Rend. Lincei Mat. Appl. 25 (2014), 309–330
DOI 10.4171/RLM/681

Mathematical Analysis — *Surface measures in infinite dimension*, by GIUSEPPE DA PRATO, ALESSANDRA LUNARDI and LUCIANO TUBARO, communicated on 26 June 2014.

ABSTRACT. — We construct surface measures associated to Gaussian measures in separable Banach spaces, and we prove several properties including an integration by parts formula.

KEY WORDS: Infinite dimensional analysis, surface measures, Gaussian measures.

MATHEMATICS SUBJECT CLASSIFICATION: 28C20.

1. INTRODUCTION

Let X be a separable Banach space with norm $\|\cdot\|$, endowed with a nondegenerate centered Gaussian measure μ, with covariance Q and associated Cameron–Martin space H.

We will construct surface measures, defined on level sets $\{x \in X : G(x) = r\}$ for suitable $G : X \mapsto \mathbb{R}$, and prove several properties including an integration by parts formula for Sobolev functions, that involves a surface integral.

Surface measures in Banach spaces are not a novelty. The first steps were made in the case of Hilbert spaces, for instance in the book [22] where a class of smooth surfaces was considered. To our knowledge, the earliest results in Banach spaces are due to Uglanov [23], about surface measures on (unions of) graphs of smooth functions, and Hertle [13], that deals only with hyperplanes and spherical surfaces.

The first systematic treatment for a more general class of surfaces was done by Airault and Malliavin in [1], that concerns level sets of functions $G \in \bigcap_{k\in\mathbb{N},p\geq 1} W^{k,p}(X,\mu)$ satisfying a nondegeneracy condition, where X is the classical Wiener space. They considered also manifolds of arbitrary codimension $n \in \mathbb{N}$.

The approach of [1] is through the study of the densities of suitable image measures. The same approach was considered in the books [3, 17] in more general contexts, and in [16] in the special case where X is the space of the tempered distributions in $\mathbb{R}$. The aim of this paper is to give an alternative simpler and clearer construction and description of surface measures through the image measures approach, under less regularity assumptions on G with respect to [1, 3, 17]. Considering only 1-codimensional manifolds allows us to avoid several complications.

A completely different approach was followed by Feyel and de La Pradelle in [11], who defined a Hausdorff–Gauss measure ρ on the Borel sets of X by finite dimensional approximations. Another very general surface measure is the perimeter measure, defined as the variation measure of the characteristic function of $\{x \in X : G(x) < r\}$ in the case that such characteristic function is of bounded variation, see [12] and the following papers [2, 14]. It is known that under some regularity assumption on G, the perimeter measure coincides with ρ on the level surfaces of G, and they coincide with the Airault–Malliavin surface measure under further assumptions ([5, 4]).

After the construction of the surface measures σ_r^G, we show several properties of them, under minimal assumptions: they are non trivial (namely, $\sigma_r^G(X) > 0$) for every $r \in (\operatorname{ess\,inf} G, \operatorname{ess\,sup} G)$, the support of σ_r^G is contained in $G^{-1}(r)$, Borel sets with null Gaussian capacity are negligible with respect to σ_r^G for every r, and the integration by parts formula

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu = \int_{G^{-1}(r)} \varphi D_k G\, d\sigma_r^G, \quad k \in \mathbb{N},$$

holds for functions $\varphi \in C_b^1(X;\mathbb{R})$. Here we use standard notation: we fix any orthonormal basis $\{v_k : k \in \mathbb{N}\}$ of H contained in $Q(X^*)$, $D_k\varphi$ denotes the derivative of φ along v_k, and $\hat{v}_k = Q^{-1}v_k \in X^*$. The integration by parts formula holds also for Sobolev functions, provided $\varphi D_k G$ in the surface integral is meant in the sense of traces. Indeed, traces of Sobolev functions on the level hypersurfaces $G^{-1}(r)$ are readily defined through this approach.

At the end of the paper we show that, under suitable assumptions, the measures constructed here coincide with weighted Hausdorff–Gauss surface measures, namely for every $r \in \mathbb{R}$ and for every Borel set $B \subset X$ we have

$$\sigma_r^G(B) = \int_{B\cap G^{-1}(r)} \frac{1}{|D_H G|_H}\, d\rho,$$

where ρ is the above mentioned measure of [11], $D_H G$ is the generalized gradient of G along H, and $|\cdot|_H$ is the H-norm, see Sect. 2. This formula clarifies the dependence of σ_r^G on G. Moreover, more refined properties of surface integrals and traces of Sobolev functions are consequences of the results of [5]. Also, the examples contained in [5] serve as examples here.

2. Notation and preliminaries

We denote by X^* the dual space of X, and by $Q : X^* \mapsto X$ the covariance of μ. The Cameron–Martin space is denoted by H, its scalar product by $\langle \cdot, \cdot \rangle_H$ and its norm by $|\cdot|_H$. The closed ball in H centered at h_0 with radius r is denoted by $B_H(h_0, r)$.

We fix once and for all an orthonormal basis $\mathscr{V} = \{v_k : k \in \mathbb{N}\}$ of H, contained in $Q(X^*)$. For every $k \in \mathbb{N}$ we set $\hat{v}_k = Q^{-1}(v_k)$.

We recall that if X is a Hilbert space and X^* is canonically identified with X, then Q is a compact self–adjoint operator with finite trace, and we can choose a basis $\{e_k : k \in \mathbb{N}\}$ of X consisting of eigenvectors of Q, $Qe_k = \lambda_k e_k$. The space H is just $Q^{1/2}(X)$ with the scalar product $\langle h_1, h_2\rangle_H = \langle Q^{-1/2}h_1, Q^{-1/2}h_2\rangle_X$, the set $\{v_k := \sqrt{\lambda_k}e_k : k \in \mathbb{N}\}$ is an orthonormal basis of H and we have $\hat{v}_k(x) = \langle x, v_k\rangle_X/\lambda_k$ for each $k \in \mathbb{N}$ and $x \in X$.

Let us recall the definition of Gaussian Sobolev spaces $W^{k,p}(X,\mu)$ for $k = 1, 2$, $p \geq 1$.

We say that a function $f : X \mapsto \mathbb{R}$ is H-differentiable at x if there is $v \in H$ such that $f(x+h) - f(x) = \langle v, h\rangle_H + o(|h|_H)$, for every $h \in H$. In this case v is unique, and we set $D_H f(x) := v$. Moreover for every $k \in \mathbb{N}$ the directional derivative $D_k f(x) := \lim_{t\to 0}(f(x+tv_k) - f(x))/t$ exists and coincides with $\langle D_H f(x), v_k\rangle_H$. It is easy to see that if f is Fréchet differentiable at x (as a function from X to $\mathbb{R}$), then it is H-differentiable. In particular, the smooth cylindrical functions, namely the functions of the type $f(x) = \varphi(l_1(x), \dots, l_n(x))$, for some $\varphi \in C_b^\infty(\mathbb{R}^n)$, $l_1, \dots, l_n \in X^*$, $n \in \mathbb{N}$, are H-differentiable at each x.

$W^{1,p}(X,\mu)$ and $W^{2,p}(X,\mu)$ are the completions of the smooth cylindrical functions in the norms

$$\begin{aligned}\|f\|_{W^{1,p}(X,\mu)} &:= \|f\|_{L^p(X,\mu)} + \Big(\int_X \Big(\sum_{k=1}^\infty (D_k f(x))^2\Big)^{p/2}\mu(dx)\Big)^{1/p}\\ &= \|f\|_{L^p(X,\mu)} + \Big(\int_X |D_H f(x)|_H^p \mu(dx)\Big)^{1/p},\end{aligned}$$

$$\|f\|_{W^{2,p}(X,\mu)} := \|f\|_{W^{1,p}(X,\mu)} + \Big(\int_X \Big(\sum_{h,k=1}^\infty (D_{hk} f(x))^2\Big)^{p/2}\mu(dx)\Big)^{1/p}.$$

Such spaces are in fact identified with subspaces of $L^p(X,\mu)$, since if (f_n) and (g_n) are Cauchy sequences in the norm of $W^{1,p}(X,\mu)$ (respectively, in the norm of $W^{2,p}(X,\mu)$), and converge to f in $L^p(X,\mu)$, then the sequences $(D_H f_n)$, $(D_h g_n)$ (respectively, $(D_H^2 f_n)$, $(D_H^2 g_n)$) have equal limits in $L^p(X,\mu;H)$ (respectively, in $L^p(X,\mu;\mathscr{H}_2)$, where $\mathscr{H}_2$ is the set of all Hilbert-Schmidt bilinear forms in H). In other words, the operators D_H and D_H^2, defined in the set of the smooth cylindrical functions with values in $L^p(X,\mu;H)$ and in $L^p(X,\mu;\mathscr{H}_2)$, are closable in $L^p(X,\mu)$. We still denote by D_H and D_H^2 their closures, that are called H-gradient and H-Hessian.

