three non-negative Sobolev functions such that*

$$\Delta u_i + 1 \geq 0, \quad \forall i = 1, 2, 3, \quad \textit{and} \quad u_i u_j = 0 \textit{ a.e. in } B_1, \ \forall i \neq j.$$

Then there are dimensional constants $\varepsilon > 0$ and $C_d > 0$ such that for each $r \in (0, 1)$ we have

$$\prod_{i=1}^{3}\Big(\frac{1}{r^{2+\varepsilon}}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx\Big) \leq C_d\Big(1+\sum_{i=1}^{3}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx\Big)^3. \tag{4.1}$$

As a corollary, we obtain the following result.

Corollary 4.2 (Multiphase monotonicity formula). *Let $m \geq 2$ and $B_1 \subset \mathbb{R}^d$ be the unit ball in $\mathbb{R}^d$. Let $u_i \in H^1(B_1)$, $i = 1, \dots, m$, be m non-negative Sobolev functions such that*

$$\Delta u_i + 1 \geq 0, \quad \forall i = 1, \dots, m, \quad \textit{and} \quad u_i u_j = 0 \textit{ a.e. in } B_1, \ \forall i \neq j.$$

Then there are dimensional constants $\varepsilon > 0$ and $C_d > 0$ such that for each $r \in (0, 1)$ we have

$$\prod_{i=1}^{m}\Big(\frac{1}{r^{2+\varepsilon}}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx\Big) \leq C_d\Big(1+\sum_{i=1}^{m}\int_{B_1}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx\Big)^m. \tag{4.2}$$

Remark 4.3. We note that the additional decay $r^{-\varepsilon}$ provided by the presence of a third phase is not optimal. Indeed, at least in dimension two, we expect that $\varepsilon = m - 2$, where m is the number of phases involved. In our proof the constant ε cannot exceed $2/3$ in any dimension.

We now proceed with the proof of the three-phase formula. Before we start with the proof of Theorem 4.1 we will need some preliminary results, analogous to Lemma 3.2 and Lemma 3.3.

We recall that, for u_1, u_2 and u_3 as in Theorem 4.1, we use the notation

$$A_i(r) = \int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx, \quad \text{for } i = 1, 2, 3. \tag{4.3}$$

The following Lemma 4.4 and Lemma 4.5 were proved in [6] and concern the function

$$\Phi(r) = r^{-(6+3\varepsilon)} A_1(r) A_2(r) A_3(r).$$

We report here the proofs for convenience of the reader.

Lemma 4.4. *Let u_1, u_2 and u_3 be as in Theorem* 4.1. *Then there are dimensional constants $C_d > 0$ and $\varepsilon > 0$ such that if $A_i(1/4) \geq C_d$, for every $i = 1, 2, 3$, then*

$$\Phi'(r) \geq -C_d\Big(\frac{1}{\sqrt{A_1(r)}} + \frac{1}{\sqrt{A_2(r)}} + \frac{1}{\sqrt{A_3(r)}}\Big)\Phi(r),$$

for Lebesgue almost every $r \in [1/4, 1]$.

Proof. We set, for $i = 1, 2, 3$ and $r > 0$,

$$B_i(r) = \int_{\partial B_r} |\nabla u_i|^2 \, d\mathscr{H}^{d-1}.$$

Since A_i, for $i = 1, 2, 3$, are increasing functions they are differentiable almost everywhere on $\mathbb{R}$ and $A_i'(r) = r^{2-d} B_i + \mu_i$ in sense of distributions. Thus, the function $\Phi(r)$ is differentiable a.e. and we have

$$\Phi'(r) = \Big(-\frac{6+3\varepsilon}{r} + \frac{r^{2-d} B_1(r)}{A_1(r)} + \frac{r^{2-d} B_2(r)}{A_2(r)} + \frac{r^{2-d} B_3(r)}{A_3(r)} \Big)\Phi(r).$$

Thus, it is sufficient to prove that for almost every $r \in [1/4, 1]$ we have

$$-\frac{6+3\varepsilon}{r} + r^{2-d}\Big(\frac{B_1(r)}{A_1(r)} + \frac{B_2(r)}{A_2(r)} + \frac{B_3(r)}{A_3(r)}\Big) \geq -C_d\Big(\frac{1}{\sqrt{A_1(r)}} + \frac{1}{\sqrt{A_2(r)}} + \frac{1}{\sqrt{A_3(r)}}\Big). \tag{4.4}$$

Using again the rescaling $u_{i,r}(x) := r^{-2} u_i(rx)$, we have that (3.5) holds and so, we may assume that $r = 1$. We consider two cases.

