



Topology — *The genus of the configuration spaces for Artin groups of affine type*, by D. MORONI, M. SALVETTI and A. VILLA, communicated on 11 April 2014.

ABSTRACT. — Let (\mathbf{W}, S) be a Coxeter system, S finite, and let $\mathbf{G}_{\mathbf{W}}$ be the associated Artin group. One has *configuration spaces* $\mathbf{Y}, \mathbf{Y}_{\mathbf{W}}$, where $\mathbf{G}_{\mathbf{W}} = \pi_1(\mathbf{Y}_{\mathbf{W}})$, and a natural \mathbf{W} -covering $f_{\mathbf{W}} : \mathbf{Y} \rightarrow \mathbf{Y}_{\mathbf{W}}$. The *Schwarz genus* $g(f_{\mathbf{W}})$ is a natural topological invariant to consider. In [DS00] it was computed for all finite-type Artin groups, with the exception of case A_n (for which see [Vas92], [DPS04]). In this paper we generalize this result by computing the Schwarz genus for a class of Artin groups, which includes the affine-type Artin groups. Let $K = K(\mathbf{W}, S)$ be the simplicial scheme of all subsets $J \subset S$ such that the parabolic group \mathbf{W}_J is finite. We introduce the class of groups for which $\dim(K)$ equals the homological dimension of K , and we show that $g(f_{\mathbf{W}})$ is always the maximum possible for such class of groups. For affine Artin groups, such maximum reduces to the rank of the group. In general, it is given by $\dim(\mathbf{X}_{\mathbf{W}}) + 1$, where $\mathbf{X}_{\mathbf{W}} \subset \mathbf{Y}_{\mathbf{W}}$ is a well-known *CW-complex* which has the same homotopy type as $\mathbf{Y}_{\mathbf{W}}$.

KEY WORDS: Configuration spaces, Schwarz genus, Artin groups, cohomology of groups.

MATHEMATICS SUBJECT CLASSIFICATION: 55R80, 20F36.

1. INTRODUCTION

To any Coxeter system (\mathbf{W}, S) , S finite, one can naturally associate (see section 2):

- a space \mathbf{Y} with a natural free action of \mathbf{W} , and an *orbit space* $\mathbf{Y}_{\mathbf{W}}$ such that the projection onto the quotient $f_{\mathbf{W}} : \mathbf{Y} \rightarrow \mathbf{Y}_{\mathbf{W}}$ is a regular covering with group \mathbf{W} ;
- an explicitly constructed *CW-complex* $\mathbf{X} \subset \mathbf{Y}$ which is a deformation retract of \mathbf{Y} , whose cells are permuted under the action of \mathbf{W} , and a *finite orbit CW-complex* $\mathbf{X}_{\mathbf{W}} \subset \mathbf{Y}_{\mathbf{W}}$ which is a deformation retract of the orbit space, such that $f_{\mathbf{W}}$ restricts to a regular \mathbf{W} -covering $f_{\mathbf{W}} : \mathbf{X} \rightarrow \mathbf{X}_{\mathbf{W}}$.

Let $K = K(\mathbf{W})$ be the simplicial scheme with vertex set S and simplices all $\Gamma \subset S$ such that the parabolic subgroup \mathbf{W}_{Γ} is finite. Then the complex \mathbf{X} is union of finitely many polyhedra, one for each maximal simplex in K , and $\mathbf{X}_{\mathbf{W}}$ is obtained by explicit identifications on the faces of these polyhedra (see [Sal94]).

The fundamental group of $\mathbf{Y}_{\mathbf{W}}$ (resp. of \mathbf{Y}) is the *Artin group* $\mathbf{G}_{\mathbf{W}}$ of type \mathbf{W} (resp. the *pure Artin group* $\mathbf{PG}_{\mathbf{W}}$). The quotient $\mathbf{G}_{\mathbf{W}}/\mathbf{PG}_{\mathbf{W}}$ is isomorphic to \mathbf{W} . For example, if the group \mathbf{W} is the symmetric group Σ_n with set of Coxeter generators $S := \{(i, i + 1) : i = 1, \dots, n - 1\}$ then $\mathbf{G}_{\mathbf{W}}$ is the *braid group* Br_n and $\mathbf{PG}_{\mathbf{W}}$ is the pure braid group P_n . The spaces $\mathbf{Y}_{\mathbf{W}}$ and \mathbf{Y} generalize the classical

configuration spaces for the braid group and the pure braid group respectively, so they can be called the *configuration spaces* for \mathbf{G}_W and for \mathbf{PG}_W respectively.

The *Schwarz genus* $g(f)$ of a locally trivial fibration $f : Y \rightarrow X$ is the minimum cardinality of an open covering \mathcal{U} of X such that there exists a section of f over each open set $U \in \mathcal{U}$ ([Sch61]; see section 3 below). The Schwarz genus $g(f_W)$ of the above covering can be considered as a natural topological invariant of the Artin group.

The main result of this paper is the computation of $g(f_W)$ for a class of Artin groups including the *affine type groups* (i.e., when (W, S) is an affine Coxeter system).

