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Mathematics — *On special covariants in the exterior algebra of a simple Lie algebra*, by Corrado De Concini, Pierluigi Möseneder Frajria, Paolo Papi and Claudio Procesi, communicated on 26 June 2014.

Abstract. — We study the subspace of the exterior algebra of a simple complex Lie algebra linearly spanned by the copies of the little adjoint representation or, in the case of the Lie algebra of traceless matrices, by the copies of the n-th symmetric power of the defining representation. As main result we prove that this subspace is a free module over the subalgebra of the exterior algebra generated by all primitive invariants except the one of highest degree.

Key words: Exterior algebra, invariants, little adjoint representation, small representation.

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1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank r. In [3], the isotypic component $A = \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \bigwedge \mathfrak{g})$ of the adjoint representation in the exterior algebra of $\mathfrak{g}$ has been studied. Recall that the invariant algebra $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is an exterior algebra $\bigwedge(P_1, \dots, P_r)$ over primitive generators P_i of degree $2m_i + 1$, where the integers m_i (with $m_1 \leq \cdots \leq m_r$) are the exponents of $\mathfrak{g}$. The main result of [3] states that A is a free algebra of rank $2r$ over the algebra $\bigwedge(P_1, \dots, P_{r-1})$. The purpose of this short paper is to single out other instances of this special behavior. We prove that the space $\operatorname{Hom}_{\mathfrak{g}}(L, \bigwedge \mathfrak{g})$ is a free algebra of dimension twice the dimension of the 0-weight space of L in the following remarkable cases:

(1) L is the *little adjoint* representation $L(\theta_s)$, i.e. the $\mathfrak{g}$-module with highest weight the highest short root of $\mathfrak{g}$;
(2) $\mathfrak{g}$ is of type A_{n-1} and $L = S^n(V)$ is the n-th symmetric power of the defining representation V. Clearly, also its dual representation shares this property.

In order to build up free generators in the little adjoint case we are going to use a result of Broer [2]. Once we have the correct candidates, the proof of the statement will follow by slight modifications of the machinery developed in [3] for the adjoint representation. The case of $S^n(V)$ is dealt with by using classical invariant theory.

The adjoint representation, the little adjoint representation, $S^n(V)$ and its dual are examples of small representations (see Section 2). For a small representation, and in fact only for a small representation, one has (see [9]) that its multiplicity in $\bigwedge \mathfrak{g}$ equals 2^r times the dimension of its zero weight space, a fact that we are going to use below. It is natural to ask whether covariants of small modules have the nice behavior described above. It is easy to provide counterexamples. In this respect, using a result of Stembridge, we are able to show that in type A the adjoint representation, $S^n(V)$ and $S^n(V)^*$ are the only small modules whose covariants are freely generated over $\bigwedge(P_1, \ldots, P_{r-1})$. Computer computations show that no other example arises among small modules for any Lie algebra of rank at most 5.

The analysis of covariants of the little adjoint representation when $\mathfrak{g}$ is of classical type has been also performed in [4] using classical invariant theory.

2. The little adjoint module

Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank r. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Let Δ be the corresponding root system, W the Weyl group, Δ^+ a positive system and Π the corresponding simple system. Let Δ_l, Δ_s denote the sets of long and short roots, respectively; set also $\Delta_s^+ = \Delta_s \cap \Delta^+$, $\Delta_l^+ = \Delta_l \cap \Delta^+$, $\Pi_s = \Delta_s \cap \Pi$, $\Pi_l = \Delta_l \cap \Pi$, $r_s = |\Pi_s|$, $r_l = |\Pi_l|$. Let $(\cdot, \cdot)$ denote the Killing form. If $\alpha \in \mathfrak{h}^*$, we let h_α be the unique element of $\mathfrak{h}$ such that $(h_\alpha, h) = \alpha(h)$ for all $h \in \mathfrak{h}$. We use this form $(\cdot, \cdot)$ to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ when convenient.

Assume that $\mathfrak{g}$ is not simply laced. Let θ_s be the highest (w.r.t. Δ^+) short root. The irreducible $\mathfrak{g}$-module $L(\theta_s)$ of highest weight θ_s is called the little adjoint representation.

We are interested in the study of

$$LA := \mathrm{Hom}_{\mathfrak{g}}(L(\theta_s), \textstyle\bigwedge \mathfrak{g}).$$

LA is the space of $\mathfrak{g}$-equivariant maps from $L(\theta_s)$ to the space of multilinear alternating functions on $\mathfrak{g}$. Clearly LA is a (left or right) module over $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$.

If L is a $\mathfrak{g}$-module, we denote by L_0 its zero weight subspace. We shall prove the following:

Theorem 2.1. *LA is freely generated over $\bigwedge(P_1, \ldots, P_{r-1})$ by $2 \dim L(\theta_s)_0$ generators, which can be explicitly described.*

As an application, we can recover the following result of Bazlov [1].

Corollary 2.2. *The Poincaré polynomial $GM_{\theta_s}(q)$, describing the dimension of LA in each degree, is given by*

$$GM_{\theta_s}(q) = (1 + q^{-1}) \prod_{i=1}^{r-1} (1 + q^{2m_i+1}) q^{m_r+1-2(r_s-1)r_l} \frac{1 - q^{4r_l r_s}}{1 - q^{4r_l}}. \tag{2.1}$$

As explained in the Introduction, our main tools are a result of Broer on covariants of small modules and the machinery developed in [3] to prove the analogue of Theorem 2.1 for the adjoint representation.

Let us describe Broer's result. Recall that a finite dimensional $\mathfrak{g}$-module L is called *small* if twice a root is not a weight of L.

THEOREM 2.3 [2, Theorem 1]. *Let L be a small $\mathfrak{g}$-module. Then $\mathrm{Hom}_{\mathfrak{g}}(L, S(\mathfrak{g}))$ is isomorphic by restriction to $\mathrm{Hom}_W(L_0, S(\mathfrak{h}))$ as a module for $S(\mathfrak{h})^W \simeq S(\mathfrak{g})^{\mathfrak{g}}$.*

It is easy to check (see [9]) that an highest weight module $L(\lambda)$ is small if and only if $\lambda \not\geq 2\eta$ for any dominant root η of $\mathfrak{g}$. In particular, both the adjoint and the little adjoint representations are small. In order to apply Theorem 2.3, we start with a simple observation. Let H be the subgroup of W generated by the reflections s_α with α long.