(A) Suppose that there is some $i = 1, 2, 3$, say $i = 1$, such that $(6 + 3\varepsilon)A_1(1) \leq B_1(1)$. Then we have

$$-(6+3\varepsilon) + \frac{B_1(1)}{A_1(1)} + \frac{B_2(1)}{A_2(1)} + \frac{B_3(1)}{A_3(1)} \geq -(6+3\varepsilon) + \frac{B_1(1)}{A_1(1)} \geq 0,$$

which proves (4.4) and the lemma.

(B) Suppose that for each $i = 1, 2, 3$ we have $(6 + 3\varepsilon)A_i(1) \geq B_i(1)$. Since, for every $i = 1, 2, 3$ we have $A_i(1) \geq C_d$, we can choose C_d large enough and $\varepsilon > 0$ small enough such that, by Lemma 2.3 with the notation $\alpha_i = \alpha(u_i, 1)$ and $\lambda_i = \lambda(u_i, 1)$, we get

$$(2 - \varepsilon)A_i(1) \leq C_d\sqrt{B_i(1)/\lambda_i} + B_i(1)/\alpha_i \leq C_d\sqrt{A_i(1)/\lambda_i} + B_i(1)/\alpha_i.$$

Moreover, $\alpha_i^2 \leq \lambda_i$, implies

$$(2 - \varepsilon)\alpha_i A_i(1) \leq C_d\sqrt{A_i(1)} + B_i(1). \tag{4.5}$$

Dividing both sides by $A_i(1)$ and summing for $i = 1, 2, 3$, we obtain

$$(2 - \varepsilon)(\alpha_1 + \alpha_2 + \alpha_3) \leq C_d \sum_{i=1}^{3} \frac{1}{\sqrt{A_i(1)}} + \sum_{i=1}^{3} \frac{B_i(1)}{A_i(1)},$$

and so, in order to prove (4.4), it is sufficient to prove that

$$\alpha_1 + \alpha_2 + \alpha_3 \geq \frac{6 + 3\varepsilon}{2 - \varepsilon}. \tag{4.6}$$

Let $\Omega_1^*, \Omega_2^*, \Omega_3^* \subset \partial B_1$ be the optimal partition of the sphere ∂B_1 for the characteristic constant α, i.e. the triple $\{\Omega_1^*, \Omega_2^*, \Omega_3^*\}$ is a solution of the problem

$$\begin{aligned} \min\{\alpha(\Omega_1) + \alpha(\Omega_2) + \alpha(\Omega_3) : \Omega_i \subset \partial B_1, \forall i; \\ \mathscr{H}^{d-1}(\Omega_i \cap \Omega_j) = 0, \forall i \neq j\}. \end{aligned} \tag{4.7}$$

We recall that for a set $\Omega \subset \partial B_1$, the characteristic constant $\alpha(\Omega)$ is the unique positive real number such that $\lambda(\Omega) = \alpha(\Omega)(\alpha(\Omega) + d - 2)$, where

$$\lambda(\Omega) = \min\left\{ \frac{\int_{\partial B_1} |\nabla v|^2 \mathscr{H}^{d-1}}{\int_{\partial B_1} v^2 \mathscr{H}^{d-1}} : v \in H^1(\partial B_1), \mathscr{H}^{d-1}(\{u \neq 0\} \backslash \Omega) = 0 \right\}.$$

We note that, by [14], $\alpha(\Omega_i^*) + \alpha(\Omega_j^*) \geq 2$, for $i \neq j$ and so summing on i and j, we have

$$3 \leq \alpha(\Omega_1^*) + \alpha(\Omega_2^*) + \alpha(\Omega_3^*) \leq \alpha_1 + \alpha_2 + \alpha_3.$$