The spaces $W^{1,p}(X,\mu;H)$ are defined similarly, using H-valued, instead of real valued, cylindrical functions. The latter are the elements of the linear span of functions such as $\psi(x) = \varphi(x)h$, with any smooth cylindrical $\varphi : X \mapsto \mathbb{R}$ and $h \in H$.

For $p > 1$ the Gaussian divergence div_μ is defined in $W^{1,p}(X,\mu;H)$ by

$$\mathrm{div}_\mu \Psi(x) = \sum_{k=1}^\infty (D_k\psi_k - \hat{v}_k\psi_k),$$

where $\psi_k(x) = \langle \Psi(x), v_k \rangle_H$, and the series converges in $L^p(X,\mu)$ if $1 < p < \infty$. See [3, Prop. 5.8.8]. It coincides with (minus) the formal adjoint of the H-gradient, since we have the integration by parts formula

$$\int_X \langle D_H\varphi, \Psi\rangle_H \, d\mu = -\int_X \varphi \, \mathrm{div}_\mu \Psi \, d\mu, \quad \varphi \in W^{1,p'}(X,\mu), \quad \Psi \in W^{1,p}(X,\mu;H),$$

with $p' = p/(p-1)$. Taking in particular $\Psi(x) = v_k$ (constant) for any $k \in \mathbb{N}$, we obtain $\mathrm{div}_\mu \Psi = Q^{-1}v_k = \hat{v}_k$, and the integration formula reads as

$$\int_X D_k\varphi \, d\mu = \int_X \hat{v}_k \varphi \, d\mu, \quad k \in \mathbb{N}. \tag{2.1}$$

We refer to [3, Ch. 5] for equivalent definitions and properties.

The surfaces taken into consideration in this paper are level sets $\{x \in X : G(x) = r\}$ of a sufficiently regular function G. Namely, our $G : X \mapsto \mathbb{R}$ is a C_p-quasicontinuous function that satisfies

$$\frac{D_H G}{|D_H G|_H^2} \in W^{1,p}(X,\mu;H) \tag{2.2}$$

for some $p > 1$. Let us recall that a function $G : X \mapsto \mathbb{R}$ is C_p-quasicontinuous if for every $\varepsilon > 0$ there exists an open set A_ε with Gaussian capacity $C_p(A_\varepsilon) < \varepsilon$, such that G is continuous at any $x \notin A_\varepsilon$. The Gaussian capacity of an open set $A \subset X$ is defined as

$$C_p(A) := \inf\{\|g\|_{W^{1,p}(X,\mu)} : g \geq 1 \ \mu\text{-}a.e. \text{ on } A, g \in W^{1,p}(X,\mu)\}.$$

We recall that every element of $W^{1,p}(X,\mu)$ with $p > 1$ has a C_p-quasicontinuous version ([3, Thm. 5.9.6]).

In addition to the Sobolev spaces, we shall consider the space $BUC(X;\mathbb{R})$ of the uniformly continuous and bounded functions from X to $\mathbb{R}$, endowed with the sup norm $\|\cdot\|_\infty$, and the space $C_b^1(X;\mathbb{R})$ of the bounded continuously Fréchet differentiable functions with bounded derivative.

For any Borel function $f : X \mapsto \mathbb{R}$ we denote by $\mu \circ f^{-1}$ the image measure on the Borel sets of $\mathbb{R}$ defined by $(\mu \circ f^{-1})(I) = \mu(f^{-1}(I))$. More generally, if $\varphi : X \mapsto \mathbb{R}$ is another Borel function in $L^1(X,\mu)$, we define the signed measure $(\varphi\mu \circ f^{-1})(I) := \int_{f^{-1}(I)} \varphi \, d\mu$ on the Borel sets I of $\mathbb{R}$.

3. Construction of surface measures

Throughout the paper, $p > 1$ and $G : X \mapsto \mathbb{R}$ is a fixed version of an element of $W^{1,p}(X,\mu)$ (still denoted by G), that satisfies (2.2). As pointed out in [20], for $p = 2$ an easy sufficient condition for G to satisfy (2.2) is

$$G \in W^{2,4}(X,\mu), \quad \frac{1}{|D_H G|_H} \in L^8(X,\mu),$$

which is generalized to

$$G \in W^{2,s}(X,\mu), \quad \frac{1}{|D_H G|_H} \in L^q(X,\mu), \quad \frac{1}{s} + \frac{2}{q} \le \frac{1}{p}$$

if p is any number > 1.

In the following, we shall make some further summability assumptions on the derivatives of G. All of them are satisfied if G fulfills next condition (3.20).

3.1. *Continuity of densities*

The starting point is the following well known lemma. See [20, Proposition 2.1.1].

LEMMA 3.1. *$\mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure dr in $\mathbb{R}$. Moreover the density $q_1 := \frac{d(\mu \circ G^{-1})}{dr}$ is given by*

$$q_1(r) = \int_{G^{-1}(-\infty,r)} \operatorname{div}_\mu\Big(\frac{D_H G}{|D_H G|_H^2}\Big)\, d\mu, \quad r \in \mathbb{R}, \tag{3.1}$$

and it is continuous and bounded.

Lemma 3.1 implies that every level surface $G^{-1}(r)$ is negligible, and that for every $\varphi \in L^1(X,\mu)$ the signed measure $\varphi\mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure. In the following we shall need some properties of the density q_φ of $\varphi\mu \circ G^{-1}$ when φ belongs to a Sobolev space. They are provided by the following lemma, which is a generalization of Lemma 3.1.

LEMMA 3.2. *Let $\varphi \in W^{1,p'}(X,\mu)$. Then $\varphi\mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure, and the corresponding density q_φ is given by*

$$q_\varphi(r) = \int_{G^{-1}(-\infty,r)} \left(\varphi \operatorname{div}_\mu\Big(\frac{D_H G}{|D_H G|_H^2}\Big) + \Big\langle D_H\varphi, \frac{D_H G}{|D_H G|_H^2}\Big\rangle_H\right) d\mu, \tag{3.2}$$

and it is continuous and bounded. There is $C > 0$ independent of φ such that

$$|q_\varphi(r)| \le C\|\varphi\|_{W^{1,p'}(X,\mu)}, \quad r \in \mathbb{R}. \tag{3.3}$$

PROOF. Fix $\alpha < \beta \in \mathbb{R}$ and set

$$f(r) := \mathbb{1}_{[\alpha,\beta]}(r), \quad h(r) = \int_{-\infty}^r f(s)\, ds = \begin{cases} 0 & \text{if } r \le \alpha, \\ r-\alpha & \text{if } \alpha \le r \le \beta, \\ \beta - \alpha & \text{if } r \ge \beta \end{cases}$$

Since h is Lipschitz continuous, then $h \circ G \in W^{1,p}(X,\mu)$ and

$$D_H(h \circ G) = (f \circ G) D_H G.$$

Therefore

$$f \circ G = \mathbb{1}_{[\alpha,\beta]} \circ G = \frac{\langle D_H(h \circ G), D_H G\rangle_H}{|D_H G|_H^2}$$

Assume that $\varphi \in C_b^1(X)$. Multiplying by φ and integrating both sides yields

$$\begin{aligned}\int_{G^{-1}(\alpha,\beta)} \varphi \, d\mu &= \int_X \varphi \frac{\langle D_H(h \circ G), D_H G\rangle_H}{|D_H G|_H^2} d\mu \\ &= -\int_X (h \circ G) \operatorname{div}_\mu\Big(\varphi \frac{\langle D_H(h \circ G), D_H G\rangle_H}{|D_H G|_H^2}\Big) d\mu.\end{aligned}$$

Set as before $\psi = D_H G |D_H G|_H^{-2}$. Then, $\operatorname{div}_\mu(\varphi\psi) = \varphi \operatorname{div}_\mu(\psi) + \langle D_H \varphi, \psi\rangle_H$. Therefore,

$$\begin{aligned}\int_{G^{-1}(\alpha,\beta)} \varphi \, d\mu &= -\int_X (h \circ G) \operatorname{div}_\mu(\varphi\psi) \, d\mu \\ &= -\int_X \Big(\int_{\mathbb{R}} \mathbb{1}_{\{-\infty, G(x)\}}(r) f(r) \, dr\Big) \operatorname{div}_\mu(\varphi\psi)(x) \, d\mu \\ &= -\int_{\mathbb{R}} f(r) \Big(\int_X \mathbb{1}_{G^{-1}(r,+\infty)} \operatorname{div}_\mu(\varphi\psi) \, d\mu\Big) dr \\ &= -\int_\alpha^\beta dr \int_X \mathbb{1}_{G^{-1}(r,+\infty)} \operatorname{div}_\mu(\varphi\psi) \, d\mu \\ &= \int_\alpha^\beta dr \int_X \mathbb{1}_{G^{-1}(-\infty,r)} (\varphi \operatorname{div}_\mu \psi + \langle D_H \varphi, \psi\rangle_H) \, d\mu\end{aligned}$$