For any finite Coxeter group W of rank n , it was shown in [DS00] that the Schwarz genus $g(f_W)$ reaches the upper bound $n + 1$ (coming from an easy argument in obstruction theory) except when $W = A_n$ and $n + 1$ is not a prime power (see [Vas92]). For the first non prime power $n + 1 = 6$, in [DPS04] it is shown that $g(f_{A_5}) = 5$, lower than $n + 1$. Next, in [Aro05] it is shown by different methods that $g(f_{A_n}) < n + 1$ when $n + 1 \neq p^k, 2p^k$ for p prime (so, the case $2p^k$ is still open, as well as the precise value of the genus).

We introduce the class of *affine-like Artin groups*, as those groups such that the homological dimension of $K(W)$ equals the topological dimension of $K(W)$. For affine type Artin groups, $K(W)$ turns out to be a sphere, so this condition is fulfilled. For this class of groups, we show that the genus coincides again with $\dim(\mathbf{X}_W) + 1$. For an affine type Artin group of rank $n + 1$, $\dim(\mathbf{X}_W) = n$ (but for other affine-like groups, the rank is bigger than $\dim(\mathbf{X}_W) + 1$).

Notice how the above condition on K contrasts what happens for the finite type cases, where K is contractible.

2. GENERAL PICTURES

2.1. Topological constructions for Artin groups

We will consider a finitely generated *Coxeter system* (W, S) (S finite), so

$$(1) \quad W = \langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle$$

where $m(s, s') \in \mathbb{N} \cup \{\infty\}$, $m(s, s') = m(s', s)$, $m(s, s) = 1$ (all results in this part which are not explicitly referred to the literature are taken from [Bou68], [Hum90], [Vin71]). We recall some general pictures.

The group W can be realized as a group generated by (in general, non-orthogonal) reflections in \mathbb{R}^n , $n = |S|$. Let \mathcal{A} be the *reflection arrangement*, i.e.

$$\mathcal{A} = \{H \subset \mathbb{R}^n \mid H \text{ is fixed by some reflection in } W\}.$$

Consider also the stratification into *facets* $\Phi := \{F\}$ of \mathbb{R}^n induced by \mathcal{A} . The codimension-0 facets, which are the connected components of the complement to the arrangement, are called *chambers*. All the chambers are simplicial cones, the group acting transitively over the set of all them. The Coxeter generator set

S corresponds to the set of reflections with respect to the *walls* of a fixed base-chamber C_0 .

Let $U := \mathbf{W} \cdot \bar{C}_0$ be the orbit of the closure of the base chamber. U is called the *Tits cone* of the Coxeter system.

Notice that the closure of the chamber \bar{C}_0 is endowed with a natural stratification into facets (which are still relatively open cones with vertex 0). When C_0 is the standard positive octant, each facet is given by imposing some coordinates equal to 0, and the remaining coordinates positive.

Each reflection in \mathbf{W} is conjugated to a reflection with respect to a wall of C_0 . So, the arrangement \mathcal{A} consists of the orbits of the walls of C_0 . Each chamber contained inside U is of the shape $w.C_0$ for a unique $w \in \mathbf{W}$. Of course, \mathcal{A} is not locally finite if \mathbf{W} is infinite (e.g. 0 is contained in all the hyperplanes). The orbits of the facets of C_0 give a “stratification” of U into relatively open cells, also called facets (in general, U is neither open nor closed in \mathbb{R}^n).

Recall also:

1. U is a convex cone in \mathbb{R}^n with vertex 0.
2. $U = \mathbb{R}^n$ iff \mathbf{W} is finite
3. The stabilizer of a facet F in U is the subgroup \mathbf{W}_F generated by all the reflections with respect to hyperplanes (in \mathcal{A}) containing F . So, in general \mathbf{W}_F is not finite.
4. $U^0 := \text{int}(U)$ is open in \mathbb{R}^n and a (relatively open) facet $F \subset \bar{C}_0$ is contained in U^0 iff the stabilizer \mathbf{W}_F is finite.

By property 4 the arrangement is locally finite in the interior part U^0 .

Let $H_{\mathbb{C}} := H + iH \subset \mathbb{C}^n$ be the *complexification* of the hyperplane H , and set

$$\mathbf{Y} := [U^0 + i\mathbb{R}^n] \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$$

which corresponds to complexifying only the interior part of the Tits cone. The group \mathbf{W} acts (as before) diagonally onto \mathbf{Y} , and one shows easily (by property 3 above) that the action is free. Therefore, one has an *orbit space*

$$\mathbf{Y}_{\mathbf{W}} := \mathbf{Y}/\mathbf{W}$$

which is still a manifold, and a regular covering

$$f_{\mathbf{W}} : \mathbf{Y} \rightarrow \mathbf{Y}_{\mathbf{W}}$$

with group \mathbf{W} .

DEFINITION 2.1. *Define*

$$\mathbf{G}_{\mathbf{W}} := \pi_1(\mathbf{Y}_{\mathbf{W}})$$

as the Artin group of type \mathbf{W} .

(see [Bri71, Del70] for the case when \mathbf{W} is finite). We recall here some topological constructions from [Sal94] (see also [Sal87], [CMS08a], [CMS08b], [CMS10], [Sal05], [MSV12]).

Take $x_0 \in C_0$ and let Q be the *finite CW-complex* constructed as follows. For all maximal subsets $\Gamma \subset S$ such that \mathbf{W}_Γ is finite, construct a $|\Gamma|$ -cell Q_Γ in U^0 as the convex hull of the \mathbf{W}_Γ orbit of x_0 . Each Q_Γ is a finite convex polyhedron which contains the point x_0 .