Moreover, the first inequality is strict. Indeed, if this is not the case, then $\alpha(\Omega_1^*) + \alpha(\Omega_2^*) = 2$, which in turn gives that Ω_1^* and Ω_2^* are two opposite hemispheres (see for example [9]). Thus $\Omega_3^* = \emptyset$, which is impossible since it contradicts the equality $\alpha(\Omega_1^*) + \alpha(\Omega_3^*) = 2$, which in turn is implied by the contradiction assumption. Choosing ε to be such that $\frac{6 + 3\varepsilon}{2 - \varepsilon}$ is smaller than the minimum in (4.7), the proof is concluded. □

LEMMA 4.5. *Let u_1, u_2 and u_3 be as in Theorem* 4.1. *Then, there are dimensional constants $C_d > 0$ and $\varepsilon > 0$ such that the following implication holds: if for some $r > 0$*

$$\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx \geq C_d, \quad \textit{for all } i = 1,2,3,$$

then we have the estimate

$$4^{(6+3\varepsilon)}A_1\Big(\frac{r}{4}\Big)A_2\Big(\frac{r}{4}\Big)A_3\Big(\frac{r}{4}\Big) \leq (1+\delta_{123}(r))A_1(r)A_2(r)A_3(r), \tag{4.8}$$

where

$$\delta_{123}(r) := C_d\sum_{i=1}^{3}\Big(\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_i|^2}{|x|^{d-2}}dx\Big)^{-1/2}. \tag{4.9}$$

PROOF. We first note that the (4.8) is invariant under the rescaling $u_r(x) = r^{-2}u(xr)$. Thus, we may suppose that $r = 1$. We consider two cases:

(A) Suppose that for some $i = 1,2,3$, say $i = 1$, we have $4^{6+3\varepsilon}A_1(1/4) \leq A_1(1)$. Then we have

$$4^{6+3\varepsilon}A_1(1/4)A_2(1/4)A_3(1/4) \leq A_1(1)A_2(1)A_3(1).$$

(B) Suppose that for every $i = 1,2,3$, we have $4^{6+3\varepsilon}A_i(1/4) \geq A_i(1)$. Then $A_i(1/4) \geq C_d$ for some C_d large enough and so, we can apply Lemma 4.4, obtaining that

$$\begin{aligned}\Phi(1) - \Phi(1/4) &\geq -C_d\int_{1/4}^{1}\Big(\sum_{i=1}^{3}\frac{1}{\sqrt{A_i(r)}}\Big)A_1(r)A_2(r)A_3(r)\,dr\\ &\geq -C_d\frac{3}{4}\Big(\sum_{i=1}^{3}\frac{1}{\sqrt{A_i(1/4)}}\Big)A_1(1)A_2(1)A_3(1)\\ &\geq -3C_d4^{2+\frac{3}{2}\varepsilon}\Big(\sum_{i=1}^{3}\frac{1}{\sqrt{A_i(1)}}\Big)A_1(1)A_2(1)A_3(1),\end{aligned}$$

which gives the claim. □

In what follows we give two proofs of Theorem 4.1. The first one follows precisely the proof of Theorem 3.1, while the second one is more direct and is contained in [6].

PROOF I OF THEOREM 4.1. For $i = 1, 2, 3$, we adopt the notation

$$A_i^k := A_i(4^{-k}), \quad b_i^k := 4^{4k} A_i(4^{-k}) \quad \text{and} \quad \delta_k := \delta_{123}(4^{-k}), \tag{4.10}$$

where A_i was defined in (3.3) and δ_{123} in (4.9).

Let $M > 0$ and let

$$S_1(M) = \{k \in \mathbb{N} : 4^{(6+3\varepsilon)k} A_1^k A_2^k A_3^k \le M(1 + A_1^0 + A_2^0 + A_3^0)^3\},$$
$$S_2 = \{k \in \mathbb{N} : 4^{6+3\varepsilon} A_1^{k+1} A_2^{k+1} A_3^{k+1} \le A_1^k A_2^k A_3^k\}.$$

