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Mechanics — *Heat and mass transfer by convection in multicomponent Navier-Stokes mixtures: absence of subcritical instabilities and global nonlinear stability via the Auxiliary System Method*, by Salvatore Rionero, communicated on 9 May 2014.

Abstract. — Because of its great geophysical relevance (engineering geology, volcanism, subsurface fluid motions, ...) and the frequent applications (industrial processes, crystal growth, thermal engineering, air and water pollution, ...) in the past as nowadays, the heat and mass transfer by convection in horizontal layers has attracted the attention of many scientists. In the present paper, this problem is investigated in the general case of a horizontal layer L—filled by a Navier-Stokes multicomponent fluid mixture—heated from below and salted (partly from below and partly from above) by $m \in \mathbb{N}$ salts $S_1, S_2, \ldots, S_m$. Generalizing the Auxiliary System Method (AS Method), recently introduced for the Darcy fluid mixtures in porous layers [32]–[34], it is shown that: i) for each Fourier component of the perturbation fields there exists an own nonlinear evolution system (auxiliary system); ii) via the auxiliary system, a linearization principle can be obtained; iii) the absence of subcritical instabilities and the property of the linear stability conditions to guarantee also the global nonlinear L^2-stability hold; iv) the Routh-Hurwitz stability conditions are characterized $\forall m \in \mathbb{N}$ and handled for $m \leq 2$; v) the looking for hidden symmetries and skew-symmetries allows to guarantee—via simple algebraic conditions in closed form—the global nonlinear stability.

Key words: Convection, global stability, Auxiliary System Method.

Mathematics Subject Classification: 76D05, 76RXX, 35B35.

Contents

1. Introduction

Let L be a horizontal layer, filled by a multicomponent Navier-Stokes fluid mixture, heated from below and salted by m chemical species ("salts") $S_1, S_2, \dots, S_m$, partly from above and partly from below. For its importance in the geophysical and industrial applications, a great attention has been payed, in the past as nowadays, to the onset of convection in L {see [1]–[16], [38] and the references therein}. In particular, for $m \leq 2$ very relevant results have been obtained investigating, either linearly or nonlinearly, the stability of the thermal conduction solution {see [1]–[31] and the references therein}. But, for $m > 1$, only rarely it has been obtained that the linear instability captures completely the physics of the phenomenon i.e. the: a) absence of subcritical instabilities; b) global nonlinear stability guaranteed by the conditions of linear stability.

In fact, although many efforts and relevant procedures have been introduced, the absence of subcritical instabilities has generally been obtained under restrictive conditions, especially on the initial data {see, for instance, [3], [7]–[9], [31], [38]}. On the contrary, in the case of porous layers filled by Darcy fluid mixtures, recently, it has been obtained that the linear instability captures completely the physics of the onset of convection. This result is due to the introduction, {see [32]–[34]}, of a new approach named Auxiliary System Method (AS Method). Our aim is to generalize the AS Method to the Navier-Stokes fluid mixtures in order to obtain that, also for them (for any number of salts dissolved in), the physics of the problem is completely captured by the linear instability. We begin by considering, in the present paper, the free-free case.

2. Preliminaries

Let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors **i**, **j**, **k** (**k** pointing vertically upwards). We assume that m different chemical species S_α $(\alpha = 1, 2, \dots, m)$, have dissolved in the fluid and have concentrations C_α $(\alpha = 1, 2, \dots, m)$, respectively and that the equation of state is

$$\rho = \rho_0 \left[1 - A(T - T_0) + \sum_{\alpha=1}^{m} A_\alpha (C_\alpha - \hat{C}_\alpha) \right],$$

where ρ_0, T_0, $\hat{C}_\alpha$ $(\alpha = 1, 2, \ldots, m)$, are reference values of the density, temperature and salt concentrations, while the constants A, A_α denote the thermal and solute S_α expansion coefficients respectively. Combining the Navier-Stokes law with energy and mass balance together with the Boussinesq approximation, one obtains the fundamental Navier-Stokes equations governing the isochoric motions given by