Define

$$(2) \quad Q := \bigcup Q_\Gamma$$

(a finite union of convex polyhedra).

The k -faces of each Q_Γ are also polyhedra, each of them corresponding to a coset of a *parabolic* subgroup \mathbf{W}_Δ , where $\Delta \subset S$ is a k -subset of Γ . The correspondence

$$\{\text{faces of } Q_\Gamma\} \leftrightarrow \{w \cdot \mathbf{W}_\Delta, w \in \Gamma, \Delta \subset \Gamma\}$$

is obtained by taking the polyhedron given by the convex-hull of the orbit $\mathbf{W}_\Delta \cdot x_0$ and translating it by w .

One also has:

PROPOSITION 2.2. *Inside each coset $w \cdot \mathbf{W}_\Delta$ there exists an unique element of minimal length.*

Here the length is the minimal number of letters (coming from S) in a reduced expression.

For every face e of Q_Γ , which corresponds to a coset $w \cdot \mathbf{W}_\Delta$, let $\beta(e) \in w \cdot \mathbf{W}_\Delta$ be the element of minimal length. Notice that \mathbf{W}_Γ permutes faces of the same dimension. Let $\mathbf{X}_{\mathbf{W}_\Gamma}$ be obtained from Q_Γ by identifications on its faces defined as: each pair of faces e, e' belonging to the same orbit is identified by using the homeomorphism $\beta(e)\beta(e')^{-1}$. Define also the finite complex

$$(3) \quad \mathbf{X}_{\mathbf{W}} := \bigcup_{\Gamma} \mathbf{X}_{\mathbf{W}_\Gamma}.$$

Notice that $\mathbf{X}_{\mathbf{W}}$ is well defined because of the following easy fact.

REMARK 2.3. *For any common cell $e \subset Q_\Gamma \cap Q_{\Gamma'}$ the minimal element $\beta(e)$ is the same when computed in \mathbf{W}_Γ and in $\mathbf{W}_{\Gamma'}$.*

For reader convenience, we also recall the following fact, whose proof is the same as in [Sal94].

THEOREM 1. *The CW-complex $\mathbf{X}_{\mathbf{W}}$ is deformation retract of the orbit space $\mathbf{Y}_{\mathbf{W}}$.*

PROOF. First, there exists a regular CW-complex $\mathbf{X} \subset \mathbf{Y}$ which is deformation retract of \mathbf{Y} , and \mathbf{X} is constructed as in [Sal87]. The construction used there

(where the starting point was an affine arrangement of hyperplanes) works in general because one reduces to the finite case around faces with finite stabilizer.

The construction of \mathbf{X} can be chosen invariantly with respect to the action of \mathbf{W} , which permutes cells of the same dimension.

The action on \mathbf{X} being free, we look at the orbit space \mathbf{X}/\mathbf{W} . By remark 2.3, this reduces to finite cases. □

As an immediate corollary of this construction one has a presentation of the group $\mathbf{G}_{\mathbf{W}}$ which generalizes the finite type case (see also [vdL83])

$$(4) \quad \mathbf{G}_{\mathbf{W}} = \langle g_s, s \in \mathcal{S} \mid g_s g_{s'} g_s \dots = g_{s'} g_s g_{s'} \dots \rangle$$

(same number $m(s, s')$ of factors on each side) where we have to consider only pairs such that $m(s, s')$ is finite.

2.2. Topological constructions for Coxeter groups

We refer here essentially to [DS00].

Consider the subspace arrangement in $\mathbb{R}^{nd} \cong (\mathbb{R}^n)^d$ given by

$$\mathcal{A}^{(d)} := \{H^{(d)}\}$$

where $H^{(d)}$ is the codimensional- d subspace given by “ d -complexification” of the hyperplane $H \in \mathcal{A}$:

$$H^{(d)} := \{(X_1, \dots, X_d) : X_i \in \mathbb{R}^n, X_i \in H\}.$$

If U^0 is the interior of the Tits cone, as above, then we consider the space

$$\mathcal{U}_{\mathbf{W}}^{(d)} := U_0 \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$$

($d - 1$ factors equal to \mathbb{R}^n), and the configuration space

$$(5) \quad \mathbf{Y}^{(d)} := \mathcal{U}_{\mathbf{W}}^{(d)} \setminus \bigcup_{H \in \mathcal{A}} H^{(d)}.$$

As before, the group \mathbf{W} acts freely on $\mathbf{Y}^{(d)}$ and we consider the orbit space

$$\mathbf{Y}_{\mathbf{W}}^{(d)} := \mathbf{Y}^{(d)} / \mathbf{W}.$$

One has:

THEOREM 2. *The space*

$$\mathbf{Y}_{\mathbf{W}}^{(\infty)} := \left[\varinjlim_d \mathbf{Y}^{(d)} \right] / \mathbf{W} = \left[\varinjlim_d \mathbf{Y}_{\mathbf{W}}^{(d)} \right]$$

is a space of type $k(\mathbf{W}, 1)$. □

So, different from the case of Artin groups, we always get a $k(\pi, 1)$ space here. Recall also:

