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Partial Differential Equations — *The supercritical Lane-Emden equation and its gradient flow*, by Michael Struwe, communicated on 14 March 2014.

Abstract. — The following text is a summary of the author's talk at the conference on "Nonlinear problems with singular data" at the Accademia dei Lincei, Rome, Novemver 26, 2013.

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1. The Lane-Emden equation

1.1. A question of Paul Rabinowitz. Consider for $p > 2$ the boundary value problem

$$-\Delta u = |u|^{p-2}u \quad \text{on } \Omega \subset\subset \mathbb{R}^n,\ n \geq 3,\ u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Recall that for $2 < p < 2^* := \frac{2n}{n-2}$ there exists a solution $u > 0$ of equation (1.1), either obtained from $0 \leq v \in H_0^1(\Omega)$ with $\|v\|_{L^p} = 1$ and minimizing the Sobolev quotient

$$Q(v) = \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^p}^2} = \min_{0 \neq w \in H_0^1(\Omega)} Q(w) = S_p(\Omega),$$

or obtained as a mountain-pass critical point of the functional

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx, \quad u \in H_0^1 \cap L^p(\Omega).$$

For $p = 2^*$ a dichotomy occurs. On the one hand, by a result of Pohozaev [30] on any star-shaped domain $\Omega \subset\subset \mathbb{R}^n$ any (smooth) solution u of (1.1) necessarily vanishes. Moreover, the Sobolev quotient $S_{2^*}(\Omega)$ is never attained on a domain $\Omega \neq \mathbb{R}^n$.

On the other hand, when $\Omega = B_{R_2} \backslash B_{R_1}(0)$ for some $0 < R_1 < R_2$, then there exists a radial solution $u > 0$ of (1.1), which can be obtained as a minimizer of the Sobolev quotient among radially symmetric functions. In addition we have the following general existence result of Coron [7].

Theorem 1.1. *Suppose* $0 \notin \Omega \supset B_{R_2} \backslash B_{R_1}(0)$, *where* $R_2/R_1 \geq R \gg 1$. *Then there exists a solution* $u > 0$ *of* (1.1).

Coron's proof uses Struwe's [33] quantization of the energy levels where the Palais-Smale condition for E_{2^*} fails, coupled with a clever minimax argument. See also [34] for further background.

Prompted by Coron's result, as recalled by Brezis [4], Rabinowitz asked: Suppose Ω is as in Coron's theorem. Does there exist a solution $u > 0$ of (1.1) for *every* $p > 2^*$?—Partial results were obtained by Dancer, del Pino, Felmer, Ge, Jing, Molle, Musso, Pacard, Passaseo, Wei, and others (see [8], [9], [10], [15], [24], [29]), but so far there is no general theory.

1.2. Good notions of "solution". Should one insist on classical solutions or should one also admit solutions with "mild" singularities?—Note that when $n \geq 7$ by a result of Jäger-Kaul [19] the singular weakly harmonic map $\bar{u} : x \mapsto x/|x|$ is absolutely energy-minimizing among maps $u : B_1^n(0) \to S^n = \partial B_1^{n+1}(0)$ with boundary data $u(x) = x$ for $|x| = 1$, where we let $B_r^n(x_0) = \{x \in \mathbb{R}^n; |x - x_0| < r\}$ for any $r > 0$, $x_0 \in \mathbb{R}^n$.

Jointly with Melanie Rupflin [31] we investigated this question in the simple case of radially symmetric solutions of (1.1). Refining results on the asymptotic decay of radial solutions of (1.1) due to Fowler [12] and Ni-Serrin [26] we showed that on the one hand the weak solution (with a suitable constant $b_* = b_*(n, p) > 0$)

$$u_*(x) = b_*|x|^{-\frac{2}{p-2}} \in H^1_{loc} \cap L^p_{loc}(\mathbb{R}^n)$$

of (1.1) on $\mathbb{R}^n$ is in the $H^1_{loc} \cap L^p_{loc}$-closure of the set of smooth solutions.