$$(2.1) \qquad \begin{cases} \rho_0(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p + \rho_0 \nu \Delta \mathbf{v} \\ \qquad -\mathbf{g}\rho_0\left[1 - A(T - T_0) + \displaystyle\sum_{\alpha=1}^{m} A_\alpha(C_\alpha - \hat{C}_\alpha)\right], \\ \nabla\cdot\mathbf{v} = 0, \\ T_t + \mathbf{v}\cdot\nabla T = k\Delta T, \\ C_{\alpha t} + \mathbf{v}\cdot\nabla C_\alpha = k_\alpha \Delta C_\alpha, \quad \alpha = 1, 2, \ldots, m, \end{cases}$$

where $\mathbf{v}$, p, ν, g, k, k_α represent velocity, pressure, viscosity, gravity, thermal diffusivity and solutal diffusivity respectively. To (2.1) we append the boundary conditions

$$(2.2) \qquad \begin{cases} T(0) = T_l, \quad T(d) = T_u, \\ C_\alpha(0) = C_{\alpha_l}, \quad C_\alpha(d) = C_{\alpha_u}, \quad \alpha = 1, 2, \ldots, m \\ \mathbf{v}\cdot\mathbf{k} = 0, \quad \text{on } z = 0, d. \end{cases}$$

The boundary value problem (2.1)–(2.2) admits the thermal conduction solution $(\bar{\mathbf{v}} = \mathbf{0})$

$$(2.3) \qquad \begin{cases} \bar{T} = T_l - \dfrac{\delta T}{d} z, \quad \bar{C}_\alpha = C_{\alpha_l} - \dfrac{(\delta C_\alpha)}{d} z, \\ \alpha = 1, 2, \ldots, m, \quad \delta T = T_l - T_u, \quad \delta C_\alpha = C_{\alpha_l} - C_{\alpha_u}, \end{cases}$$

to which it is associated the pressure $\bar{p}$ determined from the equation

$$(2.4) \qquad \frac{d\bar{p}}{dz} = -\rho_0 g\left[1 - A(\bar{T} - T_0) + \sum_{\alpha=1}^{m} A_\alpha(\bar{C}_\alpha - \hat{C}_\alpha)\right].$$

We set

$$(2.5) \qquad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{u}, \quad p = \bar{p} + \pi, \quad T = \bar{T} + \theta, \quad C_\alpha = \bar{C}_\alpha + \Phi_\alpha$$

and introduce the non dimensional scalings

$$
(2.6)\quad \begin{cases} t = t^*\dfrac{d^2}{k}, \quad \mathbf{u} = \mathbf{u}^*\dfrac{\nu}{d}, \quad \pi = \pi^*\dfrac{\nu^2\rho_0}{d^2}, \quad \mathbf{x} = \mathbf{x}^* d, \quad \theta = \theta^* T^{\#}, \\ \Phi_\alpha = (\Phi_\alpha)^*(\Phi_\alpha)^{\#}, \quad T^{\#} = \Big(\dfrac{\nu^3|\delta T|}{Agkd^3}\Big)^{1/2}, \quad (\Phi_\alpha)^{\#} = \Big(\dfrac{\nu^3|\delta C_\alpha|P_\alpha}{A_\alpha gkd^3}\Big)^{1/2}, \\ R = \Big(\dfrac{Agd^3|\delta T|}{\nu k}\Big)^{1/2}, \quad R_\alpha = \Big(\dfrac{A_\alpha gd^3|\delta C_\alpha|P_\alpha}{\nu k}\Big)^{1/2}, \\ P_r = \dfrac{\nu}{k}, \quad P_\alpha = \dfrac{k}{k_\alpha}, \quad H = \mathrm{sgn}(\delta T), \quad H_\alpha = \mathrm{sgn}(\delta C_\alpha), \end{cases}
$$

where R and R_α are thermal and solute Rayleigh numbers while P_r and P_α are the fluid and salts Prandtl numbers. The non dimensional non linear perturbation equations are then (dropping the asterisks)