(in the last equality we used the fact that the divergence of any vector field in $W^{1,p}(X,\mu;H)$ has zero mean value). Now, let $\varphi \in W^{1,p'}(X,\mu)$ and approach it by a sequence of smooth cylindrical functions φ_n. After applying the above formula to each φ_n, we may let $n \to \infty$ in both sides, since $\varphi_n \to \varphi$ and $\varphi_n \operatorname{div}_\mu \psi + \langle D_H \varphi_n, \psi\rangle_H \to \varphi \operatorname{div}_\mu \psi + \langle D_H \varphi, \psi\rangle_H$ in $L^1(X,\mu)$. Therefore we get

$$(\varphi\mu \circ G^{-1})((\alpha,\beta)) = \int_\alpha^\beta dr \int_X \mathbb{1}_{G^{-1}(-\infty,r)} (\varphi \operatorname{div}_\mu \psi + \langle D_H \varphi, \psi\rangle_H) \, d\mu,$$

namely $\varphi\mu \circ G^{-1}$ has density q_φ given by

$$q_\varphi(r) = \int_{G^{-1}(-\infty,r)} (\varphi \operatorname{div}_\mu \psi + \langle D_H \varphi, \psi\rangle_H) \mu(dx),$$

which is continuous and bounded, since $\varphi \operatorname{div}_\mu \psi + \langle D_H \varphi, \psi\rangle_H \in L^1(X,\mu)$ and $\mu(G^{-1}(r_0)) = 0$ for every $r_0 \in \mathbb{R}$. Estimate (3.3) follows just applying the Hölder inequality. □

3.2. Smoothness of densities

This § is devoted to show that for every uniformly continuous and bounded $\varphi : X \mapsto \mathbb{R}$, the function

$$F_\varphi(r) := \int_{G^{-1}(-\infty,r)} \varphi \, d\mu \tag{3.4}$$

is continuously differentiable.

A useful tool will be the following disintegration formula, whose proof is given for the reader's convenience in the appendix.

Theorem 3.3. *Let X be a Polish space endowed with a Borel probability measure μ. Let $\Gamma : X \to \mathbb{R}$ be a Borel function, and set $\lambda = \mu \circ \Gamma^{-1}$. Then there exists a family of Borel probability measures $\{m_s : s \in \mathbb{R}\}$ on X such that*

$$\int_X \varphi(x)\mu(dx) = \int_{\mathbb{R}} \Big(\int_X \varphi(x) m_s(dx) \Big) \lambda(ds), \tag{3.5}$$

for all $\varphi : X \to \mathbb{R}$ bounded and Borel measurable.

Moreover the support of m_s is contained in $\Gamma^{-1}(s)$ for λ-almost all $s \in \mathbb{R}$.

Proposition 3.4. *Let $\varphi \in BUC(X;\mathbb{R})$. Then $F_\varphi \in C^1_b(\mathbb{R})$.*

Proof. To begin with, let $\varphi : X \mapsto \mathbb{R}$ be Lipschitz continuous. By Lemma 3.2, for each $r \in \mathbb{R}$ we have

$$F_\varphi(r) = \int_{-\infty}^r q_\varphi(s)\, ds,$$

where the function $q_\varphi \in L^1(\mathbb{R})$ is continuous and bounded. Hence, $F_\varphi \in C^1_b(\mathbb{R})$.

Taking $\Gamma = G$ and replacing φ by $\varphi \mathbb{1}_{G^{-1}(-\infty,r)}$ we write the disintegration formula (3.5) as

$$F_\varphi(r) = \int_{-\infty}^r \Big(\int_X \varphi(x) m_s(dx) \Big) q_1(s)\, ds, \quad r \in \mathbb{R}. \tag{3.6}$$

Therefore, there is a Borel set $I_\varphi \subset \mathbb{R}$ such that $(\mu \circ G^{-1})(I_\varphi) = 0$ and

$$F'_\varphi(r) = q_1(r) \int_X \varphi(x) m_r(dx), \quad r \notin I_\varphi,$$

so that

$$|F'_\varphi(r)| \le q_1(r) \|\varphi\|_\infty, \quad r \notin I_\varphi.$$

Since both F'_φ and q_1 are continuous,

$$|F'_\varphi(r)| \le q_1(r) \|\varphi\|_\infty, \quad r \in \mathbb{R}. \tag{3.7}$$

Let now $\varphi \in BUC(X;\mathbb{R})$. By [19, Thm. 1], there is a sequence of Lipschitz continuous and bounded functions φ_n that converge uniformly to φ on X. Recalling that $|F_\varphi(r)| \le \|\varphi\|_{L^1(X,\mu)} \le \|\varphi\|_\infty$ for every $r \in \mathbb{R}$, estimate (3.7) yields that (F_{φ_n}) is a Cauchy sequence in $C_b^1(\mathbb{R})$, and the conclusion follows. □

For every $\varphi \in BUC(X;\mathbb{R})$ we still set

$$q_\varphi(r) := F_\varphi'(r), \quad r \in \mathbb{R}. \tag{3.8}$$

Of course, q_φ is given by (3.2) only if $\varphi \in W^{1,p'}(X,\mu)$.

3.3. Surface measures

Now we are ready to prove the existence of measures on *every* level surface $G^{-1}(r)$.

THEOREM 3.5. *For every $r \in \mathbb{R}$ there exists a unique Borel measure σ_r^G on $\mathscr{B}(X)$ such that*

$$q_\varphi(r) = \int_X \varphi(x)\sigma_r^G(dx), \quad \varphi \in BUC(X;\mathbb{R}). \tag{3.9}$$

Moreover, the support of σ_r^G is contained in $G^{-1}(r)$, and $\sigma_r^G(X) = q_1(r)$. Therefore, σ_r^G is nontrivial iff $q_1(r) > 0$.

PROOF. Fix $r \in \mathbb{R}$ and set

$$L(\varphi) := q_\varphi(r) = F_\varphi'(r), \quad \varphi \in BUC(X;\mathbb{R}) \cup W^{1,p'}(X,\mu). \tag{3.10}$$

Since F_φ is an increasing function for every φ with nonnegative values, then $L(\varphi) \ge 0$ if $\varphi(x) \ge 0$ a.e. Linear positive functionals defined on $BUC(X;\mathbb{R})$ have not necessarily an integral representation such as (3.9). To show that this is the case, we use the following procedure. We approach X by a sequence of compact sets K_n such that $\lim_{n\to\infty} \mu(K_n) = 1$ and we consider suitable restrictions L_n of L to $C(K_n;\mathbb{R})$. By the Riesz Theorem, such restrictions are represented by measures defined on the Borel sets of K_n, readily extended to measures λ_n on all Borel sets of X. Since $(\lambda_n(B))$ is an increasing sequence for every Borel set B, we set $\sigma_r^G(B) := \lim_{n\to\infty} \lambda_n(B)$ and we prove that σ_r^G is a measure, that satisfies (3.9).

Let K be a compact subset of X with positive measure. Since the embedding $H \subset X$ is compact, we may assume that K contains $B_H(0,1)$. Moreover, replacing K by its absolutely convex hull, we may assume that K is symmetric (namely, $K = -K$) and convex. The linear span E of K is a measurable subspace of X with positive measure; by the $0-1$ law (e.g., [3, Thm. 2.5.5]) it has measure 1. Therefore, setting

$$K_n := nK,$$

we have

$$\lim_{n\to\infty} \mu(K_n) = 1.$$

Now we follow a classical procedure in measure theory, see e.g. [6, Ch. 6]. For any $n \in \mathbb{N}$ we consider the restriction L_n of L to K_n defined for all $\varphi \geq 0$ as

$$L_n(\varphi) = \inf\{L(\psi) : \psi \in BUC(X;\mathbb{R}), \psi = \varphi \text{ on } K_n, \psi \geq 0 \text{ on } X\},$$

while if φ takes both positive and negative values, $L_n\varphi$ is defined by

$$L_n\varphi = L_n\varphi^+ - L_n\varphi^-,$$

where φ^+ and φ^- denote the positive and the negative part of φ. The set used in the definition of L_n is not empty, for instance it contains the extension studied in [18],

$$\psi(x) = \begin{cases} \varphi(x), & x \in K_n, \\ \inf_{u \in K_n} \frac{\varphi(u)}{\|x-u\|} \operatorname{dist}(x, K_n), & x \notin K_n. \end{cases}$$

Then, L_n is a positive linear functional in $C(K_n;\mathbb{R})$. Positivity follows immediately from the positivity of L, linearity is not immediate although elementary, it may be proved as in Lemma 6.4 of [6]. Then, there exists a Borel measure λ_n on K_n such that

$$L_n(\varphi) = \int_{K_n} \varphi \, d\lambda_n, \quad \varphi \in C(K_n;\mathbb{R}).$$

The obvious extension of λ_n to $\mathscr{B}(X)$, $B \mapsto \lambda_n(B \cap K_n)$, is still denoted by λ_n.