On the other hand we observe that for $2^* < p < 2\frac{2n-1}{n-2}$ there are oscillating distribution solutions $u \in W_0^{1,\frac{n}{n-1}} \cap L^{p-1}(B_1(0))$ of (1.1) on $B_1(0)$. The latter is in striking contrast with Pohozaev's [30] result. A "good" nootion of weak solution should therefore allow the former but rule out the latter behavior.

1.3. "Stationary" solutions. Prompted by work of Evans [11], Frank Pacard [27] proposed the following definition. A weak solution $u \in H^1 \cap L^p(\Omega)$ of (1.1) is *stationary*, if there holds

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E_p(u \circ (id + \varepsilon\tau)) = 0, \quad \forall \tau \in C_0^\infty(\Omega).$$

He obtained the following result. For given $p > 2^*$ let $\mu = \mu(p) = \frac{2p}{p-2} < n$.

THEOREM 1.2 (Pacard [27]). *Let $u \geq 0$ be a "stationary" weak solution of* (1.1), *and suppose that $2^* < p < 2^+ = \frac{2(n-1)}{n-3}$. Then $u \in C^2(\Omega \backslash S)$, where the singular set S is closed with $\mathscr{H}^{n-\mu}(S) = 0$.*

REMARK 1.3. i) In [31] we observe that the weak solution $u_*(x) = b_*|x|^{-\frac{2}{p-2}}$ of (1.1) on $\mathbb{R}^n$ is stationary.

ii) Alternatively, one could restrict attention to non-negative weak solutions of class $H^1 \cap L^p(\Omega)$. Such a restriction would also seem natural from the point of view of applications in geometry such as in the Yamabe problem. See also [32].

1.4. Scaling. The exponent μ in Pacard's result reflects the fact that whenever u is a solution of (1.1), then so is

$$u_R(x) = R^{\frac{2}{p-2}}u(x_0 + Rx), \quad R > 0,\ x_0 \in \mathbb{R}^n.$$

This scaling property distinguishes a particular Morrey exponent. Recall that a function f belongs to the Morrey space $L^{p,\lambda}(\Omega)$ with $0 < \lambda < n$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in R^n, r>0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p\, dx < \infty.$$

Note the invariance of the Morrey norm $\|u\|^p_{L^{p,\mu}(\mathbb{R}^n)} = \|u_R\|^p_{L^{p,\mu}(\mathbb{R}^n)}$, $R > 0$.

Pacard's results in particular yield that any "stationary" solution of (1.1) lies in the space $L^{p,\mu}(\Omega)$.

1.5. Monotonicity. Pacard's key tool is a novel monotonicity formula; he shows that for a stationary weak solution u of (1.1) the map

$$r \mapsto r^{\mu-n} \int_{B_r(x_0)} \Big(\frac{|\nabla u|^2}{2} + \frac{|u|^p}{p}\Big)\, dx + \frac{1}{p-2}\frac{d}{dr}\Big(r^{\mu-n}\int_{\partial B_r(x_0)} u^2 do\Big)$$

is non-decreasing. Note that the expected minus-sign in the volume integral is magically changed into a plus-sign at the expense of adding the derivative of a non-negative term of lower order.

2. Gradient flow

For given smooth initial data $u_0 \in H^1_0 \cap L^p(\Omega)$, $T_* \le \infty$, consider the Cauchy problem

$$\begin{aligned} u_t - \Delta u &= |u|^{p-2}u \quad \text{on } \Omega \times [0, T_*[, \\ u &= 0 \quad \text{on } \partial\Omega \times [0, T_*[, \\ u|_{t=0} &= u_0. \end{aligned} \tag{2.1}$$

2.1. Energy identity. Multiplying (2.1) by u_t and integrating, with $E = E_p$ we obtain the equation

$$E(u(T)) + \int_0^T \int_\Omega |u_t|^2\, dx\, dt = E(u_0), \quad 0 < T < T_*. \tag{2.2}$$

This fundamental identity shows that the flow (2.1) may be regarded as the L^2-gradient flow for the energy $E = E_p$.