$$
(2.7)\quad \begin{cases} P_r^{-1}\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi + \Delta\mathbf{u} + \Big(R\theta - \sum_{\alpha=1}^{m} R_\alpha\Phi_\alpha\Big)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \theta_t + P_r\mathbf{u}\cdot\nabla\theta = HRw + \Delta\theta, \\ P_\alpha(\Phi_{\alpha t} + P_r\mathbf{u}\cdot\nabla\Phi_\alpha) = H_\alpha R_\alpha w + \Delta\Phi_\alpha, \quad \alpha = 1,2,\dots,m, \end{cases}
$$

under the boundary conditions

$$
(2.8)\quad w = \theta = \Phi_\alpha = 0 \quad \text{on } z = 0,1, \quad \alpha = 1,2,\dots,m,
$$

with $w = \mathbf{u}\cdot\mathbf{k}$. We assume (as usually done, in stability problems in layers) that

i) the perturbations $(u, v, w, \theta, \Phi_1, \Phi_2, \dots, \Phi_m)$ are periodic in the x and y directions, respectively of periods $2\pi/a_x, 2\pi/a_y$;
ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0,1]$ is the periodicity cell;
iii) $\mathbf{u}$, $\Phi_1, \Phi_2, \dots, \Phi_m$, θ are such that together with all their first derivatives and second spatial derivatives are square integrable in Ω, $\forall t \in \mathbb{R}^+$ and can be expanded in a Fourier series uniformly convergent in Ω

and denote by $L_2^*(\Omega)$ the set of functions such that

1) $\Phi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \Phi(\mathbf{x}, t) \in \mathbb{R}$, $\Phi \in W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$, Φ is periodic in the x and y directions of period $\dfrac{2\pi}{a_x}, \dfrac{2\pi}{a_y}$ respectively and $(\Phi)_{z=0} = (\Phi)_{z=1} = 0$;
2) Φ, together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$.

Since the sequence $\{\sin n\pi z\}$, $(n = 1, 2, \dots)$ is a complete orthogonal system for $L^2(0,1)$, by virtue of periodicity, it turns out that $\forall\Phi \in L_*^2(\Omega)$, there exists a sequence $\{\tilde{\Phi}_n(x, y, t)\}$ such that

$$(2.9)\qquad \begin{cases} \Phi = \sum_{n=1}^{\infty} \tilde{\Phi}_n \sin n\pi z, & \dfrac{\partial \Phi}{\partial t} = \sum_{n=1}^{\infty} \dfrac{\partial \tilde{\Phi}_n}{\partial t} \sin n\pi z, \\ \Delta_1 \Phi = -a^2 \Phi, & \Delta \Phi = -\sum_{n=1}^{\infty} \xi_n \tilde{\Phi}_n \sin n\pi z, \end{cases}$$

with

$$(2.10)\qquad \xi_n = a^2 + n^2\pi^2, \quad a^2 = a_x^2 + a_y^2, \quad \Delta = \Delta_1 + \frac{\partial^2}{\partial z^2}, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and the series being absolutely uniformly convergent.

Finally, setting

$$(2.11)\qquad \zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

the horizontal components of $\mathbf{u}$ are given by {see [1], p. 24}

$$(2.12)\qquad u = \frac{1}{a^2}(w_{xz} + \zeta_y), \qquad v = \frac{1}{a^2}(w_{yz} - \zeta_x)$$

and in view of $\mathbf{u} = \sum_{n=1}^{\infty} \mathbf{u}_n$, $\zeta_n = \dfrac{\partial v_n}{\partial x} - \dfrac{\partial u_n}{\partial y}$, it follows that

$$(2.13)\qquad \begin{cases} u_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial x \partial z} + \dfrac{\partial \zeta_n}{\partial y}\right), \quad v_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial y \partial z} - \dfrac{\partial \zeta_n}{\partial x}\right), \\ \nabla \cdot \mathbf{u}_n = \left(\dfrac{1}{a^2}\Delta_1 w_n + w_n\right)_z = 0. \end{cases}$$