LEMMA 2.4. *H acts trivially on $L(\theta_s)_0$.*

PROOF. The weights of $L(\theta_s)$ are precisely $\Delta_s \cup \{0\}$. It follows that, if α is a long root, and e_α, f_α are root vectors in $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$, respectively, then $\exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha)$ acts trivially on $L(\theta_s)_0$. □

Let W_s be the reflection subgroup of W generated by the reflections s_α with $\alpha \in \Pi_s$.

LEMMA 2.5.

(1) *$W = W_s \ltimes H$ so W/H is canonically isomorphic to W_s.*
(2) *The isomorphism in (1) turns $L(\theta_s)_0$ into a W_s-module isomorphic to the reflection representation of W_s.*

PROOF. The proof of (1) is given in [7], Proposition 2.1. We now prove (2). By Lemma 2.4, $L(\theta_s)_0$ is a W/H-module. The isomorphism given in (1) is the one induced by the embedding of W_s in W. To prove our claim we need to provide a bijective map $\mathrm{Span}(\Pi_s) \to L(\theta_s)_0$ and check that this map intertwines the action of W_s. We realize $L(\theta_s)_0$ explicitly as follows: $\mathfrak{g}$ is the fixed point subalgebra of a diagram automorphism σ of a larger simple Lie algebra $\mathfrak{a}$. Let k be the order of σ ($k = 2$ or 3). Let ξ be a primitive k-th root of unity. Then $L(\theta_s)$ is the ξ-eigenspace of σ in $\mathfrak{a}$. Let $\mathfrak{h}'$ be a σ-stable Cartan subalgebra of $\mathfrak{a}$ containing $\mathfrak{h}$. Since there is no root α of $\mathfrak{a}$ such that $\alpha_{|\mathfrak{h}} = 0$, we have that $L(\theta_s)_0$ is the ξ-eigenspace of $\sigma_{|\mathfrak{h}'}$. Let Π' be the set of simple roots of $\mathfrak{a}$ and Π'_0 the subset of simple roots fixed by σ. Let Π'_c be a connected component of $\Pi' \backslash \Pi'_0$. Then the map $\alpha \mapsto \alpha_{|\mathfrak{h}}$ identifies $\mathrm{Span}(\Pi'_c)$ with $\mathrm{Span}(\Pi_s)$. Let π be the orthogonal projection $\mathfrak{h}' \to L(\theta_s)_0$. We define a map $\mathrm{Span}(\Pi_s) \to L(\theta_s)_0$ by

$$\alpha_{|\mathfrak{h}} \mapsto \pi(h_\alpha) = \frac{1}{k} \sum_{i=0}^{k-1} \xi^i \sigma^{k-i}(h_\alpha).$$

If $\alpha \in \mathrm{Span}(\Pi'_c)$, then the $\sigma^i(\alpha)$ are pairwise orthogonal, hence the above formula implies that the map is injective. Since $\dim L(\theta_s)_0 = \frac{1}{k-1}(\mathrm{rank}(\mathfrak{a}) - \mathrm{rank}(\mathfrak{g}))$, it is easy to check that $\dim \mathrm{Span}(\Pi_s) = \dim L(\theta_s)_0$. It follows that our map is bijective.

If $\gamma \in \Pi_s$ then $\gamma = \alpha_{|\mathfrak{h}}$ with $\alpha \in \Pi'_c$. Then $e_\gamma = \sum_{i=0}^{k-1} e_{\sigma^i(\alpha)}$ and $f_\gamma = \sum_{i=0}^{k-1} f_{\sigma^i(\alpha)}$. Since the roots in the σ-orbit of α are orthogonal, we see that

$$s_\gamma = \exp(e_\gamma)\exp(-f_\gamma)\exp(e_\gamma) = \prod_{i=0}^{k-1} s_{\sigma^i(\alpha)},$$

so the action of s_γ on $\mathfrak{h}'$ commutes with σ. It follows that, if $\beta \in \mathrm{Span}(\Pi_s)$ and $\beta = \beta'_{|\mathfrak{h}}$ with $\beta' \in \Pi'_c$, then

$$s_\gamma(\beta) \mapsto \pi(h_{s_\gamma(\beta')}) = s_\gamma(\pi(h_{\beta'})). \qquad \square$$

By [7], $\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_l} \mathfrak{g}_\alpha$ is a semisimple equal rank subalgebra of $\mathfrak{g}$ whose Weyl group is H. Obviously, the action of H on $\mathfrak{h}$ is the reflection representation. Let $J_H = S(\mathfrak{h})^H$. Since H is a normal subgroup of W, it is clear that J_H is W-stable.

PROPOSITION 2.6. $\dim \mathrm{Hom}_W(L(\theta_s)_0, J_H/J_H^2) = 1$ *and* $\dim(J_H/J_H^2)^W = r_l$.

PROOF. The proof is a case by case check. In each case we will provide an explicit realization of the reflection representation of W_s in a suitable W-stable space of basic invariants for H.

Type C_r. In this case $\mathfrak{l}$ is the product of r copies of A_1, so $H = (\mathbb{Z}/2\mathbb{Z})^r$. Let $\Delta_l^+ = \{\beta_1, \ldots, \beta_r\}$. Clearly $J_H/J_H^2 \simeq \mathrm{Span}(h_{\beta_1}^2, \ldots, h_{\beta_r}^2)$. It is easy to check that $W_s \simeq S_r$ and its action on $\mathfrak{h}$ is given by the permutation representation on the basis $\{h_{\beta_i}\}$. It follows that J_H/J_H^2 is the sum of the reflection representation of S_r and a 1-dimensional invariant space.