2.2. Finite-time blow-up and global existence. Again a dichotomy occurs. On the one hand, as observed by Kaplan [20] and Fujita [14], for data u_0 with $E(u_0) < 0$

the solution to (2.1) blows up in finite-time. To see this, simply multiply (2.1) by u and use (2.2) to obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\int_{\Omega\times\{t\}}(|\nabla u|^2 - |u|^p)\,dx = -2E(u(t)) + \frac{p-2}{p}\|u(t)\|_{L^p}^p$$
$$\geq -2E(u_0) + c_0\|u(t)\|_{L^2}^p \geq c_0\|u(t)\|_{L^2}^p$$

for some constant $c_0 > 0$. Hence

$$\|u(t)\|_{L^2} \geq (\|u_0\|_{L^2}^{(2-p)/2} - c_0(p-2)t)^{-1/(p-2)},$$

and $u(t)$ must blow up at the latest at time $T = c_0^{-1}(p-2)^{-1}\|u_0\|_{L^2}^{(2-p)}$.

On the other hand, for data u_0 which are small in the C^1-norm the maximum principle gives global existence, as can be seen by comparing a solution u of (2.1) with a sufficiently small multiple $\bar{u} = s\varphi_1$ of the first eigenfunction $\varphi_1 > 0$ of the Laplacian on Ω, satisfying

$$-\Delta\varphi_1 = \mu_1\varphi_1 \quad \text{on } \Omega,\ \varphi_1 = 0 \text{ on } \partial\Omega.$$

Note that for sufficiently small $s > 0$ we have $s^{p-2}\|\varphi_1\|_{L^\infty}^{p-2} < \mu_1$ and there holds

$$-\Delta\bar{u} - \bar{u}^{p-1} = (\mu_1 - s^{p-2}\varphi_1^{p-2})\bar{u} > 0 \quad \text{on } \Omega,\ \bar{u} = 0 \text{ on } \partial\Omega.$$

Hence if $-\bar{u} \leq u_0 \leq \bar{u}$ initially, the inequality $|u| \leq \bar{u}$ will be preserved by the flow.

2.3. "Borderline" solutions. Following Ni-Sacks-Tavantzis [25], for data $0 \leq u_0 \in C^1(\overline{\Omega})$ not vanishing identically and any $\lambda > 0$ let u^λ be the solution of (2.1) with $u^\lambda|_{t=0} = \lambda u_0$. Noting that $E(\lambda u_0) \to -\infty$ as $\lambda \to \infty$ from the results in the preceding section we then conclude

$$0 < \lambda^* := \sup\{\lambda > 0; u^\lambda \text{ is global}\} < \infty.$$

By the maximum principle, moreover, as $\lambda \uparrow \lambda^*$ we have monotone convergence $u^\lambda \uparrow u^* \leq \infty$: a "borderline" (weak) solution of (2.1).

Chou-Du-Zheng [6] showed partial regularity of these "borderline" solutions u^*. Their proof uses the (parabolic) monotonicity formulas of Giga-Kohn [16] for (2.1) and Struwe [34] for the heat flow of harmonic maps, respectively, and the partial regularity theory from [34].

One might hope that for suitable u_0 we have convergence $u^*(t) \to u_\infty$ as $t \to \infty$, where u_∞ is a non-trivial solution of (1.1). However, Matano-Merle [23] observe that radial solutions of (2.1) on a ball or on $\mathbb{R}^n$ for compactly supported initial data u_0 always either blow up in finite time or uniformly decay to 0 as $t \to \infty$. Since we already suspect that "interesting" solutions in general need not be smooth, we are thus led to consider also solutions u to (2.1) that blow up in finite time.