For every $\varphi \in BUC(X;\mathbb{R})$ with nonnegative values the sequence $(L_n(\varphi))$ is increasing, since $\{L(\psi) : \psi \in BUC(X;\mathbb{R}), \psi = \varphi \text{ on } K_{n+1}, \psi \geq 0 \text{ on } X\} \subset \{L(\psi) : \psi \in BUC(X;\mathbb{R}), \psi = \varphi \text{ on } K_n, \psi \geq 0 \text{ on } X\}$ for every $n \in \mathbb{N}$. It follows that for every $B \in \mathscr{B}(X)$ the sequence $(\lambda_n(B))$ is increasing. Setting

$$\sigma_r^G(B) := \lim_{n\to\infty} \lambda_n(B),$$

we claim that σ_r^G is a measure on $\mathscr{B}(X)$ and that (3.9) holds.

Note that if A, B are Borel sets such that $A \subset B$, then $\sigma_n(A) \leq \sigma_n(B)$ for every n, and consequently $\sigma_r^G(A) \leq \sigma_r^G(B)$. Let now B, $B_m \in \mathscr{B}(X)$ be such that $B_m \uparrow B$. Then

$$\begin{aligned} \lim_{m\to\infty} \sigma_r^G(B_m) &= \lim_{m\to\infty} \left(\lim_{n\to\infty} \lambda_n(B_m) \right) = \sup_{m\in\mathbb{N}} \left(\sup_{n\in\mathbb{N}} \lambda_n(B_m) \right) \\ &= \sup_{n\in\mathbb{N}} \left(\sup_{m\in\mathbb{N}} \lambda_n(B_m) \right) = \sup_{n\in\mathbb{N}} \lambda_n(B) = \sigma_r^G(B). \end{aligned}$$

So, σ_r^G is a measure. As a next step, we prove that (3.9) holds for $\varphi \equiv 1$. To this aim we construct a sequence of $W^{1,p'}(X,\mu)$ functions θ_n, such that $\theta_n \equiv 1$ on K_n and $\theta_n \equiv 0$ outside K_{2n}. The starting point is the Minkowsky functional of K,

$$\mathfrak{m}(x) := \inf\{\lambda \geq 0 : x \in \lambda K\}, \quad x \in E,$$

which is positively homogeneous, sub-additive, and H-Lipschitz since K contains the unit ball of H. Indeed, for any $h \in H$, $h \neq 0$, we have

$$\frac{h}{|h|_H} \in B_H(0,1) \subset K,$$

that is, $h \in |h|_H K$, that implies $\mathfrak{m}(h) \leq |h|_H$. As a consequence, for any $x \in E$, $h \in H$,

$$\mathfrak{m}(x+h) \leq \mathfrak{m}(x) + \mathfrak{m}(h) \leq \mathfrak{m}(x) + |h|_H,$$

and

$$\mathfrak{m}(x) = \mathfrak{m}(x+h-h) \leq \mathfrak{m}(x+h) + \mathfrak{m}(-h) \leq \mathfrak{m}(x+h) + |h|_H,$$

so that

$$|\mathfrak{m}(x+h) - \mathfrak{m}(x)| \leq |h|_H.$$

The null extension f of $\mathfrak{m}$ to the whole of X is also H-Lipschitz, because $H \subset E$ so that if $x \ni E$ then $x+h \ni E$ for every $h \in H$, and $f(x) = f(x+h) = 0$. Therefore, it belongs to $W^{1,q}(X,\mu)$ for every $q > 1$ (e.g., [3, Ex. 5.4.10]). Now, let $\alpha \in C_c^\infty(\mathbb{R})$ be such that $\alpha \equiv 1$ in $[0,1]$, $\alpha \equiv 0$ in $[2,+\infty)$, $0 \leq \alpha \leq 1$, and set

$$\theta_n(x) \begin{cases} = \alpha(\mathfrak{m}(x/n)), & x \in E, \\ = 0, & x \notin E. \end{cases}$$

Then $\theta_n \in W^{1,q}(X,\mu)$ for every $q > 1$. Recalling that for every $x \in E$ we have $\mathfrak{m}(x/n) \leq 1$ iff $x \in K_n$, we obtain $\theta_n \equiv 1$ in K_n, $\theta_n \equiv 0$ outside K_{2n}, and

$$\lim_{n\to\infty} \|\theta_n - 1\|_{W^{1,p'}(X,\mu)} = 0.$$

Indeed, $\lim_{n\to\infty} \theta_n(x) = 1$ for every $x \in E$ and $0 \leq \theta_n(x) \leq 1$, so that $\lim_{n\to\infty} \theta_n = 1$ in $L^{p'}(X,\mu)$. Moreover,

$$D_H\theta_n(x) = \frac{1}{n}\alpha'(\mathfrak{m}(x/n))D_H\mathfrak{m}(x/n),$$

so that $\lim_{n\to\infty} D_H\theta_n = 0$ in $L^{p'}(X,\mu;H)$.

Then,

$$\sigma_r^G(X) = \lim_{n\to\infty} \lambda_n(X) = \lim_{n\to\infty} \int_X 1\, d\lambda_n = \lim_{n\to\infty} L_n(1) = \lim_{n\to\infty} L_{2n}(1).$$

On the other hand, for every $\psi \in BUC(X)$ such that $\psi \geq 1$ in K_{2n}, $\psi \geq 0$ in X, we have $\psi \geq \theta_n$ and therefore $L\psi \geq L(\theta_n)$, since $L(\psi) - L(\theta_n)$ is the derivative at r of the increasing function $\xi \mapsto \mu\{x : \psi(x) - \theta_n(x) \leq \xi\}$. Taking the infimum, we get $L_{2n}(1) \geq L(\theta_n)$. Since θ_n goes to 1 in $W^{1,p'}(X,\mu)$, by Lemma 3.2 $L(\theta_n)$ goes to $L(1) = q_1(r)$ as $n \to \infty$. This shows that

$$\sigma_r^G(X) = q_1(r) = L(1). \tag{3.11}$$

Now we show that (3.9) holds for any $\varphi \in BUC(X;\mathbb{R})$. It is sufficient to prove that it holds for every $\varphi \in BUC(X;\mathbb{R})$ with values in $[0,1]$. In this case, by definition,

$$L\varphi \geq L_n(\varphi_{|K_n}) = \int_X \varphi\, d\lambda_n$$

where the right–hand side converges to $\int_X \varphi\, d\sigma_r^G$ as $n \to \infty$, since the sequence (λ_n) weakly converges to σ_r^G. Therefore,

$$L\varphi \geq \int_X \varphi\, d\sigma_r^G$$

Now we remark that $1 - \varphi$ has positive values, and using (3.11) and the above inequality we get

$$q_1(r) - L\varphi = L(1-\varphi) \geq \int_X (1-\varphi)\, d\sigma_r^G = q_1(r) - \int_X \varphi\, d\sigma_r^G$$

so that

$$L\varphi \leq \int_X \varphi\, d\sigma_r^G,$$

and (3.9) follows.

It remains to prove that σ_r^G has support in $G^{-1}(r)$. To this aim, we remark that for every $\varepsilon > 0$ and $\varphi \in BUC(X;\mathbb{R})$ with support contained in $G^{-1}(-\infty, r-\varepsilon) \cup G^{-1}(r+\varepsilon, +\infty)$, the function F_φ is constant in $(r-\varepsilon, r+\varepsilon)$, and therefore $F'_\varphi(r) = 0$. By (3.9), $\int_X \varphi\, d\sigma_r^G = 0$. So, the support of σ_r^G is contained in $\bigcap_{\varepsilon>0} G^{-1}[r-\varepsilon, r+\varepsilon] = G^{-1}(r)$. □

Remark 3.6. Let m_r be the probability measures given by the disintegration Theorem 3.3. For a.e. $r \in \mathbb{R}$ such that $q_1(r) > 0$ we have

$$\sigma_r^G = q_1(r) m_r.$$

PROOF. Fix any $\varphi \in BUC(X;\mathbb{R})$. As we already observed, applying formula (3.5) to the function $\varphi \mathbb{1}_{G^{-1}(-\infty,r)}$ we obtain

$$F_\varphi(r) = \int_{-\infty}^{r} \Big(\int_X \varphi(x) m_s(dx) \Big) q_1(s)\, ds, \quad r \in \mathbb{R}.$$

On the other hand, by (3.9) we have

$$F_\varphi(r) = \int_{-\infty}^{r} \Big(\int_X \varphi(x) \sigma_s^G(dx) \Big)\, ds, \quad r \in \mathbb{R}.$$

Therefore, there exists a negligible $I_\varphi \subset \mathbb{R}$ such that for every $s \notin I_\varphi$ we have

$$\int_X \varphi\, d\sigma_s^G = q_1(s) \int_X \varphi(x) m_s(dx).$$

We recall that X^* is separable with respect to the weak* topology (e.g., [9, Cor. 3.25]). Let $\mathscr{F} = \{f_n : n \in \mathbb{N}\}$ be any dense subset of X^*, and set $I = \bigcup_{n\in\mathbb{N}} I_{e^{if_n}}$. For every $r \notin I$ we have

$$\int_X e^{if_n(x)}\, d\sigma_r^G = q_1(r) \int_X e^{if_n(x)}\, dm_r.$$

Every $f \in X^*$ is the pointwise limit of a sequence of elements of $\mathscr{F}$. Using the Dominated Convergence Theorem, we obtain that if $q_1(r) \neq 0$ the probability measures $\sigma_r^G/q_1(r)$ and m_r have the same Fourier transform, so that they coincide. □

The following proposition shows a class of sets that are negligible with respect to all measures σ_r^G.