Type B_r. In this case $\mathfrak{l}$ is of type D_r, so $J_H = \mathbb{C}[p_0, p_1, \ldots, p_{r-1}]$, where p_i is a basic invariant for B_r of degree $2i$ if $i = 1, 2, \ldots, r-1$ and $p_0 = \prod_{\alpha \in \Delta_s^+} h_\alpha$. Since W_s has order 2, generated by the reflection s_α w.r.t. the unique short simple root α, we see that $\mathbb{C}p_0$ affords the reflection representation of W_s and that $(J_H/J_H^2)^W \simeq \mathrm{Span}(p_1, \ldots, p_{r-1})$.

Type G_2. In this case $\mathfrak{l}$ is of type A_2, so there are basic invariants p_1, p_2 for H in degree 2 and 3 respectively. We can choose p_1 to be the basic invariant of degree 2 for W. In this case $W_s \simeq \mathbb{Z}/2\mathbb{Z}$. Since $J_H \cap S^3(\mathfrak{h}) = \mathbb{C}p_2$ we see that $\mathbb{C}p_2$ is W_s-stable. Since p_2 is not W-invariant, we see that W_s acts on $\mathbb{C}p_2$ by its reflection representation.

Type F_4. In this case $\mathfrak{l}$ is of type D_4 and $W_s \simeq S_3$. Let h_1, f_1, f_2, h_2 be basic invariants for H of degree 2, 4, 4, 6 respectively. The basic invariants for W occur in degrees 2, 6, 8, 12. We can choose h_1, h_2 to be basic invariants for W. We claim that the action of W_s on $\mathrm{Span}(f_1, f_2)$ is given by its reflection representation. Indeed, since $\mathrm{Span}(f_1, f_2)$ cannot contain invariants for W_s, the only other possibility is that W_s acts on f_1, f_2 by the sign representation. If this were the case, we would have that $\dim S^8(\mathfrak{h})^W \geq 5$. But we know that $\dim S^8(\mathfrak{h})^W = 3$. $\square$

Proof of Theorem 2.1. Choose $q \in \mathrm{Hom}_W(L(\theta_s)_0, J_H)$ so that q induces the embedding of $L(\theta_s)_0$ in J_H/J_H^2 provided by Proposition 2.6. We can choose q to be homogeneous and we let n_0 be the degree of q. We can write

$$J_H/J_H^2 = q(L(\theta_s)_0) \oplus (J_H/J_H^2)^W$$

as a W-module. Using the fact that $J_H = S(J_H/J_H^2)$ and Lemma 2.4, we can write

$$\begin{aligned} \text{(2.2)} \quad \mathrm{Hom}_W(L(\theta_s)_0, S(\mathfrak{h})) &= \mathrm{Hom}_W(L(\theta_s)_0, J_H) \\ &= S((J_H/J_H^2)^W) \otimes \mathrm{Hom}_{W_s}(L(\theta_s)_0, S(q(L(\theta_s)_0))). \end{aligned}$$

Since the action of W_s on $L(\theta_s)_0$ is the reflection representation and W_s is a reflection group of type A, it is known (see [5]) that $\mathrm{Hom}_{W_s}(L(\theta_s)_0, S(L(\theta_s)_0))$ is freely generated over $S(L(\theta_s)_0)^{W_s}$ by r_s homogeneous generators $g_1, \dots, g_{r_s}$ in degrees $1, 2, \dots, r_s$. It follows from (2.2) that $q(g_i)$ $(i = 1, \dots, r_s)$ are free generators for $\mathrm{Hom}_W(L(\theta_s)_0, S(\mathfrak{h}))$ over $S(\mathfrak{h})^W$ in degrees $n_0, 2n_0, \dots, r_s n_0$.

Theorem 2.3 now provides free generators $F_1, \dots, F_{r_s}$ for $\mathrm{Hom}_{\mathfrak{g}}(L(\theta_s), S(\mathfrak{g}))$ over $S(\mathfrak{g})^{\mathfrak{g}}$ in degrees $n_0, 2n_0, \dots, r_s n_0$. Let $\delta : \bigwedge^i \mathfrak{g} \to \bigwedge^{i+1} \mathfrak{g}$ be the Koszul differential. Let $s : S(\mathfrak{g}) \to \bigwedge \mathfrak{g}$ be the map extending $\delta_{|\bigwedge^1 \mathfrak{g}} : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ to $S(\mathfrak{g})$. Since s is a $\mathfrak{g}$-equivariant map, composing with s defines a map $\mathrm{Hom}_{\mathfrak{g}}(L, S(\mathfrak{g})) \to \mathrm{Hom}_{\mathfrak{g}}(L, \bigwedge \mathfrak{g})$. Set $f_i = s \circ F_i$ and $u_i = \partial \circ f_i$. Here $\partial = {}^t\delta$.

We claim that f_i, u_i are free generators for $\mathrm{Hom}_{\mathfrak{g}}(L(\theta_s), \bigwedge \mathfrak{g})$ over $\bigwedge(P_1, \dots, P_{r-1})$. From now on, we may proceed as in [3]. Let us sketch the main steps. By [9, Corollary 4.2], we have that $\dim LA = 2^r \dim L(\theta_s)_0$, hence it suffices to prove that f_i, u_i are linearly independent over $\bigwedge(P_1, \dots, P_{r-1})$. Writing a linear combination of f_i, u_i with coefficients in $\bigwedge(P_1, \dots, P_{r-1})$ and applying δ one readily reduces to prove that the f_i are independent. Identify $\mathrm{Hom}_{\mathfrak{g}}(L(\theta_s), \bigwedge \mathfrak{g})$ with $(\bigwedge \mathfrak{g} \otimes L(\theta_s))^{\mathfrak{g}}$ and fix a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ on $L(\theta_s)$. For $a, b \in \bigwedge \mathfrak{g}$, $x, y \in L(\theta_s)$ we set

$$e(a \otimes x, b \otimes y) = \langle x, y \rangle a \wedge b.$$

If instead $a, b \in S(\mathfrak{g})$, then we set

$$(a \otimes x, b \otimes y) = \langle x, y \rangle ab.$$

Now, as in [3, Lemma 2.6], the claim about the independence of the f_i boils down to showing that