3. Recent results

3.1. Results for the flow (2.1). Jointly with Simon Blatt in a recent paper [2] we obtain a Pacard-type monotonicity formula for the flow (2.1), thereby using the framework of [34]. More precisely, let $\varphi = \varphi(|x|) \in C^\infty(\mathbb{R}^n)$ be a compactly supported cut-off function such that $0 \le \varphi \le 1$ and let

$$G(x,t) = \frac{1}{(4\pi|t|)^{n/2}} e^{-\frac{|x|^2}{4|t|}}, \quad x \in \mathbb{R}^n,\ t < 0,$$

be the fundamental solution to the heat equation with singularity at $(0,0)$. For any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ also set

$$G_{(x_0,t_0)}(x,t) = G(x - x_0, t - t_0).$$

Given $x_0 \in \Omega$, $t_0 > 0$ define

$$D^\varphi(R) = D^\varphi_{(x_0,t_0)}(R) = \frac{R^\mu}{2} \int_{\Omega \times \{t_0 - R^2\}} |\nabla u|^2 \varphi^2(x - x_0) G_{(x_0,t_0)}\, dx$$

and for $q \ge 2$ also let

$$F^\varphi_q(R) = \frac{R^\mu}{q} \int_{\Omega \times \{t_0 - R^2\}} |u|^q \varphi^2(x - x_0) G_{(x_0,t_0)}\, dx.$$

Then setting

$$H^\varphi(R) = \frac{p-2}{p+2}(D^\varphi(R) + F^\varphi_p(R)) + \frac{1}{p+2}\Big(\frac{d}{dR}(RF^\varphi_2(R)) - A^\varphi_2(R)\Big), \tag{3.1}$$

we show that for any $x_0 \in \Omega$ with a suitable cut-off function φ for all $R > 0$ there holds

$$\begin{aligned} R\frac{d}{dR} H^\varphi(R) \ge \frac{R^\mu}{4} &\int_{\Omega \times \{t_0 - R^2\}} \frac{|x \cdot \nabla u + 2(t - t_0)u_t + au|^2}{|t_0 - t|} \varphi^2 G_{(x_0,t_0)}\, dx \\ &+ A^\varphi_0(R) + B^\varphi(R), \end{aligned} \tag{3.2}$$

where B^φ is a boundary term and where A^φ_0, A^φ_2 are error terms induced by localization, containing derivatives of the cut-off function φ and involving combinations of u and ∇u of lower order with respect to $D^\varphi + F^\varphi_p$.

This result improves the Giga-Kohn [16] monotonicity result (which involves the difference $D^\varphi(R) - F^\varphi_p(R)$ instead of the sum) and shows conservation of the scale invariant Morrey norm up to blow-up time on domains of size proportional to remaining time. If Ω is not convex the boundary term B^φ may also be negative.

For this reason, on a general domain in [2] we derive these Morrey estimates only locally away from $\partial\Omega$; their extension up to the boundary will be treated in our forthcoming paper [3].

Moreover, by adapting the potential theoretic arguments of Adams [1] we obtain an ε-regularity result for weak solutions $u \in L^{p,\mu}$ of (2.1) with small Morrey norm, which improves the regularity estimates of Chou-Du-Zheng [6] for smooth solutions.

We also show that at any first blow-up point $(x_0, T) \in \Omega \times]0, \infty[$ for a suitable cut-off function φ and any sufficiently small $R > 0$ there holds $H^{\varphi}_{(x_0,T)}(R) \geq \varepsilon_0$, where $\varepsilon_0 > 0$ is an absolute constant. Together with (3.2) this allows to show the existence of a non-trivial, partially regular, self-similar tangent map at any first blow-up point of type I; moreover, at any first blow-up point of type II (that is, not of type I) by arguing similar to Hamilton [17] we obtain a non-trivial tangent map which is a smooth eternal solution of the flow (2.1). This result extends results of Matano-Merle [23] for radially symmetric solutions to the general case.