We first note that if $k \notin S_1$, then we have

$$\begin{aligned} M(1 + A_1^0 + A_2^0 + A_3^0)^3 &\le 4^{(6+3\varepsilon)k} A_1^k A_2^k A_3^k \\ &\le 4^{-(2-3\varepsilon)k} b_1^k 4^{4k} A_2^k A_3^k \\ &\le b_1^k C_d (1 + A_1^0 + A_2^0 + A_3^0)^2, \end{aligned}$$

where the last inequality is due to the two-phase monotonicity formula (Theorem 3.1). Choosing $M > 0$ big enough, we have that

$$(k \notin S_1(M)) \Rightarrow (b_i^k \ge C_d, \forall i = 1, 2, 3).$$

Fix $L \in \mathbb{N}$ and suppose that $L \notin S_1(M)$. Let $l \in \{0, \dots, L\}$ be the largest index such that $l \in S_1(M)$. We now consider two cases for the interval $[l + 1, L]$.

(*Case* 1) If $\{l + 1, \dots, L\} \subset S_2$, then we have

$$\begin{aligned} 4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L \le \dots &\le 4^{(6+3\varepsilon)(l+1)} A_1^{l+1} A_2^{l+1} A_3^{l+1} \\ &\le 4^{6+3\varepsilon} M (1 + A_1^0 + A_2^0 + A_3^0)^2, \end{aligned}$$

and so $L \in S_1(4^{6+3\varepsilon} M)$.

(*Case* 2) If $\{l + 1, \dots, L\} \setminus S_2 \ne \emptyset$, then we choose k_1 to be the largest index in $\{l + 1, \dots, L\} \setminus S_2$. Then we define the sequence

$$l + 1 \le k_m < \dots < k_1 \le L,$$

by induction as

$$k_{j+1} := \max\{k \in \{l + 1, \dots, k_j - 1\} \setminus S_2 : b_i^{k_{j+1}+1} \le (1 + \delta_{k_{j+1}}) b_i^{k_j}, \ \forall i = 1, 2, 3\}.$$

The proof now proceeds in four steps.

Step 1. $4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L \le 4^{(6+3\varepsilon)(k_1+1)} A_1^{k_1} A_2^{k_1} A_3^{k_1}$.

Indeed, since $\{k_1 + 1, \dots, L\} \subset S_2$, we have

$$\begin{aligned} 4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L \le \dots &\le 4^{(6+3\varepsilon)(k_1+1)} A_1^{k_1+1} A_2^{k_1+1} A_3^{k_1+1} \\ &\le 4^{6+3\varepsilon} 4^{(6+3\varepsilon)k_1} A_1^{k_1} A_2^{k_1} A_3^{k_1}. \end{aligned}$$

Step 2. $4^{(6+3\varepsilon)k_m} A_1^{k_m} A_2^{k_m} A_3^{k_m} \le 4^{6+3\varepsilon} M(1 + A_1^0 + A_2^0 + A_3^0)^3$.

Let $\tilde{k} \in \{l+1, \dots, k_m - 1\}$ be the smallest index such that $\tilde{k} \notin S_2$. If no such $\tilde{k}$ exists, then we have

$$\begin{aligned} 4^{(6+3\varepsilon)k_m} A_1^{k_m} A_2^{k_m} A_3^{k_m} &\le \dots \le 4^{(6+3\varepsilon)(l+1)} A_1^{l+1} A_2^{l+1} A_3^{l+1} \\ &\le 4^{6+3\varepsilon} 4^{(6+3\varepsilon)l} A_1^l A_2^l A_3^l \\ &\le 4^{6+3\varepsilon} M(1 + A_1^0 + A_2^0 + A_3^0)^3. \end{aligned}$$

Otherwise, since k_m is the last index in the sequence constructed above, there exists $i \in \{1, 2, 3\}$ such that

$$b_i^{\tilde{k}+1} > (1 + \delta_{\tilde{k}}) b_i^{k_m}. \tag{4.11}$$

Assuming, without loss of generality that $i = 1$, we get

$$4^{(6+3\varepsilon)k_m} A_1^{k_m} A_2^{k_m} A_3^{k_m} = 4^{(-2+3\varepsilon)k_m} b_1^{k_m} 4^{4k_m} A_2^{k_m} A_3^{k_m}$$