THEOREM 3. *The space $\mathbf{Y}_{\mathbf{W}}^{(d)}$ contracts over a CW-complex $\mathbf{X}_{\mathbf{W}}^{(d)}$ such that*

$$\{k - \text{cells of } \mathbf{X}_{\mathbf{W}}^{(d)}\} \leftrightarrow \left\{ \text{flags } \Gamma := (\Gamma_1 \supset \dots \supset \Gamma_d) : \Gamma_1 \subset S, \mathbf{W}_{\Gamma_1} \text{ finite, } \sum_{i=1}^d |\Gamma_i| = k \right\}.$$

Passing to the limit, $\mathbf{Y}_{\mathbf{W}}^{(\infty)} = k(W, 1)$ contracts over a CW-complex $\mathbf{X}_{\mathbf{W}}^{(\infty)}$ such that

$$\{k - \text{cells of } \mathbf{X}_{\mathbf{W}}^{(\infty)}\} \leftrightarrow \left\{ \text{flags } \Gamma := (\Gamma_1 \supset \Gamma_2 \supset \dots) : \Gamma_1 \subset S, \mathbf{W}_{\Gamma_1} \text{ finite, } \sum_{i \geq 1} |\Gamma_i| = k \right\}. \quad \square$$

Notice that $\mathbf{X}_{\mathbf{W}}^{(\infty)}$ does not have finite dimension but the number of k -cells is finite, bounded by $\binom{n+k-1}{k}$.

2.3. Algebraic complexes for Artin groups

We refer here mainly to [Sal94], [DS96].

We consider the algebraic complex related to the cell structure of $\mathbf{X}_{\mathbf{W}}$. It is given in the following way.

Let $\mathbb{Z}[\mathbf{G}_{\mathbf{W}}]$ be the group algebra of $\mathbf{G}_{\mathbf{W}}$. Let (C_*, ∂_*) be the algebraic complex of free $\mathbb{Z}[\mathbf{G}_{\mathbf{W}}]$ -modules such that in degree k it is free with basis e_J corresponding to subsets $J \subset S$ such that \mathbf{W}_J is finite:

$$(6) \quad C_k := \bigoplus_{\substack{J \subset S \\ |J|=k \\ \mathbf{W}_J \text{ finite}}} \mathbb{Z}[\mathbf{G}_{\mathbf{W}}]e_J$$

Let

$$(7) \quad \partial(e_J) := \sum_{\substack{I \subset J \\ |I|=k-1}} [I : J] T_I^J \cdot e_I$$

where $[I : J]$ is the incidence number ($= 0, 1$ or -1) of the cells in $\mathbf{X}_{\mathbf{W}}$ and

$$T_I^J := \sum_{\beta \in W_I^J} (-1)^{\ell(\beta)} g_{\beta}$$

where

- $\mathbf{W}_I^J := \{\beta \in \mathbf{W}_J : \ell(\beta s) > \ell(\beta), \forall s \in \mathbf{W}_I\}$ is the set of elements of minimal length for the cosets $\frac{\mathbf{W}_J}{\mathbf{W}_I}$ (prop. 2.2);

- 2. if $\beta \in \mathbf{W}_I^J$ and $\beta = s_{i_1} \dots s_{i_k}$ is a reduced expression then $\ell(\beta) = k$;
- 3. if β is as in 2 then $g_\beta := g_{s_{i_1}} \dots g_{s_{i_k}}$. One shows that this map

$$(8) \quad \psi : \mathbf{W} \rightarrow \mathbf{G}_\mathbf{W}$$

is a well-defined section (not a homomorphism) of the standard surjection $\mathbf{G}_\mathbf{W} \rightarrow \mathbf{W}$.

REMARK 2.4. *When $\mathbf{X}_\mathbf{W}$ is a space of type $k(\mathbf{G}_\mathbf{W}, 1)$ then (C_*, ∂_*) gives a free $\mathbb{Z}[\mathbf{G}_\mathbf{W}]$ resolution of \mathbb{Z} as a trivial module. In any case, (C_*, ∂_*) corresponds to the cellular complex structure of the universal covering of $\mathbf{X}_\mathbf{W}$, so they can be used to compute local systems over $\mathbf{X}_\mathbf{W}$.*

Let $R := A[q, q^{-1}]$ be the ring of Laurent polynomials over a ring A . One can represent $\mathbf{G}_\mathbf{W}$ by

$$(9) \quad g_s \mapsto [\text{multiplication by } -q] \quad \forall s \in S$$

($\in \text{Aut}(R)$).

The tensor product $C_* \otimes R$ has boundary

$$(10) \quad \partial(e_J) = \sum_{\substack{I \subset J \\ |I|=|J|-1}} [I : J] \frac{\mathbf{W}_J(q)}{\mathbf{W}_I(q)} e_I$$

where

$$\mathbf{W}_J(q) := \sum_{w \in W_J} q^{\ell(w)}$$

is the Poincaré series of the group \mathbf{W}_J (here, a polynomial since the stabilizers are finite). The denominator $\mathbf{W}_I(q)$ divides the numerator $\mathbf{W}_J(q)$ so the quotient is still a polynomial.

2.4. Algebraic complexes for Coxeter groups

Consider the algebraic complex (C_*, ∂) of free $\mathbb{Z}[\mathbf{W}]$ -modules, where

$$C_k := \bigoplus_{\substack{\Gamma: \sum_{i \geq 1} |\Gamma_i| = k \\ |W_{\Gamma_1}| < \infty}} \mathbb{Z}[\mathbf{W}]e(\Gamma)$$

The generators of C_* are in one to one correspondence with the cells of $\mathbf{X}_\mathbf{W}^{(\infty)}$, so with the flags $\Gamma = (\Gamma_1 \supset \Gamma_2 \supset \dots)$, $\Gamma_1 \subset S$, \mathbf{W}_{Γ_1} finite.