$$\text{(2.3)} \qquad e(f_i, u_{r_s - i + 1}) = c_i P_r, \quad c_i \neq 0.$$

Let $d : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes \mathfrak{g}$ be the usual differential on functions and $m : \bigwedge \mathfrak{g} \otimes \mathfrak{g} \to \bigwedge \mathfrak{g}$ the multiplication map. Define $t : S(\mathfrak{g}) \to \bigwedge \mathfrak{g}$ setting $m \circ (s \otimes 1) \circ d$. The argument given in [3] to prove formula (2.21) therein shows that, up to a nonzero constant,

$$e(f_i, u_{r_s - i + 1}) = t((F_i, F_{r_s - i + 1})).$$

Now observe that, by inspection, we have

$$n_0 = \begin{cases} \frac{m_r+1}{2} & \text{if } r_s = 1, \\ \frac{m_r+1}{2} - (r_s-1)r_l = 2r_l & \text{if } r_s > 1. \end{cases} \tag{2.4}$$

This implies that $2n_0 i + 2n_0(r_s - i + 1) - 1 = 2n_0(r_s+1) - 1 = 2m_r + 1$.

Recall that the range of the map t, when restricted to $S(\mathfrak{g})^{\mathfrak{g}}$, is, by e.g. [6, Theorem 64], the space of primitive elements in $\bigwedge \mathfrak{g}$, so it is enough to check that $t((F_i, F_{r_s-i+1})) \neq 0$. This is equivalent to checking that, if J^+ is the ideal in $S(\mathfrak{g})^{\mathfrak{g}}$ of elements of positive degree, then $(F_i, F_{r_s-i+1}) \notin (J^+)^2$. As in Lemma 2.8 in [3], we see that the restriction of (F_i, F_{r_s-i+1}) to $\mathfrak{h}$ is $(q(g_i), q(g_{r_s-i+1}))$. In the proof of Proposition 2.6, we identified J_H/J_H^2 with $\{(x_1, \ldots, x_{r_s+1}) \in \mathbb{C}^{r_s+1} \mid \sum_i x_i = 0\}$ in such a way that the action of W_s on J_H/J_H^2 intertwines with the standard action of the symmetric group S_{r_s+1} on the latter space. With this identification, the generators g_i can be chosen to correspond precisely to the differentials of normalized Newton polynomials $\psi_{[i+1]} := \frac{1}{i+1}\sum_{k=1}^{r_s+1} x_k^{i+1}$. We can conclude using the formula

$$(d\psi_{[k]}, d\psi_{[g]}) = \sum_{i=1}^{r_s+1} x_i^{k+g-2} = (k+g-2)\psi_{[k+g-2]}$$

(see [3]). □

Proof of Corollary 2.2. The proof of Theorem 2.1 shows that

$$GM_{\theta_s}(q) = (1+q^{-1})\prod_{i=1}^{r-1}(1+q^{2m_i+1})q^{2n_0}(1+q^{2n_0}+\cdots+q^{2(r_s-1)n_0}).$$

Now formula (2.1) follows from (2.4). □

3. The module $S^n(V)$

In this section, V is a n-dimensional complex vector space and $\mathfrak{g} = \mathrm{sl}(V)$. We sometimes assume to have chosen a trivialization $\bigwedge^n V \simeq \mathbb{C}$, although for a formal step it is better not to think in this form.

We are interested in studying the isotypic component of type $S^n(V)$ (resp. $S^n(V^*)$) in $\bigwedge \mathfrak{g}^*$, or the $\mathfrak{g}$-invariants of $S^n(V^*) \otimes \bigwedge \mathfrak{g}^*$, (resp. $S^n(V) \otimes \bigwedge \mathfrak{g}^*$). As we will see in the next Section, $S^n(V)$ is a small representation, hence we can use [9, Corollary 4.2] to deduce that

$$\dim((S^n(V^*) \otimes \textstyle\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}) = \dim((S^n(V) \otimes \textstyle\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}) = 2^{n-1}. \tag{3.1}$$

We think of $\bigwedge^i \mathfrak{g}^*$ as the space of multilinear alternating functions in i variables from $\mathfrak{g}$ to $\mathbb{C}$ and of $\bigwedge^i \mathfrak{g}^* \otimes \bigwedge^n V$ as the space of multilinear alternating functions in i variables from $\mathfrak{g}$ to $\bigwedge^n V$ (similarly for $\bigwedge^n V^*$).

Recall that the primitive generators of the ring of invariants $(\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ are the functions T_i defined by

$$T_i := \mathrm{tr}(St_{2i+1}(A_1, A_2, \ldots, A_{2i}, A_{2i+1})),$$

where $St_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \varepsilon_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$ is the standard polynomial.

We introduce equivariant maps

$$\Phi : S^n(V) \to \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V, \quad \Psi : S^n(V) \to \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V$$

by assigning homogeneous polynomial maps (cf. [8, §5, 2.3])

$$v \mapsto \Phi(v) \in \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V, \quad v \mapsto \Psi(v) \in \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V$$

defined, for $v \in V$, as

$$\begin{aligned}\Phi(v)(A_1, \ldots, A_n) &:= A_1 v \wedge A_2 v \wedge \cdots \wedge A_{n-1} v \wedge A_n v,\\ \Psi(v)(A_1, \ldots, A_{n-1}) &:= A_1 v \wedge A_2 v \wedge \cdots \wedge A_{n-1} v \wedge v.\end{aligned}$$

A similar formula holds for maps $\Phi^* : S^n(V^*) \to \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V^*$, $\Psi^* : S^n(V^*) \to \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V^*$: when $\gamma \in V^*$ we set

$$\begin{aligned}\Phi^*(\gamma)(A_1, \ldots, A_n) &:= A_1^t \gamma \wedge A_2^t \gamma \wedge \cdots \wedge A_{n-1}^t \gamma \wedge A_n^t \gamma,\\ \Psi^*(\gamma)(A_1, \ldots, A_{n-1}) &:= A_1^t \gamma \wedge A_2^t \gamma \wedge \cdots \wedge A_{n-1}^t \gamma \wedge \gamma.\end{aligned}$$