$$\le \frac{4^{(-2+3\varepsilon)k_m}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} (1 + \delta_{23}(4^{-k_m+1})) 4^{4(k_m-1)} A_2^{k_m-1} A_3^{k_m-1} \tag{4.12}$$

$$\le \frac{4^{(-2+3\varepsilon)(k_m-1)}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} 4^{4(k_m-1)} A_2^{k_m-1} A_3^{k_m-1} \tag{4.13}$$

$$\dots$$

$$\le \frac{4^{(-2+3\varepsilon)(\tilde{k}+1)}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} 4^{4(\tilde{k}+1)} A_2^{\tilde{k}+1} A_3^{\tilde{k}+1} \tag{4.14}$$

$$= \frac{4^{(6+3\varepsilon)(\tilde{k}+1)}}{1+\delta_{\tilde{k}}} A_1^{\tilde{k}+1} A_2^{\tilde{k}+1} A_3^{\tilde{k}+1}$$

$$\le 4^{(6+3\varepsilon)\tilde{k}} A_1^{\tilde{k}} A_2^{\tilde{k}} A_3^{\tilde{k}} \le \dots \le 4^{(6+3\varepsilon)(l+1)} A_1^{l+1} A_2^{l+1} A_3^{l+1} \tag{4.15}$$

$$\le 4^{6+3\varepsilon} 4^{(6+3\varepsilon)l} A_1^l A_2^l A_3^l \le 4^{6+3\varepsilon} M(1 + A_1^0 + A_2^0 + A_3^0)^3, \tag{4.16}$$

where in order to obtain (4.12) we used (4.11) and the two-phase estimate from Lemma 3.3; for (4.13), we absorb the term that appears after applying Lemma 3.3, using that if M is large enough and $\varepsilon < 2/3$, then $(1 + \delta_{23}(4^{-k_m+1}))4^{-2+3\varepsilon} \le 1$; repeating the same estimate as above we obtain (4.14); for (4.15), we use the three-phase Lemma 4.5 and then the fact that $\{l+1, \dots, \tilde{k}\} \subset S_2$; for the last inequality (4.16) we just observed that $l \in S_1(M)$.

Step 3. $4^{(6+3\varepsilon)k_j} A_1^{k_j} A_2^{k_j} A_3^{k_j} \le (1 + \delta_{\tilde{k}_{j+1}}) 4^{(6+3\varepsilon)k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}} A_3^{k_{j+1}}$.

We reason as in **Step 2** choosing $\tilde{k} \in \{k_{j+1} + 1, \dots, k_j - 1\}$ to be the smallest index such that $\tilde{k} \notin S_2$. If no such $\tilde{k}$ exists, then $\{k_{j+1} + 1, \dots, k_j - 1\} \subset S_2$ and so we have

$$4^{(6+3\varepsilon)k_j} A_1^{k_j} A_2^{k_j} A_3^{k_j} \le \cdots \le 4^{(6+3\varepsilon)(k_{j+1}+1)} A_1^{k_{j+1}+1} A_2^{k_{j+1}+1} A_3^{k_{j+1}+1}$$
$$\le (1+\delta_{k_{j+1}}) 4^{(6+3\varepsilon)k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}} A_3^{k_{j+1}},$$

where the last inequality is due to Lemma 3.3. Suppose now that $\tilde{k}$ exists. Since k_j and k_{j+1} are consecutive indices, there exists some $i \in \{1,2,3\}$ such that

$$b_i^{\tilde{k}+1} > (1+\delta_{\tilde{k}}) b_i^{k_j}. \tag{4.17}$$

Without loss of generality we may assume that $i = 1$.

$$4^{(6+3\varepsilon)k_j} A_1^{k_j} A_2^{k_j} A_3^{k_j} = 4^{(-2+3\varepsilon)k_j} b_1^{k_j} 4^{4k_j} A_2^{k_j} A_3^{k_j}$$

$$\le \frac{4^{(-2+3\varepsilon)k_j}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} (1+\delta_{23}(4^{-k_j+1})) 4^{4(k_j-1)} A_2^{k_j-1} A_3^{k_j-1} \tag{4.18}$$

$$\le \frac{4^{(-2+3\varepsilon)(k_j-1)}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} 4^{4(k_j-1)} A_2^{k_j-1} A_3^{k_j-1} \tag{4.19}$$