The expression of the boundary is the following:

$$(11) \quad \partial e(\Gamma) = \sum_{\substack{i \geq 1 \\ |\Gamma_i| > |\Gamma_{i+1}|}} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in \mathbf{W}_{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1} \Gamma_{i+1} \beta \subset \Gamma_i \setminus \{\tau\}}} (-1)^{z(\Gamma, i, \tau, \beta)} \beta e(\Gamma')$$

where

$$\Gamma' = (\Gamma_1 \supset \dots \supset \Gamma_{i-1} \supset \Gamma_i \setminus \{\tau\} \supset \beta^{-1}\Gamma_{i+1}\beta \supset \beta^{-1}\Gamma_{i+2}\beta \supset \dots)$$

and $(-1)^{\alpha(\Gamma, i, \tau, \beta)}$ is an incidence index. To get a precise expression for $\alpha(\Gamma, i, \tau, \beta)$, fix a linear order on S and let

$$\begin{aligned} \mu(\Gamma_i, \tau) &:= |\{j \in \Gamma \text{ s.t. } j \leq \tau\}| \\ \sigma(\beta, \Gamma_j) &:= |\{(a, b) \in \Gamma_j \times \Gamma_j \text{ s.t. } a < b \text{ and } \beta(a) > \beta(b)\}| \end{aligned}$$

in other words, $\mu(\Gamma_i, \tau)$ is the number of reflections in Γ_i less or equal to τ and $\sigma(\beta, \Gamma_j)$ is the number of inversions operated by β on Γ_j . Then we define:

$$(12) \quad \alpha(\Gamma, i, \tau, \beta) = i\ell(\beta) + \sum_{j=1}^{i-1} |\Gamma_j| + \mu(\Gamma_i, \tau) + \sum_{j=i+1}^d \sigma(\beta, \Gamma_j)$$

where ℓ is the length function in the Coxeter group.

THEOREM 4. *For any finitely generated \mathbf{W} , the algebraic complex (C_*, ∂_*) gives a free resolution of the trivial $\mathbb{Z}[\mathbf{W}]$ -module \mathbb{Z} .*

The proof follows straightforward from the remark that the limit space $\mathbf{Y}_{\mathbf{W}}^{(\infty)}$, so $\mathbf{X}_{\mathbf{W}}^{(\infty)}$, is a space of type $k(\mathbf{W}, 1)$.

3. THE GENUS PROBLEM FOR ARTIN GROUPS

Our main application here is the extension of some of the results found in [DS00], [DPS04] about the genus of the covering associated to an Artin group.

3.1. Schwarz genera and homological genera

We start recalling the definition of Schwarz genus and discussing briefly some of its properties (we refer to [Sch61], [Vas92] for details).

DEFINITION 3.1. *For a locally trivial fibration $f : \mathbf{Y} \rightarrow \mathbf{X}$, the Schwarz genus $g(f)$ is the minimal cardinality of an open cover \mathcal{U} of \mathbf{X} such that f admits a section over each set $U \in \mathcal{U}$.*

REMARK. The Schwarz genus is the extension to fibrations of the Lusternik-Schnirelmann category of a topological space; indeed the category of a path connected topological space coincides with the Schwarz genus of its Serre fibration.

When \mathbf{X} has the homotopy type of a finite dimensional CW complex, we have an upper bound for the genus of any fibration:

THEOREM 5. *If X has the homotopy type of a CW complex of dimension N , then $g(f) \leq N + 1$. □*

Let now $f : \mathbf{Y} \rightarrow \mathbf{X}$ be a regular G -covering. Then $f = a^*(p)$, where $a : \mathbf{X} \rightarrow BG$ is a classifying map into the classifying space BG for G and $p : EG \rightarrow BG$ is the universal G -bundle.

Let M be an arbitrary G -module and a^*M be the local system on X induced by the map a .

DEFINITION 3.2. *The homological M -genus of $f : \mathbf{Y} \rightarrow \mathbf{X}$ is the smallest integer $h_M(f)$ such that the induced map in cohomology:*

$$a^* : H^j(BG; M) \rightarrow H^j(\mathbf{X}; a^*M)$$

is zero in degree j for $j \geq h_M(f)$.

The homological genus is defined as the maximum $h(f) = \max_M h_M(f)$ of the homological M -genera.

Homological genus provides a lower bound for Schwarz genus:

THEOREM 6. *For any regular covering $f : \mathbf{Y} \rightarrow \mathbf{X}$, we have $g(f) \geq h(f)$. □*

3.2. The genus problem for Artin groups

Let \mathbf{W} be a Coxeter group and consider the regular covering $f_{\mathbf{W}} : \mathbf{Y} \rightarrow \mathbf{Y}_{\mathbf{W}}$ between the configuration spaces introduced in part 2.1. We are interested in the genus $g(f_{\mathbf{W}})$ of $f_{\mathbf{W}}$.

We start by some general remarks on the algebraic complexes for $\mathbf{G}_{\mathbf{W}}$.