We use the same symbols Φ, Ψ to denote the corresponding elements in $(S^n(V^*) \otimes \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$. Notice that we have an equivariant pairing $S^n(V) \times S^n(V^*) \to \mathbb{C}$, which gives, by duality, a canonical map $I : \mathbb{C} \to S^n(V) \otimes S^n(V^*)$, and which induces an equivariant pairing

$$\begin{aligned}(\cdot, \cdot) : \mathrm{Hom}\Big(S^n(V), \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V\Big) &\times \mathrm{Hom}\Big(S^n(V^*), \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V^*\Big)\\ &\to \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^n V^* = \bigwedge \mathfrak{g}^*\end{aligned}$$

in the following way. We let $\langle \cdot | \cdot \rangle$ denote the natural pairing between V and V^*. We extend this pairing to define the canonical trivialization $\bigwedge^n V \otimes \bigwedge^n V^* \to \mathbb{C}$ by setting

$$\langle v_1 \wedge v_2 \wedge \cdots \wedge v_n \,|\, \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \rangle = \det(\langle v_i \,|\, \gamma_j \rangle). \tag{3.2}$$

The pairing (a, b) is then defined by computing in 1 the composition

$$\begin{aligned}\mathbb{C} \xrightarrow{I} S^n(V) \otimes S^n(V^*) &\xrightarrow{a \otimes b} \bigwedge^i \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^j \mathfrak{g}^* \otimes \bigwedge^n V^*\\ &\xrightarrow{m} \bigwedge^{i+j} \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^n V^* \xrightarrow{\eta} \bigwedge^{i+j} \mathfrak{g}^*.\end{aligned}$$

Here m is exterior multiplication and the isomorphism η is given by the canonical trivialization (3.2).

Restricting to invariants we have finally a pairing

$$\mathrm{Hom}\Big(S^n(V^*), \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V^*\Big)^{\mathfrak{g}} \times \mathrm{Hom}\Big(S^n(V), \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V\Big)^{\mathfrak{g}} \to (\bigwedge \mathfrak{g}^*)^{\mathfrak{g}}.$$

We want to compute (Ψ, Φ^*), so we want to understand the composed map

$$\mathbb{C} \xrightarrow{I} S^n(V) \otimes S^n(V^*) \xrightarrow{\Psi\otimes\Phi^*} \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V^*$$
$$\xrightarrow{\eta} \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n \mathfrak{g}^*.$$

For this we can polarize, getting the following commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "I"] \arrow[d, "1"'] & S^n(V) \otimes S^n(V^*) \arrow[r, "\Psi\otimes\Phi^*"] \arrow[d, "p"'] & \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V^* \arrow[r, "\eta"] \arrow[d, "1"'] & \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n \mathfrak{g}^* \arrow[d, "i"'] \\
\mathbb{C} \arrow[r, "I"] & V^{\otimes n} \otimes (V^*)^{\otimes n} \arrow[r, "\psi\otimes\phi^*"] & \bigwedge^{n-1} \mathfrak{g}^* \otimes \bigwedge^n V \otimes \bigwedge^n \mathfrak{g}^* \otimes \bigwedge^n V^* \arrow[r, "\pi"] & (\mathfrak{g}^{\otimes 2n-1})^*
\end{tikzcd}.$$

The map $p(v^n \otimes \gamma^n) := v^{\otimes n} \otimes \gamma^{\otimes n}$ is polarization, the map i is the embedding of multilinear functions alternating in two blocks of variables into multilinear functions, the map π is the (external) multiplication of multilinear functions composed with the canonical trivialization. The polarized maps ϕ and ψ are given by

$$\begin{aligned}
&\psi(v_1, \ldots, v_n)(A_1, \ldots, A_{n-1}) \\
&\qquad := (n!)^{-1} \sum_{\sigma \in S_n} A_1 v_{\sigma(1)} \wedge A_2 v_{\sigma(2)} \wedge \cdots \wedge A_{n-1} v_{\sigma(n-1)} \wedge v_{\sigma(n)}, \\
&\phi^*(\gamma_1, \ldots, \gamma_n)(B_1, \ldots, B_n) \\
&\qquad := (n!)^{-1} \sum_{\tau \in S_n} B_1^t \gamma_{\tau(1)} \wedge B_2^t \gamma_{\tau(2)} \wedge \cdots \wedge B_{n-1}^t \gamma_{\tau(n-1)} \wedge B_n^t \gamma_{\tau(n)},
\end{aligned}$$

thus

$$\begin{aligned}
&\pi \circ (\psi \otimes \phi^*)(v_1, \ldots, v_n, \gamma_1, \ldots, \gamma_n)(A_1, \ldots, A_{n-1}, B_1, \ldots, B_n) \\
&\quad = \langle \psi(v_1, \ldots, v_n)(A_1, \ldots, A_{n-1}) \,|\, \phi^*(\gamma_1, \ldots, \gamma_n)(B_1, \ldots, B_n) \rangle \\
&\quad = (n!)^{-2} \sum_{\sigma, \tau \in S_n} \langle A_\sigma v \,|\, B_\tau \gamma \rangle
\end{aligned}$$

where for shortness we have set $A_\sigma v = A_1 v_{\sigma(1)} \wedge A_2 v_{\sigma(2)} \wedge \cdots \wedge A_{n-1} v_{\sigma(n-1)} \wedge v_{\sigma(n)}$, and $B_\tau \gamma = B_1^t \gamma_{\tau(1)} \wedge B_2^t \gamma_{\tau(2)} \wedge \cdots \wedge B_{n-1}^t \gamma_{\tau(n-1)} \wedge B_n^t \gamma_{\tau(n)}$.