PROPOSITION 3.7. *Let $B \subset X$ be a Borel set with $C_{p'}(B) = 0$. Then $\sigma_r^G(B) = 0$, for every $r \in \mathbb{R}$.*

PROOF. We partly follow the argument used in [3, Lemma 6.10.1]. For every $\varepsilon > 0$ let $O_\varepsilon \supset B$ be an open set such that $C_{p'}(O_\varepsilon) < \varepsilon$. Then there exists $f_\varepsilon \in W^{1,p'}(X,\mu)$ such that $\|f_\varepsilon\|_{W^{1,p'}(X,\mu)} \le \varepsilon$ and $f_\varepsilon \ge 1$ a.e. in O_ε. Replacing f_ε by $\max\{f_\varepsilon, 0\}$ we may assume that $f_\varepsilon \ge \mathbb{1}_{O_\varepsilon}$, μ-a.e.

Let us fix a sequence of BUC functions that converge to $\mathbb{1}_{O_\varepsilon}$ pointwise. For instance, we can take

$$\theta_n(x) = \begin{cases} 0, & x \in X \setminus O_\varepsilon, \\ n\,\mathrm{dist}(x, X\setminus O_\varepsilon), & 0 < \mathrm{dist}(x, X\setminus O_\varepsilon) < 1/n, \\ 1, & \mathrm{dist}(x, X\setminus O_\varepsilon) \ge 1/n \end{cases}$$

Then, $\lim_{n\to\infty} \theta_n(x) = \mathbb{1}_{O_\varepsilon}(x)$, for every $x \in X$. Using the Dominated Convergence Theorem, and then formula (3.9), we get

$$\sigma_r^G(O_\varepsilon) = \int_X \mathbb{1}_{O_\varepsilon}\, d\sigma_r^G = \lim_{n\to\infty} \int_X \theta_n\, d\sigma_r^G = \lim_{n\to\infty} q_{\theta_n}(r).$$

On the other hand, $f_\varepsilon(x) \geq \mathbb{1}_{O_\varepsilon}(x) \geq \theta_n(x)$, for μ-a.e. $x \in X$, so that the function $F_{f_\varepsilon - \theta_n}$ is increasing. In particular, $F'_{f_\varepsilon-\theta_n}(r) = q_{f_\varepsilon}(r) - q_{\theta_n}(r) \geq 0$ for every $r \in \mathbb{R}$. Therefore, for every $r \in \mathbb{R}$,

$$\sigma_r^G(O_\varepsilon) \leq q_{f_\varepsilon}(r).$$

On the other hand, by (3.3) we have

$$|q_{f_\varepsilon}(r)| \leq C\|f_\varepsilon\|_{W^{1,p'}(X,\mu)} \leq C\varepsilon,$$

with C independent of ε. Therefore, $\sigma_r^G(O_\varepsilon) \leq C\varepsilon$, which implies $\sigma_r^G(B) = 0$. □

Proposition 3.7 clarifies the dependence of the measures σ_r^G on the version of G that we have fixed. Two versions of G that coincide outside a set with null $C_{p'}$ capacity give rise to the same measures σ_r^G.

3.4. *Integration by parts formulae*

To start with, we establish an integration formula for C_b^1 functions that is a first step towards an integration by parts formula. The proof follows arguments from [7, 5] (in fact, it is a rewriting of a part of [5, Prop. 4.1] in our setting).

Proposition 3.8. *Let $k \in \mathbb{N}$ be such that either $D_k G \in W^{1,p'}(X,\mu)$ or $D_k G \in BUC(X,\mathbb{R})$. Then for every $\varphi \in C_b^1(X,\mathbb{R})$ and for every $r \in \mathbb{R}$ we have*

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu = q_{\varphi D_k G}(r). \tag{3.12}$$

Moreover, (3.12) *holds also for every $\varphi \in W^{1,q}(X,\mu)$ provided $D_k G \in W^{1,s}(X,\mu)$ and*

$$\frac{1}{q} + \frac{1}{s} + \frac{1}{p} \leq 1. \tag{3.13}$$

Proof. Fix $\varphi \in C_b^1(X;\mathbb{R})$. For $\varepsilon > 0$ we define a function θ_ε by

$$\theta_\varepsilon(\xi) := \begin{cases} 1, & \xi \leq -\varepsilon, \\ -\frac{1}{\varepsilon}\xi, & -\varepsilon < \xi < 0, \\ 0, & \xi \geq 0. \end{cases}$$

and we consider the function

$$x \mapsto \varphi(x)\theta_\varepsilon(G(x) - r),$$

which belongs to $W^{1,p'}(X,\mu)$. Its derivative along v_k is $\theta_\varepsilon'(G(x))D_kG(x)\varphi(x) + \theta_\varepsilon(G(x))D_k\varphi(x)$. Applying the integration by parts formula (2.1) we get

$$\int_X (D_k\varphi - \hat{v}_k\varphi)(\theta_\varepsilon \circ G)\,d\mu = \frac{1}{\varepsilon}\int_{G^{-1}(r-\varepsilon,r)} \varphi D_k G\,d\mu, \quad k \in \mathbb{N}. \tag{3.14}$$

As $\varepsilon \to 0$, $\theta_\varepsilon \circ G$ converges pointwise to $\mathbb{1}_{G^{-1}(-\infty,r)}$. Since $0 \le \theta_\varepsilon \circ G \le 1$, by the Dominated Convergence Theorem the left hand side converges to

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\,d\mu.$$

Concerning the right hand side, for every $\varepsilon > 0$ we have

$$\frac{1}{\varepsilon}\int_{G^{-1}(r-\varepsilon,r)} \varphi D_k G\,d\mu = \frac{1}{\varepsilon}\int_{r-\varepsilon}^{r} q_{\varphi D_k G}(\xi)\,d\xi.$$

Since $\varphi D_k G$ belongs to $W^{1,p'}(X,\mu)$ or to $BUC(X;\mathbb{R})$, by Lemma 3.2 or by Proposition 3.4 the function $q_{\varphi D_k G}$ is continuous. Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{G^{-1}(r-\varepsilon,r)} \varphi D_k G\,d\mu = q_{\varphi D_k G}(r),$$

and (3.12) follows.

Let now $\varphi \in W^{1,q}(X,\mu)$, $D_kG \in W^{1,s}(X,\mu)$, with q, s satisfying (3.13). Let $\varphi_n \in C_b^1(X,\mu)$ approach φ in $W^{1,q}(X,\mu)$, so that $\varphi_n D_k G$ approaches $\varphi D_k G$ in $W^{1,p'}(X,\mu)$. By (3.12), for every $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi_n - \hat{v}_k\varphi_n)\,d\mu = q_{\varphi_n D_k G}(r).$$

Letting $n \to \infty$, the left hand side goes to $\int_{G^{-1}(-\infty,r)} (D_k\varphi_n - \hat{v}_k\varphi_n)\,d\mu$, while the right hand side goes to $q_{\varphi D_k G}(r)$ by (3.3). $\square$

The measures σ_r^G constructed in Theorem 3.5 are trivial if $q_1(r) = 0$. So, it is important to know for which values of r we have $q_1(r) > 0$. This question was addressed in the paper [15], where it was proved that under the assumptions of [1], the set $I := \{r \in \mathbb{R} : q_1(r) > 0\}$ is an interval. Here we improve such a result, characterizing I under more general assumptions and with a different simpler proof.