First we denote by $K := K(\mathbf{W})$ the simplicial scheme, defined over S , of the subsets $J \subset S$ which generate a finite parabolic subgroup \mathbf{W}_J (we include the empty set which by definition generates the trivial subgroup). The algebraic complex which computes the simplicial homology of K with coefficients in \mathbb{Z} will be denoted here by

$$D_k^0(\mathbf{W}) := \bigoplus_{\substack{J \subseteq S \\ |J|=k \\ |\mathbf{W}_J| < \infty}} \mathbb{Z} \cdot e_J^0$$

with boundary:

$$(13) \quad \partial^0(e_J^0) = \sum_{\substack{I \subset J \\ |J|=|I|+1}} [I : J] e_I^0$$

Here we indicate by $[I : J]$ the incidence number of oriented simplices, namely ± 1 or 0 , which is the same as that of the corresponding cells appearing in (10).

Notice that we found convenient here to graduate D_*^0 according to the cardinality of the subsets, so there is a degree-1 shift isomorphism with the standard complex for the simplicial homology of $K : H_m(D_*^0) \cong H_{m-1}(K)$, $m \geq 1$.

We also denote by $D_*^0(\mathbf{W}, B) := D_*^0(\mathbf{W}) \otimes B$ the algebraic complex computing the homology with trivial coefficients a \mathbb{Z} -module B . When no module is indicated we mean $B = R$.

Let us consider the representation $\rho : \mathbf{G}_{\mathbf{W}} \rightarrow \text{Aut}(R)$ of (2.3), obtained by sending the standard generators of $\mathbf{G}_{\mathbf{W}}$ into $(-q)$ -multiplication. Let R_q be the ring R with the prescribed structure of $\mathbf{G}_{\mathbf{W}}$ -module. It is convenient to indicate here by $D_*(\mathbf{W})$ the algebraic complex of part 2.3, so

$$(14) \quad D_k(\mathbf{W}) := \bigoplus_{\substack{J \subseteq S \\ |J|=k \\ |\mathbf{W}_J| < \infty}} R \cdot e_J$$

and the boundary is given by (10).

REMARK 3.3. *We can formally rewrite the boundary map in (10) as:*

$$(15) \quad \partial \left(\frac{1}{\mathbf{W}_J(q)} e_J \right) = \sum_{\substack{I \subset J \\ |J|=|I|+1}} [I : J] \frac{1}{\mathbf{W}_I(q)} \cdot e_I$$

That means that the fractions $e_J/\mathbf{W}_J(q)$ behave like the cells of the simplicial scheme K .

Consider the diagonal map:

$$(16) \quad \Delta : D_*(\mathbf{W}) \rightarrow D_*^0(\mathbf{W}), \quad e_J \mapsto \mathbf{W}_J(q)e_J^0.$$

It is clear by the previous discussion that Δ is an injective chain-complex homomorphism, so there is an exact sequence of complexes:

$$(17) \quad 0 \rightarrow D_*(\mathbf{W}) \xrightarrow{\Delta} D_*^0(\mathbf{W}) \xrightarrow{\pi} L_*(\mathbf{W}) \rightarrow 0$$

where

$$L_k(\mathbf{W}) := \bigoplus_{\substack{J \subseteq S \\ |J|=k \\ |\mathbf{W}_J| < \infty}} \frac{R}{(\mathbf{W}_J(q))} \cdot \bar{e}_J$$

is the quotient complex.

Passing to the associated long exact sequence we get:

$$(18) \quad \xrightarrow{\pi_*} H_{k+1}(L_*) \xrightarrow{\delta} H_k(D_*) \xrightarrow{\Delta_*} H_k(D_*^0) \xrightarrow{\pi_*} H_k(L_*) \xrightarrow{\delta} H_{k-1}(D_*) \xrightarrow{\Delta_*}$$

REMARK 3.4. *It is possible to consider both the data $\frac{R}{(\mathbf{W}_J(q))}$, $J \subset S$, $|\mathbf{W}_J| < \infty$, and the complex L_* as functorial constructions associated to any (finitely generated) Artin group. In [MSV12] we introduce a class of “sheaves over posets” called*

weighted sheaves over posets, and associated weighted complexes, a particular case being that associated to an Artin group, and we used this construction for computations of the cohomology (see also [SV13]).

We need some definitions.

DEFINITION 3.5. We define the virtual dimension of an Artin group $\mathbf{G}_{\mathbf{W}}$ as

$$vd(\mathbf{G}_{\mathbf{W}}) := \dim(K) + 1$$

where K is the associated simplicial scheme and the dimension is that of a simplicial complex, so by definition of K :

$$vd(\mathbf{G}_{\mathbf{W}}) = \max\{|J| : J \subset S, |\mathbf{W}_J| < \infty\}$$

(Equivalently, $vd(\mathbf{G}_{\mathbf{W}}) = \max\{n : D_n^0(\mathbf{W}; \mathbb{Z}) \neq 0\}$)

DEFINITION 3.6. We define the homological virtual dimension of $\mathbf{G}_{\mathbf{W}}$ as:

$$hvd(\mathbf{G}_{\mathbf{W}}) := \max\{n : H_n(D_*^0(\mathbf{W}; \mathbb{Z})) \neq 0\}$$

(Equivalently: $hvd(\mathbf{G}_{\mathbf{W}}) := \max\{n : H_n(K; \mathbb{Z}) \neq 0\} + 1$, where here we use the standard graduation for the homology).