$$\cdots$$

$$\le \frac{4^{(-2+3\varepsilon)(\tilde{k}+1)}}{1+\delta_{\tilde{k}}} b_1^{\tilde{k}+1} 4^{4(\tilde{k}+1)} A_2^{\tilde{k}+1} A_3^{\tilde{k}+1} \tag{4.20}$$

$$= \frac{4^{(6+3\varepsilon)(\tilde{k}+1)}}{1+\delta_{\tilde{k}}} A_1^{\tilde{k}+1} A_2^{\tilde{k}+1} A_3^{\tilde{k}+1} \le 4^{(6+3\varepsilon)\tilde{k}} A_1^{\tilde{k}} A_2^{\tilde{k}} A_3^{\tilde{k}} \le \cdots$$

$$\le 4^{(6+3\varepsilon)(k_{j+1}+1)} A_1^{k_{j+1}+1} A_2^{k_{j+1}+1} A_3^{k_{j+1}+1} \tag{4.21}$$

$$\le (1+\delta_{k_{j+1}}) 4^{(6+3\varepsilon)k_{j+1}} A_1^{k_{j+1}} A_2^{k_{j+1}} A_3^{k_{j+1}}, \tag{4.22}$$

where for (4.18) we used (4.17) and Lemma 3.3; for (4.19) and (4.20), we use that for $M > 0$ large enough and $\varepsilon < 2/3$ we have $(1+\delta_{23}(4^{-k_m+1}))4^{-2+3\varepsilon} \le 1$; for (4.21), we apply Lemma 4.5 and then the fact that $\{l+1, \ldots, \tilde{k}\} \subset S_2$; for the last inequality (4.22) we use Lemma 4.5.

Step 4. *Conclusion*. By the steps 1, 2 and 3 we have that

$$4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L \le 4^{2(6+3\varepsilon)} M (1 + A_1^0 + A_2^0 + A_3^0)^3 \prod_{j=1}^{m} (1+\delta_{k_j}) \tag{4.23}$$

we now prove that for each $i = 1,2,3$ the sequence $b_i^{k_j}$ is majorized by a geometric progression depending on M. Indeed, since $k_j \notin S_2$, we have

$$A_1^{k_j} A_2^{k_j} A_3^{k_j} \le 4^{6+3\varepsilon} A_1^{k_j+1} A_2^{k_j+1} A_3^{k_j+1}$$
$$\le 4^{-(2-3\varepsilon)} 4^4 A_1^{k_j+1} (1+\delta_{23}(4^{-k_j})) A_2^{k_j} A_3^{k_j}$$
$$\le \sigma^2 4^4 A_1^{k_j+1} A_2^{k_j} A_3^{k_j},$$

for some dimensional constant $\sigma < 1$, where the second inequality is due to Lemma 3.3 and the last inequality is due to the choice of M large enough and $\varepsilon < 2/3$. Thus we obtain

$$b_i^{k_j} \leq \sigma^2 b_i^{k_j+1}, \quad \forall i = 1,2,3 \text{ and } \forall j = 1,\dots,m. \tag{4.24}$$

for each $i = 1,2,3$ and each $k_j \in S_3$. Now using the definition of the finite sequence k_j and (4.24), we deduce that for all $i = 1,2,3$ and $j = 2,\dots,m$ we have

$$b_i^{k_j} \leq \sigma^2 b_i^{k_j+1} \leq \sigma^2(1+\delta_{k_j}) b_i^{k_{j-1}} \leq \sigma b_i^{k_{j-1}},$$

and so, repeating the above estimate, we get

$$b_i^{k_j} \geq \sigma^{-1} b_i^{k_{j+1}} \geq \cdots \geq \sigma^{j-m} b_i^{k_m} \geq \sigma^{j-m} M,$$

and, by the definition (4.27) (and (4.9)) of δ_{k_j},

$$\delta_{k_j} \leq \frac{C_d}{M}\sigma^{\frac{m-j}{2}}, \quad \forall j = 1,\dots,m. \tag{4.25}$$

By (4.23) and (4.25) and reasoning as in (3.16) we deduce

$$4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L \leq \exp\Big(\frac{C_d}{1-\sqrt{\sigma}}\Big) 4^{2(6+3\varepsilon)} M (1 + A_1^0 + A_2^0 + A_3^0)^3, \tag{4.26}$$

which concludes the proof of Theorem 4.1. □

Proof II of Theorem 4.1. For $i = 1,2,3$, we adopt the notation

$$A_i^k := A_i(4^{-k}), \quad b_i^k := 4^{4k} A_i(4^{-k}) \quad \text{and} \quad \delta_k := \delta_{123}(4^{-k}), \tag{4.27}$$

where A_i was defined in (3.3) and δ_{123} in (4.9).