We have also, setting $A_n = 1_V$,

$$\langle A_\sigma v \mid B_\tau \gamma\rangle = \sum_{\lambda \in S_n} \varepsilon_\lambda \prod_{h=1}^{n} \langle A_h v_{\sigma(h)} \mid B^t_{\lambda(h)} \gamma_{\tau\circ\lambda(h)}\rangle \tag{3.3}$$

$$= \sum_{\lambda \in S_n} \varepsilon_\lambda \prod_{h=1}^{n} \langle B_{\lambda(h)} A_h v_{\sigma(h)} \mid \gamma_{\tau\circ\lambda(h)}\rangle.$$

Consider $\langle A_\sigma v \mid B_\tau \gamma\rangle$ as a function on $V^{\otimes n} \otimes (V^*)^{\otimes n} = \mathrm{End}(V)^{\otimes n}$. The image of the canonical element I in $\mathrm{End}(V)^{\otimes n}$ is $1_V^{\otimes n}$ and we want to compute $\langle A_\sigma v \mid B_\tau \gamma\rangle$ on this canonical element.

For this define formally matrix variables $Y_i = v_i \otimes \gamma_i$. We first compute $\langle A_\sigma v \mid B_\tau \gamma\rangle$ on all elements $Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n \in \mathrm{End}(V)^{\otimes n}$; then we set all $Y_i = 1_V$ in order to perform the desired computation.

More in detail, we proceed as follows. For $X_1, \dots, X_n \in \mathfrak{g}$, set

$$I_{\sigma,\tau} := \prod_{h=1}^{n} \langle X_i v_{\sigma(i)} \mid \gamma_{\tau(i)}\rangle = \prod_{h=1}^{n} \langle X_{\sigma^{-1}(i)} v_i \mid \gamma_{\tau\circ\sigma^{-1}(i)}\rangle.$$

In order to explicit this formula set $w_i = X_{\sigma^{-1}(i)} v_i$ and $Z_i^\sigma = X_{\sigma^{-1}(i)} \circ Y_i = X_{\sigma^{-1}(i)} v_i \otimes \gamma_i = w_i \otimes \gamma_i$. We have

$$I_{\sigma,\tau} = \prod_{i=1}^{n} \langle w_i \mid \gamma_{\tau\circ\sigma^{-1}(i)}\rangle.$$

Recall that, if we take matrix variables $W_i := w_i \otimes \gamma_i$ and a permutation μ, then $\prod_i \langle w_i \mid \gamma_{\mu(i)}\rangle$ is the multilinear invariant of n matrices $\phi_\mu(W_1, \dots, W_n) := \prod \mathrm{tr}(M_j)$, where the monomials M_j are the products of the W_i over the indices i appearing in the cycles of μ. It follows that we have the formula

$$\phi_\mu(W_{\tau(1)}, \dots, W_{\tau(n)}) = \phi_{\tau\mu\tau^{-1}}(W_1, \dots, W_n). \tag{3.4}$$

Clearly,

$$I_{\sigma,\tau} = \phi_{\tau\circ\sigma^{-1}}(Z_1^\sigma, Z_2^\sigma, \dots, Z_n^\sigma). \tag{3.5}$$

When we compute this invariant on the canonical element, this is equivalent to setting all $Y_i = 1_V$, hence $Z_i^\sigma = X_{\sigma^{-1}(i)} \circ Y_i$ becomes $X_{\sigma^{-1}(i)}$ and we get as evaluation

$$\phi_{\tau\circ\sigma^{-1}}(X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, \dots, X_{\sigma^{-1}(n)}) = \phi_{\sigma^{-1}\circ\tau}(X_1, X_2, \dots, X_n).$$

In the last equality we have used (3.4). Setting $X_i = B_{\lambda(i)} A_i$ we find

$$\langle A_\sigma v \mid B_\tau \gamma\rangle(I) = \sum_{\lambda \in S_n} \varepsilon_\lambda \phi_{\sigma^{-1}\circ\tau\circ\lambda}(B_{\lambda(1)} A_1, B_{\lambda(2)} A_2, \dots, B_{\lambda(n-1)} A_{n-1}, B_{\lambda(n)}),$$

so that

$$\text{(3.6)}\quad \pi\circ(\phi\otimes\psi^*)\circ I(1)$$
$$= (n!)^{-2}\sum_{\sigma,\tau,\lambda}\varepsilon_\lambda\phi_{\sigma^{-1}\circ\tau\circ\lambda}(B_{\lambda(1)}A_1, B_{\lambda(2)}A_2,\ldots,B_{\lambda(n-1)}A_{n-1},B_{\lambda(n)})$$

Recall that $(\Psi,\Phi^*) = (m\circ\eta\circ(\Psi\otimes\Phi^*)\circ I)(1)$. For any vector space U we identify the space $\bigwedge^k U^*$ with the subspace of $(U^*)^{\otimes k}$ formed by the alternating multilinear functions. Under this embedding, a decomposable element $\phi_1\wedge\cdots\wedge\phi_k$ corresponds to the function

$$f(x_1,\ldots,x_k) := \sum_{\sigma\in S_k}\varepsilon_\sigma\phi_1(x_{\sigma(1)})\ldots\phi_1(x_{\sigma(k)}).$$

The alternator operator on $(U^*)^{\otimes n}$ is

$$\mathrm{Alt} : y_1\otimes\cdots\otimes y_n\mapsto\frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon_\sigma y_{\sigma(1)}\otimes\cdots\otimes y_{\sigma(k)}.$$

The relation between exterior multiplication of alternating functions and of multilinear functions is given by the following commutative diagram

$$\begin{tikzcd}
\bigwedge^h U^*\otimes\bigwedge^k U^* \arrow[r, "m"] \arrow[d, "i"'] & \bigwedge^{h+k} U^* \arrow[d, "\frac{1}{\binom{h+k}{h}}i"'] \\
(U^*)^{\otimes h+k} \arrow[r, "\mathrm{Alt}"] & (U^*)^{\otimes h+k}
\end{tikzcd}$$

which in our setting reads

$$\begin{tikzcd}
\bigwedge^{n-1}\mathfrak{g}^*\otimes\bigwedge^{n}\mathfrak{g}^* \arrow[r, "m"] \arrow[d, "i"'] & \bigwedge^{2n-1}\mathfrak{g}^* \arrow[d, "C_n 1"'] \\
(\mathfrak{g}^*)^{\otimes 2n-1} \arrow[r, "\mathrm{Alt}"] & \bigwedge^{2n-1}\mathfrak{g}^*
\end{tikzcd},$$

where $C_n = \frac{(n-1)!n!}{(2n-1)!}$. Thus

$$\text{(3.7)}\qquad (\Psi,\Phi^*) = C_n^{-1}\,\mathrm{Alt}\circ\pi\circ(\phi\otimes\psi^*)\circ I(1).$$