By construction one has

$$\dim(\mathbf{X}_{\mathbf{W}}) = vd(\mathbf{G}_{\mathbf{W}})$$

and from theorem 5 it follows

$$(19) \quad g(f_{\mathbf{W}}) \leq vd(\mathbf{G}_{\mathbf{W}}) + 1$$

DEFINITION 3.7. We say that a Coxeter system (\mathbf{W}, S) is affine-like type if

$$vd(\mathbf{G}_{\mathbf{W}}) = hvd(\mathbf{G}_{\mathbf{W}})$$

REMARK 3.8. Recall that for an affine Coxeter system (\mathbf{W}, S) of rank $n + 1$, a parabolic subgroup \mathbf{W}_J is finite if and only if J is a proper subset of S . In particular the poset of finite parabolic subgroups is isomorphic to the poset of proper subsets of $I_{n+1} = \{1, \dots, n + 1\}$ (that is the boolean lattice minus its maximum). Then the homology of $D_*^0(\mathbf{W})$ is the reduced homology of a $(n - 1)$ -sphere modulo a degree shift:

$$H_k(D_*^0(\mathbf{W})) \cong \tilde{H}_{k-1}(S^{n-1}; R) \cong \begin{cases} 0 & \text{if } k \neq n \\ R & \text{if } k = n \end{cases}$$

Therefore (\mathbf{W}, S) is affine-like.

We can now state the main result of the paper.

THEOREM 7. *Let (\mathbf{W}, S) be an affine-like Coxeter system. Then for the Schwarz genus of the fibration*

$$f_{\mathbf{W}} : \mathbf{Y} \rightarrow \mathbf{Y}_{\mathbf{W}}$$

it holds

$$g(f_{\mathbf{W}}) = vd(\mathbf{G}_{\mathbf{W}}) + 1$$

PROOF. Inequality (19) gives the upper bound which we need. To obtain the lower bound we will use homological methods.

First we prove:

THEOREM 8. *Let $M = \mathbb{Z}[-1]$ be the $\mathbf{G}_{\mathbf{W}}$ -module \mathbb{Z} with the action given by the sign representation. Let*

$$F_k^0 := FH_k(D_*^0(\mathbf{W}; \mathbb{Z})) := H_k(D_*^0(\mathbf{W}; \mathbb{Z})) / \text{Tors}(H_k(D_*^0(\mathbf{W}; \mathbb{Z})))$$

be the free component of the integral k -th homology of D_^0 . Assume that*

$$F_k^0 \neq 0$$

for some k .

Then also the free part

$$FH_k(\mathbf{X}_{\mathbf{W}}; M) \neq 0.$$

PROOF OF THEOREM 8. The homology of $\mathbf{X}_{\mathbf{W}}$ in the sign representation is computed by specializing the complex (14) to $q = 1$. We obtain sequences analog to (17) and (18) respectively, where in this case $D_*^0(\mathbf{W}; \mathbb{Z})$ computes the homology of K with trivial integer coefficients.

By definition $F_k^0 \neq 0$. Let $z^0 \in D_k^0$ be any no-torsion k -cycle,

$$z^0 = \sum_{|J|=k} \varepsilon_J e_J^0, \quad \varepsilon_J \in \mathbb{Z}.$$

Notice that the map Δ is injective, being diagonal with all non-vanishing entries given by $\mathbf{W}_J(1) \neq 0, |J| = k$.

Let $\mu := \text{lcm}(\mathbf{W}_J(1))_{|J|=k}$. Then

$$\mu \cdot z^0 = \Delta(z)$$

where

$$z := \sum_{|J|=k} \varepsilon_J \frac{\mu}{\mathbf{W}_J(1)} e_J.$$

It follows that z is a no-torsion cycle in $H_n(\mathbf{X}_{\mathbf{W}}; M)$. □

From theorem 8 we deduce:

COROLLARY 3.9. *Let (\mathbf{W}, S) be an affine-like Coxeter system and let*

$$n := vd(\mathbf{G}_W) = hcd(\mathbf{G}_W).$$

Let $M = \mathbb{Z}[-1]$ be the sign representation.

Then the \mathbb{Z} -rank of $H^n(\mathbf{X}_W; M)$ is bigger than 0.

PROOF OF COROLLARY 3.9. In top dimension n we have (we drop the coefficients):

$$(20) \quad 0 \rightarrow H_n(D_*) \xrightarrow{\Delta_*} H_n(D_*^0) \xrightarrow{\pi_*} H_n(L_*) \xrightarrow{\delta} H_{n-1}(D_*) \xrightarrow{\Delta_*}$$

By definition $H_n(D_*^0) \neq 0$, and since we are in top dimension such group is a free \mathbb{Z} -module. Then we apply theorem 8 and the thesis follows by passing to cohomology. \square

Recall that we have an inclusion $i : \mathbf{X}_W \hookrightarrow \mathbf{X}_W^{(\infty)}$ and that \mathbf{X}_W may be identified with the subcomplex of $\mathbf{X}_W^{(\infty)}$ consisting of cells of type $\Gamma = (\Gamma_1 \supseteq \emptyset \supseteq \emptyset \supseteq \dots)$.