Let $M > 0$ and let

$$S(M) = \{k \in \mathbb{N} : 4^{(6+3\varepsilon)k} A_1^k A_2^k A_3^k \leq M(1 + A_1^0 + A_2^0 + A_3^0)^3\}.$$

We will prove that if $\varepsilon > 0$ is small enough, then there is M large enough such that for every $k \notin S(M)$, we have

$$4^{(6+3\varepsilon)k} A_1^k A_2^k A_3^k \leq CM(1 + A_1^0 + A_2^0 + A_3^0)^3,$$

where C is a constant depending on d and ε.

We first note that if $k \notin S(M)$, then we have

$$\begin{aligned} M(1 + A_1^0 + A_2^0 + A_3^0)^3 &\leq 4^{(6+3\varepsilon)k} A_1^k A_2^k A_3^k \\ &\leq 4^{-(2-3\varepsilon)k} b_1^k 4^{4k} A_2^k A_3^k \\ &\leq 4^{-(2-3\varepsilon)k} b_1^k C_d (1 + A_1^0 + A_2^0 + A_3^0)^2, \end{aligned}$$

and so $b_1^k \geq C_d^{-1} M 4^{(2-3\varepsilon)k}$, where C_d is the constant from Theorem 3.1. Thus, choosing $\varepsilon < 2/3$ and $M > 0$ large enough, we can suppose that, for every $i = 1, 2, 3$, $b_i^k > C_d$, where C_d is the constant from Lemma 4.5.

Suppose now that $L \in \mathbb{N}$ is such that $L \notin S(M)$ and let

$$l = \max\{k \in \mathbb{N} : k \in S(M) \cap [0, L]\} < L,$$

where we note that the set $S(M) \cap [0, L]$ is non-empty for large M, since for $k = 0, 1$, we can apply Theorem 3.1. Applying Lemma 4.5, for $k = l+1, \dots, L-1$ we obtain

$$\begin{aligned} 4^{(6+3\varepsilon)L} A_1^L A_2^L A_3^L &\leq \Big(\prod_{k=l+1}^{L-1} (1+\delta_k)\Big) 4^{(6+3\varepsilon)(l+1)} A_1^{l+1} A_2^{l+1} A_3^{l+1} \\ &\leq \Big(\prod_{k=l+1}^{L-1} (1+\delta_k)\Big) 4^{(6+3\varepsilon)(l+1)} A_1^l A_2^l A_3^l \\ &\leq \Big(\prod_{k=l+1}^{L-1} (1+\delta_k)\Big) 4^{6+3\varepsilon} M (1 + A_1^0 + A_2^0 + A_3^0)^2, \end{aligned} \tag{4.28}$$

where δ^k is the variable from Lemma 4.5.

Now it is sufficient to notice that for $k = l+1, \dots, L-1$, the sequence δ_k is bounded by a geometric progression. Indeed, setting $\sigma = 4^{-1+3\varepsilon/2} < 1$, we have that, for $k \notin S(M)$, $\delta_k \leq C\sigma^k$, which gives

$$\begin{aligned} \prod_{k=l+1}^{L-1} (1+\delta_k) &\leq \prod_{k=l+1}^{L-1} (1 + C\sigma^k) \\ &= \exp\Big(\sum_{k=l+1}^{L-1} \log(1 + C\sigma^k)\Big) \\ &\leq \exp\Big(C \sum_{k=l-1}^{L+1} \sigma^k\Big) \leq \exp\Big(\frac{C}{1-\sigma}\Big), \end{aligned} \tag{4.29}$$

which concludes the proof. □

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