3.2. Results for the time-independent problem (1.1). By applying our ε-regularity result to solutions of (1.1) we are able to improve Pacard's partial regularity result as follows.

THEOREM 3.1. *For any $2^* < p < \infty$ let $u \in H^1 \cap L^p$ be a "stationary" weak solution of* (1.1). *Then $u \in C^2(\Omega \backslash S)$, where S is closed with $\mathscr{H}^{n-\mu}(S) = 0$.*

That is, we are able to remove Pacard's additional assumption that $u \geq 0$ and his restriction on the range of admissible exponents p in Theorem 1.2.

In fact, by applying the potential theoretic approach of Adams [1] to (not necessarily stationary) weak solutions $u \in H^1 \cap L^{p,\mu}$ of (1.1) we obtain the following estimates, which together with Pacard's work imply Theorem 3.1.

THEOREM 3.2. *Let $u \in H^1 \cap L^{p,\mu}(B_2(0))$ with $\|u\|_{L^{p,\mu}(B_2(0))} \leq \varepsilon$ be a weak solution of* (1.1). *Then $u \in C^1(B_1(0))$ with*

$$\|u\|_{L^\infty(B_1(0))} + \|\nabla u\|_{L^\infty(B_1(0))} \leq C\|u\|_{L^{p,\mu}(B_2(0))}.$$

Still using ideas and techniques from Adams [1], we also have estimates reminiscent of Sobolev's embedding and the standard L^p-estimates for the Laplace operator which for any weak solution $u \in H_0^1 \cap L^{p,\mu}$ of (1.1) yield the bounds

$$\|\nabla u\|_{L^{2,\mu}(\Omega)} \leq C\|\Delta u\|_{L^{\frac{p}{p-1},\mu}(\Omega)} = C\|u\|^{p-1}_{L^{p,\mu}(\Omega)} \leq C\|\nabla u\|^{p-1}_{L^{2,\mu}(\Omega)}.$$

In particular, either $u \equiv 0$ or $\|\nabla u\|_{L^{2,\mu}(\Omega)} \geq c_1 > 0$, which gives a possibly optimal threshold result for solutions $u \not\equiv 0$ of (1.1). Related to this one might ask whether there exist minimizers of the Morrey norm ratio and if such minimizers correspond to solutions of (1.1); moreover, it would be interesting to classify all such "ground states".

4. Open problems

If Ω is convex, and if $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ with $\nabla u_0 \in L^{2,\mu}(\Omega)$ for sufficiently small $\varepsilon > 0$ and suitable $r_0 > 0$ satisfies the condition

$$\sup_{x_0 \in R^n, 0<r<r_0} r^{\lambda-n} \int_{\Omega_r(x_0)\cap\Omega} (|\nabla u_0|^2 + |u_0|^p)\, dx < \varepsilon, \tag{4.1}$$

then our results give the a-priori L^∞-bound

$$\|u_k(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{2}{p-2}} \tag{4.2}$$

on a uniform time interval $0 < t < T$ for the solutions u_k of (2.1) for smooth initial data $u_{0k} \to u_0$ in $H_0^1 \cap L^p(\Omega)$ $(k \to \infty)$ uniformly satisfying the bound (4.1). Hence, on any convex domain the Cauchy problem for (2.1) is locally well-posed for data u_0 in this class. In our forthcoming work [3] we extend this result to a general domain.

For large data $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ a-priori bounds like (4.2) are not yet available. Thus we may ask if (2.1) is locally well-posed for data $u_0 \in H_0^1 \cap L^{p,\mu}(\Omega)$ in general. See [5] for a related study.

Moreover, it would seem desirable to understand and exhaustively classify the different ways how solutions to equation (2.1) can blow up, extending the work of Friedman-McLeod [13], Herrero-Velazquez [18], Matano-Merle [21], [22], Troy [36], Weissler [37], and others.

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