LEMMA 3.9. *Assume that for every $k \in \mathbb{N}$, $D_kG \in W^{1,p'}(X,\mu) \cup BUC(X,\mathbb{R})$. Then $\{r \in \mathbb{R} : q_1(r) > 0\} = (\operatorname{ess\,inf} G, \operatorname{ess\,sup} G)$.*

PROOF. By Theorem 3.5, if $q_1(r) = 0$ then $\int_X \varphi\,d\sigma_r^G = 0$ for every $\varphi \in BUC(X,\mathbb{R})$, which implies $q_\varphi(r) = 0$ for every $\varphi \in BUC(X,\mathbb{R})$. Approaching any $\varphi \in W^{1,p'}(X,\mu)$ by a sequence of C_b^1 functions φ_n, it follows that $q_\varphi(r) = 0$

for every $\varphi \in W^{1,p'}(X,\mu)$. Taking $\varphi = D_k\psi D_k G$, with any cylindrical smooth ψ, we have $q_\varphi(r) = 0$, and formula (3.12) yields

$$\int_{G^{-1}(-\infty,r)} (D_{kk}\psi - \hat{v}_k D_k\psi)\, d\mu = 0.$$

Summing over k and using [3, Thm. 5.8.3, Rem. 5.8.7] we obtain

$$\int_{G^{-1}(-\infty,r)} L\psi\, d\mu = 0, \tag{3.15}$$

where L is the realization of the Ornstein–Uhlenbeck operator in $L^2(X,\mu)$. We recall (e.g., [3, Thm. 5.7.1]) that the domain of L is $W^{2,2}(X,\mu)$, and the graph norm of L is equivalent to the $W^{2,2}$-norm. This implies that the set of the cylindrical smooth functions is a core for L, and then (3.15) holds for every $\psi \in D(L)$. In other words, the characteristic function $\mathbb{1}_{G^{-1}(-\infty,r)}$ is orthogonal to the range of L. Since 0 is an isolated simple eigenvalue of L, the orthogonal space to the range of L consists of constant a.e. functions. Then, $\mathbb{1}_{G^{-1}(-\infty,r)}$ is constant μ-a.e., which implies that either $\mu(G^{-1}(-\infty,r)) = 0$ or $\mu(G^{-1}(-\infty,r)) = 1$. So, $q_1(r) = 0$ implies that $r \in (-\infty, \operatorname{ess\,inf} G] \cup [\operatorname{ess\,sup} G, +\infty)$.

Conversely, the function F_1 is continuously differentiable, and it is constant in $(-\infty, \operatorname{ess\,inf} G]$ and in $[\operatorname{ess\,sup} G, +\infty)$, so that for every $r \in (-\infty, \operatorname{ess\,inf} G] \cup [\operatorname{ess\,sup} G, +\infty)$ we have $F_1'(r) = q_1(r) = 0$. □

Let us go back to Proposition 3.8. We recall that $q_\varphi(r) = \int_X \varphi\, d\sigma_r^G$ if $\varphi \in BUC(X;\mathbb{R})$. Therefore, if G and φ are so smooth that $\varphi D_k G \in BUC(X;\mathbb{R})$, (3.12) yields

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu = \int_{G^{-1}(r)} \varphi D_k G\, d\sigma_r^G. \tag{3.16}$$

For more general G and φ the above formula still holds, but it is not obvious. For the right hand side of (3.16) to make sense, we need conditions guaranteeing that $\varphi D_k G$ has a trace at $G^{-1}(r)$, belonging to $L^1(X,\sigma_r^G)$. Then, $\varphi D_k G$ in the right hand side integral should be interpreted in the sense of traces.

The starting point is Lemma 3.2 and in particular formula (3.3), applied to the function $|\varphi|$, that together with (3.9) yields

$$\int_{G^{-1}(r)} |\varphi|\, d\sigma_r^G = q_{|\varphi|}(r) \le C\|\varphi\|_{W^{1,p'}(X,\mu)}, \quad \varphi \in C_b^1(X;\mathbb{R}). \tag{3.17}$$

Since $C_b^1(X;\mathbb{R})$ is dense in $W^{1,p'}(X,\mu)$, the above estimate is extended to the whole of $W^{1,p'}(X,\mu)$, and it allows to define the traces of such Sobolev functions at $G^{-1}(r)$. Indeed, approaching any $\varphi \in W^{1,p'}(X,\mu)$ by a sequence of C_b^1 functions φ_n, (3.17) implies that the sequence of the restrictions $\varphi_{n|G^{-1}(r)}$ to $G^{-1}(r)$ is a Cauchy sequence in $L^1(G^{-1}(r),\sigma_r^G)$, that converges to an element of $L^1(G^{-1}(r), \sigma_r^G)$. Still by (3.17), such element does not depend on the approximating sequence.

DEFINITION 3.10. Let $\varphi \in W^{1,p'}(X, \mu)$. The trace of φ at $G^{-1}(r)$ is the limit in $L^1(G^{-1}(r), \sigma_r^G)$ of the sequence of the restrictions $\varphi_{n|G^{-1}(r)}$ to $G^{-1}(r)$, for every sequence of C_b^1 functions φ_n that converges to φ in $W^{1,p'}(X, \mu)$. It is denoted by $\varphi_{|G^{-1}(r)}$.

By definition, $\varphi_{|G^{-1}(r)} \in L^1(G^{-1}(r), \sigma_r^G)$, and $\|\varphi_{|G^{-1}(r)}\|_{L^1(G^{-1}(r), \sigma_r^G)} \le C\|\varphi\|_{W^{1,p'}(X,\mu)}$, where C is the constant in (3.17). In other words, the trace is a bounded operator from $W^{1,p'}(X, \mu)$ to $L^1(G^{-1}(r), \sigma_r^G)$. If $\varphi \in W^{1,q}(X, \mu)$, with $q > p'$, then $|\varphi|^{q/p'} \in W^{1,p'}(X, \mu)$, and estimate (3.17) applied to $|\varphi|^{q/p'}$ yields that the trace of φ at $G^{-1}(r)$ belongs to $L^{q/p'}(G^{-1}(r), \sigma_r^G)$, and the trace operator is bounded from $W^{1,q}(X, \mu)$ to $L^{q/p'}(G^{-1}(r), \sigma_r^G)$.

The trace operator preserves positivity, as the next lemma shows.

LEMMA 3.11. *Let $\varphi \in W^{1,p'}(X, \mu)$ have nonnegative values, μ-a.e. Then for every $r \in \mathbb{R}$ the trace of φ at $G^{-1}(r)$ has nonnegative values, σ_r^G-a.e.*

PROOF. Let (φ_n) be a sequence of C_b^1 functions, converging to φ in $W^{1,p'}(X, \mu)$. Possibly replacing (φ_n) by a subsequence, we may assume that (φ_n) converges to φ pointwise μ-a.e.

We claim that the sequence (φ_n^+) (the positive parts of φ_n) still converges to φ in $W^{1,p'}(X, \mu)$. Indeed, $\|\varphi_n^+ - \varphi\|_{L^{p'}(X,\mu)} \le \|\varphi_n - \varphi\|_{L^{p'}(X,\mu)}$, while, recalling that $D_H\varphi_n^+ = D_H\varphi_n$ in the set $\{x : \varphi_n(x) > 0\}$, and $D_H\varphi_n^+ = 0$ in the set $\{x : \varphi_n(x) \le 0\}$, $D_H\varphi = 0$ in the set $\{x : \varphi(x) = 0\}$ ([3, Lemma 5.7.7]) we obtain

$$
\begin{aligned}
(3.18)\qquad \int_X |D_H\varphi_n^+ - D_H\varphi|_H^{p'}\, d\mu &= \int_{\{x:\varphi_n(x)>0\}} |D_H\varphi_n^+ - D_H\varphi|_H^{p'}\, d\mu \\
&\quad + \int_{\{x:\varphi_n(x)\le 0\}} |D_H\varphi_n^+ - D_H\varphi|_H^{p'}\, d\mu \\
&= \int_{\{x:\varphi_n(x)>0\}} |D_H\varphi_n - D_H\varphi|_H^{p'}\, d\mu \\
&\quad + \int_{\{x:\varphi_n(x)\le 0\}} |D_H\varphi|_H^{p'}\, d\mu \\
&\le \|D_H\varphi_n - D_H\varphi\|_{L^{p'}(X,\mu;H)}^{p'} \\
&\quad + \int_{\{x:\varphi_n(x)\le 0,\, \varphi(x)>0\}} |D_H\varphi|_H^{p'}\, d\mu.
\end{aligned}
$$

Setting $A_n := \{x : \varphi_n(x) \le 0, \varphi(x) > 0\}$, then $\mu(A_n)$ vanishes as $n \to \infty$. This is because for every x in the set

$$
A := \bigcap_{n=1}^{\infty} \bigcup_{k \ge n} A_k,
$$

the sequence $(\varphi_n(x))$ does not converge to $\varphi(x)$, and therefore $0 = \mu(A) = \lim_{n\to\infty} \mu(\bigcup_{k\geq n} A_k) \geq \limsup_{n\to\infty} \mu(A_n)$. Then, the right hand side of (3.18) vanishes as $n \to \infty$, and this implies that $(\varphi_n)^+$ converges to φ in $W^{1,p'}(X,\mu)$. Consequently, the traces of $(\varphi_n)^+$ at $G^{-1}(r)$ converge to the trace of φ at $G^{-1}(r)$, in $L^1(G^{-1}(r), \sigma_r^G)$. Since each $(\varphi_n)^+$ has nonnegative values at every $x \in G^{-1}(r)$, then their L^1 limit has nonnegative values, σ_r^G-a.e. □

Formula (3.9) may now be extended to elements of $W^{1,p'}(X,\mu)$.