We need therefore to apply Alt to the right hand side of (3.6). For shortness set

$$f(\sigma,\tau,\lambda) := \phi_{\sigma^{-1}\circ\tau\circ\lambda}(B_{\lambda(1)}A_1, B_{\lambda(2)}A_2,\ldots,B_{\lambda(n-1)}A_{n-1},B_{\lambda(n)}).$$

Let us apply the procedure of alternation to a term $f(\sigma,\tau,\lambda)$. If $\sigma^{-1}\circ\tau\circ\lambda$ is not a full cycle, then $\mathrm{Alt}(f(\sigma,\tau,\lambda)) = 0$. To check this we need only to find an odd permutation in S_{2n-1} that fixes the term $f(\sigma,\tau,\lambda)$. Let $c_1\ldots c_s$ be the cycle decomposition of $\sigma^{-1} \circ \tau \circ \lambda$. We can assume that $c_1 = (i_1 \dots i_k)$ is a cycle that does not contain $\lambda(n)$. It follows that, if $M_2, \dots, M_s$ are the products of matrices corresponding to cycles $c_2, \dots, c_s$,

$$\begin{aligned} f(\sigma, \tau, \lambda) &= \operatorname{tr}(B_{\lambda(i_1)} A_{i_1} B_{\lambda(i_2)} A_{i_2} \dots B_{\lambda(i_k)} A_{i_k}) \operatorname{tr}(M_2) \dots \operatorname{tr}(M_s) \\ &= \operatorname{tr}(A_{i_1} B_{\lambda(i_2)} A_{i_2} \dots B_{\lambda(i_k)} A_{i_k} B_{\lambda(i_1)}) \operatorname{tr}(M_2) \dots \operatorname{tr}(M_s) \end{aligned}$$

and the last equality gives an odd permutation (a cycle of length $2k$) in S_{2n-1} that fixes $f(\sigma, \tau, \lambda)$. If $\sigma^{-1} \circ \tau \circ \lambda$ is a full cycle $(j_1 \dots j_n)$, we can assume that $j_n = n$. Then

$$\varepsilon_\lambda f(\sigma, \tau, \lambda) = \varepsilon_\lambda \operatorname{tr}(B_{\lambda(j_1)} A_{j_1} B_{\lambda(j_2)} A_{j_2} \dots B_{\lambda(j_{n-1})} A_{j_{n-1}} B_{\lambda(n)}).$$

Let $\mu \in S_{2n-1}$ be defined by $\mu(i) = n+1$ for $i = 1, \dots, n-1$ and $\mu(i) = i - n + 1$ for $i = n, \dots, 2n-1$. If $\omega \in S_n$ we can consider ω as an element of S_{2n-1} (fixing $n+1, \dots, 2n-1$). Let $\nu \in S_n$ be defined by $\nu(i) = j_i$. Then $(\nu \circ \mu^{-1} \circ \lambda \circ \nu \circ \mu)^{-1}$ is the permutation mapping $\operatorname{tr}(B_{\lambda(j_1)} A_{j_1} B_{\lambda(j_2)} A_{j_2} \dots B_{\lambda(j_{n-1})} A_{j_{n-1}} B_{\lambda(n)})$ to $\operatorname{tr}(B_1 A_1 B_2 A_2 \dots B_{n-1} A_{n-1} B_n)$. Since the sign of $(\nu \circ \mu^{-1} \circ \lambda \circ \nu \circ \mu)^{-1}$ is ε_λ, we see that, if $\sigma^{-1} \circ \tau \circ \lambda$ is a full cycle,

$$\operatorname{Alt}(\varepsilon_\lambda f(\sigma, \tau, \lambda)) = \frac{1}{(2n-1)!} \operatorname{tr}(St_{2n-1}(B_1, A_1, B_2, A_2, \dots, B_{n-1}, A_{n-1}, B_n)). \tag{3.8}$$

We are now ready to prove the key result of this section.

THEOREM 3.1. $(\Psi, \Phi^*) = \frac{(-1)^{\binom{n}{2}}}{n!} T_{n-1}$.

PROOF. Combining (3.6), (3.7), and (3.8) we have

$$(\Psi, \Phi^*) = \frac{C}{(n!)^3 (n-1)!} \operatorname{tr}(St_{2n-1}(B_1, A_1, B_2, A_2, \dots, B_{n-1}, A_{n-1}, B_n)).$$

where C is the number of triples σ, τ, λ such that $\sigma^{-1} \circ \tau \circ \lambda$ is a full cycle. There are $(n!)^2 (n-1)!$ such triples. $\square$

THEOREM 3.2. $\operatorname{Hom}_{\mathfrak{g}}(S^n(V), \bigwedge \mathfrak{g}^* \otimes \bigwedge^n V) \cong (S^n(V^*) \otimes \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ *is a free module on the two generators* Φ, Ψ *over* $\bigwedge(T_1, \dots, T_{n-2})$.

PROOF. We first prove that Ψ and $\delta\Psi$ freely generate $(S^n(V^*) \otimes \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ over $\bigwedge(T_1, \dots, T_{n-2})$. Using the formula (3.1) it is enough to prove that the two elements are linearly independent over $\bigwedge(T_1, \dots, T_{n-2})$.