Let M be a \mathbf{W} -module and M' the local coefficient system on \mathbf{X}_W induced by M via i . Equivalently, we can consider the natural surjection

$$\mathbf{G}_W \rightarrow \mathbf{W} \rightarrow 1.$$

The associated map of cochains

$$i^* : C^*(\mathbf{W}; M) \rightarrow C^*(\mathbf{X}_W; M')$$

is described as the restriction of $c \in C^*(\mathbf{W}; M)$ to the chains for \mathbf{X}_W . Let n be the maximal cardinality of a subset $J \subset S$ s.t. $|\mathbf{W}_J| < \infty$. Then in degree n we have:

$$(21) \quad \begin{array}{ccccccc} \longrightarrow & C^{n-1}(\mathbf{W}; M) & \longrightarrow & C^n(\mathbf{W}; M) & \longrightarrow & C^{n+1}(\mathbf{W}; M) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & C^{n-1}(\mathbf{X}(\mathbf{W}); M') & \longrightarrow & C^n(\mathbf{X}_W; M') & \longrightarrow & 0 & \longrightarrow \end{array}$$

The above sign representation $\mathbb{Z}[-1]$ is clearly induced by the analog sign representation of \mathbf{W} . We have:

THEOREM 9. *Let $M = \mathbb{Z}[-1]$ be the sign representation as above. Then the map $i^* : H^k(\mathbf{W}; M) \rightarrow H^k(\mathbf{X}_W; M')$ is an epimorphism, for all $k \geq 0$.*

PROOF OF THEOREM 9. The proof is entirely analog to [DS00], where only the case $k = n$ is considered. We include it here for reader convenience.

We consider the diagram 21 at vertical level k .

Let $S_k^{(\infty)}$ be the set of (infinite) flags of total cardinality k (see part 2.4) so that we identify $C^k(\mathbf{W}; M)$ with the set of functions $f : S_k^{(\infty)} \rightarrow M$, denoted $M^{S_k^{(\infty)}}$. Analogously we identify $C^k(\mathbf{X}_\mathbf{W}; M')$ with the set of such functions defined only over the flags of length 1, i.e. such that $\Gamma_i = \emptyset$ for $i > 1$.

It is sufficient to consider functions $f \in M^{S_n^{(\infty)}}$ such that

$$f(\Gamma = (\Gamma_1 \supset \Gamma_2 \supset \dots)) = 0 \quad \text{if } |\Gamma_i| > 0 \text{ for some } i > 1$$

(equivalently, if $|\Gamma_2| > 0$). Let us compute the coboundary of f .

(a) $\delta^k(f)(\Gamma_1 \supset \Gamma_2 \supset \dots) = 0$ if $|\Gamma_2| > 1$ or $|\Gamma_3| > 0$.

In fact, by formula (11) we get a linear combination of f computed on chains with $|\Gamma_2| > 0$.

(b) If $\Gamma_1 = \{s_{j_1} < \dots < s_{j_k}\}$, $\Gamma_2 = \{s_{j_m}\}$ then

$$\begin{aligned} (22) \quad \delta^k(f)(\Gamma_1 \supset \Gamma_2 \supset \emptyset \supset \dots) &= \sum_{\beta_2 \in \mathbf{W}_{s_{j_m}}} (-1)^{\alpha(\Gamma, 2, 1, \beta_2)} \rho(\beta_2) \cdot f(\Gamma_1) \\ &= (-1)^{k+1} f(\Gamma_1) + (-1)^{k+1} \rho(s_{j_k}) \cdot f(\Gamma_1) \\ &= (-1)^{k+1} (1 + \rho(s_{j_m})) \cdot f(\Gamma_1) \end{aligned}$$

It immediately follows from (22) that each cocycle $f \in C^k(\mathbf{X}_\mathbf{W}; M')$ can be extended to a cocycle in $C^k(\mathbf{W}; M)$, which concludes the proof. □

END OF PROOF OF THEOREM 7. From Theorem 8, we know that the top-cohomology of $\mathbf{X}_\mathbf{W}$ with coefficients in the sign representation does not vanish. Using theorem 9, the homological genus $h(f_\mathbf{W})$ is greater than $vd(\mathbf{G}_\mathbf{W}) + 1$. Since $\mathbf{X}_\mathbf{W}$ has dimension $vd(\mathbf{G}_\mathbf{W})$, the result follows from theorems 6 and 5. □

We have immediately (see rmk 3.8):

COROLLARY 3.10. *Let \mathbf{W}_a be an affine Weyl group of rank $n + 1$. Then the Schwarz genus of the fibration $\mathbf{Y}_{\mathbf{W}_a} \rightarrow \mathbf{X}_{\mathbf{W}_a}$ is precisely $n + 1$.* □

In case $hvd(\mathbf{G}_\mathbf{W})$ is strictly lower than $vd(\mathbf{G}_\mathbf{W})$ we can get an estimate by slightly modifying the definitions.

DEFINITION 3.11. *We define the rational homological virtual dimension of $\mathbf{G}_\mathbf{W}$, written $rhvd(\mathbf{G}_\mathbf{W})$, as*

$$rhvd(\mathbf{G}_\mathbf{W}) := \max\{n : H_n(D_*^0(\mathbf{W}; \mathbb{Q})) \neq 0\}.$$

Then a straightforward modification of the proof of theorem 7 implies:

THEOREM 10. *For any finitely generated Coxeter system (\mathbf{W}, S) one has*

$$g(f_\mathbf{W}) \geq rhvd(\mathbf{G}_\mathbf{W}) + 1. \quad \square$$

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