LEMMA 3.12. *For every $\varphi \in W^{1,p'}(X,\mu)$ and for every $r \in \mathbb{R}$ we have*

$$q_\varphi(r) = \int_{G^{-1}(r)} \varphi_{|G^{-1}(r)} \, d\sigma_r^G. \tag{3.19}$$

PROOF. It is sufficient to approximate φ in $W^{1,p'}(X,\mu)$ by a sequence of functions $\varphi_n \in C_b^1(X;\mathbb{R})$, and to let $n \to \infty$ in the equality

$$q_{\varphi_n}(r) = \int_X \varphi_n \, d\sigma_r^G,$$

that holds by Theorem 3.5. The left hand side goes to $q_\varphi(r)$ by estimate (3.3), the right hand side goes to $\int_{G^{-1}(r)} \varphi_{|G^{-1}(r)} \, d\sigma_r^G$ by the above construction of the trace of φ. □

With the aid of Proposition 3.7 we can prove that the traces of the elements of $W^{1,p'}(X,\mu)$ at $G^{-1}(r)$ coincide with the restrictions of their $C_{p'}$-quasicontinuous versions at $G^{-1}(r)$. In particular, if φ is a continuous version of a Sobolev function, its trace is just the restriction of φ at $G^{-1}(r)$. This justifies the notation $\varphi_{|G^{-1}(r)}$ for the trace of φ at $G^{-1}(r)$.

PROPOSITION 3.13. *Let φ be a $C_{p'}$-quasicontinuous version of an element of $W^{1,p'}(X,\mu)$. Then the trace of φ at $G^{-1}(r)$ coincides with the restriction of φ at $G^{-1}(r)$, σ_r^G-a.e.*

PROOF. We use arguments similar to [5, Prop. 4.8]. Let (φ_n) be a sequence of smooth cylindrical functions that converge to φ in $W^{1,p'}(X,\mu)$. By [3, Thm. 5.9.6(ii)], applied with the operator $T = (I-L)^{-1/2}$, a subsequence (φ_{n_k}) converges to $\varphi(x)$ for every x except at most on a set with zero Gaussian capacity $C_{p'}$. By Proposition 3.7, such a subsequence converges σ_r^G-a.e to φ. On the other hand, by the definition of the trace, the restrictions of φ_n to $G^{-1}(r)$ converge to $\varphi_{|G^{-1}(r)}$ in $L^1(G^{-1}(r), \sigma_r^G)$. In particular, a subsequence of (φ_{n_k}) converges to $\varphi_{|G^{-1}(r)}$, σ_r^G-a.e. Therefore, $\varphi_{|G^{-1}(r)} = \varphi$, σ_r^G-a.e. □

To extend the integration by parts formula (3.16) to Sobolev functions we need some further assumptions on G.

COROLLARY 3.14. *Let $\varphi \in W^{1,q}(X,\mu)$ and $D_kG \in W^{1,s}(X,\mu)$, with q, s satisfying* (3.13). *Then formula* (3.16) *holds, with $\varphi D_k G$ replaced by $(\varphi D_k G)_{|G^{-1}(r)}$.*

PROOF. Note that $\varphi D_k G \in W^{1,p'}(X,\mu)$. By Lemma 3.12 we have

$$\int_{G^{-1}(r)} (\varphi D_k G)_{|G^{-1}(r)}\, d\sigma_r^G = q_{\varphi D_k G}(r).$$

On the other hand, Proposition 3.8 yields

$$q_{\varphi D_k G}(r) = \int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu,$$

and the statement follows. □

Corollary 3.14 yields an integration by parts formula for $\varphi \in W^{1,s}(X,\mu)$ for any $s > 1$, provided G is good enough. In particular, if next assumption (3.20) holds, then G satisfies (2.2) and the conditions of Corollary 3.14 with any $s > 1$, and (3.16) holds for $\varphi \in W^{1,q}(X,\mu)$ for any $q > 1$.

3.5. Dependence on G, and relationship with other surface measures

Now we are ready to compare the measures σ_r^G defined in §3.3 with the perimeter measure and with the Hausdorff-Gauss surface measure ρ of Feyel and de La Pradelle [11].

We use the notation of [2]. We recall that a subset $B \subset X$ is said to have finite perimeter if $\mathbb{1}_B$ is a bounded variation function, namely there exists a H-valued measure Γ such that for every $k \in \mathbb{N}$ and for every smooth cylindrical function φ we have

$$\int_B (D_k\varphi - \hat{v}_k\varphi)\, d\mu = \int_X \varphi\, d\gamma_k,$$

with $\gamma_k = \langle \Gamma, v_k\rangle_H$. In this case, Γ is unique, it is called perimeter measure, and denoted by $D_\mu \mathbb{1}_B$.

If $G \in W^{2,p'}(X,\mu)$, then for every $r \in \mathbb{R}$ the set $B = G^{-1}(-\infty,r)$ satisfies the above condition, with $D_\mu \mathbb{1}_{G^{-1}(-\infty,r)} = D_H G_{|G^{-1}(r)}\sigma_r^G$. Indeed, by formulae (3.12) (applied to φ) and (3.19) (applied to $\varphi D_k G$), for every smooth cylindrical φ and for every $k \in \mathbb{N}$ we have

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu = \int_X (\varphi D_k G)_{|G^{-1}(r)}\, d\sigma_r^G.$$

On the other hand, for smooth cylindrical φ the trace of $\varphi D_k G$ at the support $G^{-1}(r)$ of σ_r^G coincides σ_r^G-a.e. with the restriction of $\varphi\widetilde{D_k G}$ at $G^{-1}(r)$, where $\widetilde{D_k G}$ is any $C_{p'}$-quasicontinuous version of $D_k G$, by Proposition 3.13.

Let us now recall the assumptions of Feyel [10],

$$G \in \bigcap_{p>1} W^{2,p}(X,\mu), \quad \frac{1}{|D_H G|_H} \in \bigcap_{p>1} L^p(X,\mu), \tag{3.20}$$

under which it was proved that for every $\varphi \in W^{1,q}(X;\mathbb{R})$ for some $q > 1$, the density q_φ of the signed measure $\varphi\mu \circ G^{-1}$ with respect to the Lebesgue measure is given by

$$q_\varphi(r) = \int_{G^{-1}(r)} \frac{\varphi}{|D_H G|_H} d\rho, \quad r \in \mathbb{R}. \tag{3.21}$$

In the right hand side, φ and $|D_H G|_H$ are quasicontinuous versions of the respective Sobolev elements. More precisely, φ is C_q-quasicontinuous and $|D_H G|_H$ is C_p-quasicontinuous for every $p > 1$. See [10, 5].

Proposition 3.15. *Let G satisfy* (3.20). *Then for every Borel set $B \subset X$ we have*

$$\sigma_r^G(B) = \int_{B\cap G^{-1}(r)} \frac{1}{|D_H G|_H} d\rho, \quad r \in \mathbb{R}.$$

Proof. Note that if (3.20) holds, then G satisfies (2.2). Comparing (3.21) with (3.9) yields

$$\int_X \varphi \, d\sigma_r^G = \int_{G^{-1}(r)} \frac{\varphi}{|D_H G|_H} d\rho, \quad \varphi \in C_b^1(X;\mathbb{R}),$$

and the statement holds. □

Corollary 3.16. *Let G_1, G_2 satisfy* (3.20). *Assume that for some $r_1, r_2 \in \mathbb{R}$ we have $G_1^{-1}(r_1) = G_2^{-1}(r_2) := \Sigma$. Then*

$$|D_H G_1(r_1)|_H \, d\sigma_{r_1}^{G_1} = |D_H G_2(r_2)|_H \, d\sigma_{r_2}^{G_2} = \rho_{|\Sigma}.$$

We recall that the assumptions of [1] are

$$G \in \bigcap_{k\in\mathbb{N}, p>1} W^{k,p}(X,\mu), \quad \frac{1}{|D_H G|_H} \in \bigcap_{p>1} L^p(X,\mu), \tag{3.22}$$

and that the measures v_r constructed in [1, 3] under such assumptions, for all $r \in \mathbb{R}$ such that $q_1(r) > 0$, coincide with the restriction of ρ to $G^{-1}(r)$. This is because they satisfy the same integration by parts formula,

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi)\, d\mu = \int_{G^{-1}(r)} \frac{\varphi D_k G}{|D_H G|_H} dv_r = \int_{G^{-1}(r)} \frac{\varphi D_k G}{|D_H G|_H} d\rho, \tag{3.23}$$
$$\varphi \in \bigcap_{k\in\mathbb{N}, p>1} W^{k,p}(X,\mu),$$

and replacing φ by $\varphi D_k G/|D_H G|_H$ and summing up, we obtain

$$\int_{G^{-1}(r)} \varphi\, d\nu_r = \int_{G^{-1}(r)} \varphi\, d\rho, \quad \varphi \in \bigcap_{k\in\mathbb{N},\, p>1} W^{k,p}(X,\mu)$$

which implies the statement.