Let $\{e_i\}$ be a basis of weight vectors for V with e_1 a highest weight vector. Let $\{E_{ij}\}$ be the basis of $\operatorname{End}(V)$ of elementary matrices and $\{E^{ij}\}$ the dual basis. Then it is not hard to check that, up to a constant depending on the choice of a trivialization of $\bigwedge^n V$, we have

$$\Psi(e_1) = E^{21} \wedge \dots \wedge E^{n1}.$$

Since $[E_{i1}, E_{j1}] = 0$ if $i, j \neq 1$, we see that $\partial(\Psi(e_1)) = 0$. By equivariance, we obtain that $\partial\Psi = 0$. Recall (see [6, (94)]) that the Laplacian $\delta\partial + \partial\delta$ equals $\frac{1}{2}\sum_{i=1}^{\dim \mathfrak{g}} \theta(z_i)^2$, where $\{z_i\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form and θ is the extension of ad to $\bigwedge \mathfrak{g}$. It follows that $\partial\delta\Psi = (\delta\partial + \partial\delta)\Psi = c\Psi$ with c a non-zero scalar. We can then argue as in the previous section and deduce that is enough to prove that an identity $a \wedge \Psi = 0$, $a \in \bigwedge(T_1, \ldots, T_{n-2})$, implies $a = 0$. For this, we compute $(a \wedge \Psi, \Phi^*)$ and have, by Theorem 3.1, that $0 = (a \wedge \Psi, \Phi^*) = \frac{(-1)^{\binom{n}{2}}}{n!} a \wedge T_{n-1}$. Since the relation $a \wedge T_{n-1} = 0$ with $a \in \bigwedge(T_1, \ldots, T_{n-2})$ implies $a = 0$, we have proven that Ψ and $\delta\Psi$ freely generate $(S^n(V^*) \otimes \bigwedge \mathfrak{g}^*)^{\mathfrak{g}}$ over $\bigwedge(T_1, \ldots, T_{n-2})$. This in particular proves that $\dim \mathrm{Hom}_{\mathfrak{g}}(S^n(V), \bigwedge^n \mathfrak{g}) = 1$, thus $\delta\Psi$ is a multiple of Φ, hence the proof is complete. □

4. Small representations in type A

For $\mathrm{sl}(n, \mathbb{C})$, one can show that an highest weight module V is small if and only if the highest weight of either V or V^* comes from a partition of n. This means the following: given a partition $\lambda_1 \geq \cdots \geq \lambda_n$ of n, the corresponding highest weight λ is 0 if $\lambda_1 = \cdots = \lambda_n = 1$ or $\lambda = \sum_{i=1}^{n-1} a_i\omega_i$ where $\omega_1, \ldots, \omega_{n-1}$ are the fundamental weights and a_i is the number of columns of length i of the partition. For such weights, Stembridge has proved the following formula (cf. [10, Corollary 6.2]), yielding the graded multiplicities $M_\lambda(q)$ of the corresponding modules in $\bigwedge \mathrm{sl}(n, \mathbb{C})$. Display the Young diagram in the English way, label the boxes as matrix entries and denote by $h(i, j)$ the hook length of the box (i, j), i.e. the number of boxes strictly on the right of box (i, j) plus the number of boxes strictly below box (i, j) plus one. Set, as usual, $[n]_q = \frac{1-q^n}{1-q}$ and $[n]_q! = \prod_{i=1}^n [i]_q$. Then

$$M_\lambda(q) = \frac{[n]_{q^2}!}{1+q} \prod_{(i,j)\in\lambda} \frac{q^{2i-1} + q^{2j-2}}{1 - q^{2h(i,j)}}. \tag{4.1}$$

Notice that, since we are dealing with $\mathrm{sl}(n, \mathbb{C})$ rather than $\mathrm{gl}(n, \mathbb{C})$, there is an extra factor $1/(1+q)$ in the right hand side of (4.1) w.r.t. the formula displayed in [10].

Proposition 4.1. *If $\mathfrak{g} = \mathrm{sl}(V)$ and $V(\lambda)$ is an irreducible non-trivial representation of $\mathfrak{g}$ with λ corresponding to a partition of $n = \dim V$, then $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \bigwedge \mathfrak{g})$ is free over $\bigwedge(P_1, \ldots, P_{n-2})$ if and only if $V(\lambda)$ is either $S^n(V)$ or the adjoint representation.*

Proof. The fact that the adjoint representation and $S^n(V)$ have the desired property has been shown in [3] and in Section 3 above, respectively. Assume now that λ corresponds to a partition of n. We can assume $n \geq 4$: if $n \leq 3$ the result is trivially verified. If $\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \bigwedge \mathfrak{g})$ is free over $\bigwedge(P_1, \ldots, P_{n-2})$, the polynomial affording its graded multiplicities in $\bigwedge \mathfrak{g}$ has to be divisible by $\prod_{i=1}^{n-2}(1+q^{2i+1})$. Use now formula (4.1). Look at the highest term $1+q^{2n-3}$ in the graded multiplicities of $\bigwedge(P_1,\dots,P_{n-2})$. The only possible simplification occurs in the term $\prod_{(i,j)\in\lambda}(q^{2i-1}+q^{2j-2})$ of (4.1). This can happen just in the following three cases:

(1) $i=n,\ j=1$;
(2) $i=n-1,\ j=1$;
(3) $i=1,\ j=n$.

The first case gives the partition corresponding to the trivial representation, which is excluded. In the second case, since we are excluding the case where $\lambda_n=1$, the partition is necessarily $(2,1^{n-2})$, corresponding to the adjoint representation. In the third case the partition is necessarily (n), which corresponds to $S^n(V)$. □

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Corrado De Concini
Dipartimento di Matematica
Sapienza Università di Roma
P.le A. Moro 2, 00185, Roma, Italy
deconcin@mat.uniroma1.it

Pierluigi Möseneder Frajria
Politecnico di Milano
Polo regionale di Como
Via Valleggio 11, 22100, Como, Italy
pierluigi.moseneder@polimi.it

Paolo Papi
Dipartimento di Matematica
Sapienza Università di Roma
P.le A. Moro 2, 00185, Roma, Italy
papi@mat.uniroma1.it

Claudio Procesi
Dipartimento di Matematica
Sapienza Università di Roma
P.le A. Moro 2, 00185, Roma, Italy
procesi@mat.uniroma1.it