A. Proof of the disintegration theorem

We follow here [21]. For any $A \in \mathscr{B}(X)$ we consider the conditional expectation $\mathbb{E}[\mathbb{1}_A|\Gamma]$, which may be expressed as $f_A \circ \Gamma$ for some Borel function $f_A : \mathbb{R} \mapsto \mathbb{R}$. So, for any $I \in \mathscr{B}(\mathbb{R})$ we have

$$\int_{\Gamma^{-1}(I)} \mathbb{1}_A\, d\mu = \int_{\Gamma^{-1}(I)} (f_A \circ \Gamma)\, d\mu = \int_I f_A(r)(\mu \circ \Gamma^{-1})(dr)$$

which we rewrite as

$$\mu(A \cap \Gamma^{-1}(I)) = \int_I f_A(r)\lambda(dr). \tag{A.1}$$

Since X is separable, there exists $K \subset \mathbb{R}$ with $\lambda(K) = 0$ and for any $r \notin K$ a Borel measure m_r on $\mathbb{R}$ such that

$$f_A(r) = m_r(A), \quad \forall r \notin K.$$

See e.g. [8, Theorem 10.2.2]. Replacing in (A.1) we obtain

$$\mu(A \cap \Gamma^{-1}(I)) = \int_I m_r(A)\lambda(dr), \quad I \in \mathscr{B}(\mathbb{R}). \tag{A.2}$$

It is enough to prove that (3.5) holds for $\varphi = \mathbb{1}_{\Gamma^{-1}(J)}$ with $J \in \mathscr{B}(X)$. In this case,

$$\int_X \varphi(x) m_r(dx) = m_r(\Gamma^{-1}(J))$$

and integrating with respect to λ over $\mathbb{R}$ and taking into account of (A.2) with $I = \mathbb{R}$, yields

$$\int_{\mathbb{R}} \Big(\int_X \varphi(x) m_r(dx) \Big) \lambda(dr) = \int_{\mathbb{R}} m_r(\Gamma^{-1}(J))\lambda(dr) = \mu(\Gamma^{-1}(J)),$$

as claimed.

Let us show that for λ-a.e. $r_0 \in \mathbb{R}$, the support of m_{r_0} is contained in $\Gamma^{-1}(r_0)$. If I is any interval, setting $A = \Gamma^{-1}(\mathbb{R}\backslash I)$ in (A.2), we find

$$0 = \int_I m_r(\Gamma^{-1}(\mathbb{R}\backslash I))\lambda(dr),$$

so that $m_r(\Gamma^{-1}(\mathbb{R}\backslash I)) = 0$ for λ-almost all $r \in I$, say for every $r \in I\backslash J_I$ with $\lambda(J_I) = 0$. Now, let us consider all the open intervals with rational endpoints, $I = (a_n, b_n)$ with $a_n < b_n \in \mathbb{Q}$. The set $J := \bigcup_{n\in\mathbb{N}} J_{(a_n,b_n)}$ is still λ-negligible, and we have

$$\text{(A.3)} \qquad m_r(\Gamma^{-1}((-\infty, a_n] \cup [b_n, +\infty))) = 0, \quad n \in \mathbb{N}, \quad r \in (a_n, b_n)\backslash J.$$

For every $r_0 \in \mathbb{R}\backslash J$, fix two subsequences (a_{n_k}), (b_{n_k}) such that $a_{n_k} < r_0$, $b_{n_k} > r_0$ and $\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} b_{n_k} = r_0$. Taking $r = r_0$ and replacing a_n, b_n by a_{n_k}, b_{n_k} in (A.3), we obtain that m_{r_0} has support contained in (a_{n_k}, b_{n_k}) for every $k \in \mathbb{N}$, and the statement follows.

References

[1] H. Airault - P. Malliavin: *Intégration géométrique sur l'espace de Wiener*, Bull. Sci. Math. 112 (1988), 3–52.

[2] L. Ambrosio - M. Miranda Jr. - S. Maniglia - D. Pallara: *BV functions in abstract Wiener spaces*, J. Funct. Anal. 258 (2010), 785–813.

[3] V. I. Bogachev: *Gaussian Measures*, American Mathematical Society, Providence (1998).

[4] V. Caselles - A. Lunardi - M. Miranda Jr. - M. Novaga: *Perimeter of sublevel sets in infinite dimensional spaces*, Adv. Calc. Var. 5 (2012), 59–76.

[5] P. Celada - A. Lunardi: *Traces of Sobolev functions on regular surfaces in infinite dimensions*, J. Funct. Anal. 266 (2014), 1948–1987.

[6] G. Da Prato: *An introduction to infinite dimensional analysis*, Springer-Verlag, Berlin (2006).

[7] G. Da Prato - A. Lunardi: *On the Dirichlet semigroup for Ornstein–Uhlenbeck operators in subsets of Hilbert spaces*, J. Funct. Anal. 259 (2010), 2642–2672.

[8] R. M. Dudley: *Real Analysis and Probability*, Cambridge Univ. Press, Cambridge (2002).

[9] M. Fabian - P. Habala - P. Hajek, V. Montesinos Santalucía, J. Pelant, V. Zizler: *Functional analysis and infinite-dimensional geometry*, Springer-Verlag, New York (2001).

[10] D. Feyel: *Hausdorff-Gauss Measures*, in: Stochastic Analysis and Related Topics, VII. Kusadasi 1998, Progr. in Probab. 98, Birkhäuser (2001), 59–76.

[11] D. Feyel - A. de La Pradelle: *Hausdorff measures on the Wiener space*, Pot. Analysis 1 (1992), 177–189.

[12] M. Fukushima - M. Hino: *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal. 183 (2001), 245–268.

[13] A. Hertle: *Gaussian surface measures and the Radon transform on separable Banach spaces*, Lecture Notes in Math. 794, 513–531. Springer-Verlag (1980).

[14] M. Hino: *Sets of finite perimeter and the Hausdorff–Gauss measure on the Wiener space*, J. Funct. Anal. 258 (2010), 1656–1681.

[15] F. Hirsch - S. Song: *Properties of the set of positivity for the density of a regular Wiener functional*, Bull. Sci. Math. 122 (1998), 1–15.

[16] S. Chaari - F. Cipriano - H.-H. Kuo - H. Ouerdiane: *Surface measures on the dual space of the Schwartz space*, Comm. Stoch. Anal. 4 (2010), 467–480.

[17] P. Malliavin: *Stochastic analysis*, Springer-Verlag, Berlin (1997).

[18] M. Mandelkern: *On the uniform continuity of Tietze extensions*, Arch. Math. 55 (1990), 387–388.

[19] R. Miculescu: *Approximation by Lipschitz functions generated by extensions*, Real Anal. Exchange 28 (2002/03), 33–40.

[20] D. Nualart: *The Malliavin calculus and related topics*, Probability and its Applications, Springer-Verlag, Berlin (1995).

[21] L. Schwartz: *Surmartingales régulières à valeurs mesures et désintégration régulieѐre d'une mesure*, J. Analyse Math. 26 (1973), 1–168.

[22] A. V. Skorohod: *Integration in Hilbert Space*, Springer Verlag, New York (1974).

[23] A. V. Uglanov: *Surface integrals in a Banach space*, Matem. Sb. 110 (1979), 189–217; English transl.: Math. USSR Sb. 38 (1981), 175–199.

[24] A. V. Uglanov: *Integration on Infinite-Dimensional Surfaces and Its Applications*, Springer Science+Business Media, B.V. (2000).

Received 13 May 2014,
and in revised form 29 May 2014.

Giuseppe Da Prato
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56126 Pisa, Italy
g.daprato@sns.it

Alessandra Lunardi
Dipartimento di Matematica e Informatica
Università di Parma
Parco Area delle Scienze, 53/A
43124 Parma, Italy
alessandra.lunardi@unipr.it

Luciano Tubaro
Dipartimento di Matematica
Università di Trento
Via Sommarive 14
38123 Povo, Italy
tubaro@science.unitn.it