REMARK 2.1. Since the stability of the null solution of (2.7)–(2.8) makes sense only in a class of solutions in which it is unique, we eliminate any other rigid solution by requiring the "average velocity condition"

$$(2.14)\qquad \int_\Omega u \, d\Omega = \int_\Omega v \, d\Omega = 0.$$

3. Nonlinear system governing the nth-Fourier component of the perturbation fields

Let $(\Pi, \mathbf{u}, \theta, \Phi_1, \ldots, \Phi_\alpha)$ be solution of (2.7)–(2.8) with

$$(3.1)\qquad \begin{cases} \Pi = \sum_{n=1}^{\infty} \Pi_n, \quad \mathbf{u} = \sum_{n=1}^{\infty} \mathbf{u}_n, \quad \theta = \sum_{n=1}^{\infty} \theta_n, \quad \Phi_\alpha = \sum_{n=1}^{\infty} \Phi_{\alpha n}, \\ \nabla \cdot \mathbf{u}_n = 0, \\ w_n = \mathbf{u}_n \cdot \mathbf{k} = \theta_n = \Phi_{\alpha n} = 0, \quad z = 0, 1, \quad \alpha = 1, \ldots, m \end{cases}$$

and

$$
(3.2)\qquad \begin{cases} (\mathbf{u}_n)_{t=0} = \mathbf{u}_n^{(0)}, \quad (\mathbf{u})_{t=0} = \mathbf{u}^{(0)} = \displaystyle\sum_{n=1}^{\infty} \mathbf{u}_n^{(0)}, \\[2ex] (\theta_n)_{t=0} = \theta_n^{(0)}, \quad (\theta)_{t=0} = \theta^{(0)} = \displaystyle\sum_{n=1}^{\infty} \theta_n^{(0)}, \\[2ex] (\Phi_{\alpha n})_{t=0} = \Phi_{\alpha n}^{(0)}, \quad (\Phi_\alpha)_{t=0} = \Phi_\alpha^{(0)} = \displaystyle\sum_{n=1} \Phi_{\alpha n}^{(0)}, \end{cases}
$$

$\mathbf{u}^{(0)}$, $\theta^{(0)}$, $\Phi_\alpha^{(0)}$ being assigned arbitrary initial data such that $\nabla \cdot \mathbf{u}^{(0)} = 0$. In view of (2.7)–(2.8) and (3.1)–(3.2) the i.b.v. problem at stake can be written

$$
(3.3)\qquad \begin{cases} \displaystyle\sum_{n=1}^{\infty} \left(P_r^{-1} \frac{\partial \mathbf{u}_n}{\partial t} \right) = \sum_{n=1}^{\infty} \left[\Delta \mathbf{u}_n + \left(R\theta_n - \sum_{\alpha=1}^{m} R_\alpha \Phi_{\alpha n} \right) \mathbf{k} \right] \\[2ex] \qquad - \displaystyle\sum_{n=1}^{\infty} \nabla \Pi_n - \sum_{n=1}^{\infty} \mathbf{u} \cdot \nabla \mathbf{u}_n, \\[2ex] \displaystyle\sum_{n=1}^{\infty} \nabla \cdot \mathbf{u}_n = 0, \quad \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \theta_n = \sum_{n=1}^{\infty} (HRw_n + \Delta\theta_n) - P_r \sum_{n=1}^{\infty} \mathbf{u} \cdot \nabla \theta_n, \\[2ex] P_\alpha \displaystyle\sum_{n=1}^{\infty} \frac{\partial}{\partial t} \Phi_{\alpha n} = \sum_{n=1}^{\infty} (H_\alpha R_\alpha w_n + \Delta \Phi_{\alpha n}) - P_\alpha P_r \sum_{n=1}^{\infty} \mathbf{u} \cdot \nabla \Phi_{\alpha n}, \end{cases}
$$

under the initial boundary conditions

$$
(3.4)\qquad \begin{cases} (\mathbf{u}_n)_{t=0} = \mathbf{u}_n^{(0)}, \quad (\theta_n)_{t=0} = \theta_n^{(0)}, \quad (\Phi_{\alpha n})_{t=0} = \Phi_{\alpha n}^{(0)}, \quad \alpha = 1, \ldots, m, \\ \mathbf{u}_n \cdot \mathbf{k} = \theta_n = \Phi_{\alpha n} = 0, \quad z = 0, 1, \quad \alpha = 1, \ldots, m. \end{cases}
$$

According to the guideline of the AS Method [34], to (3.3)–(3.4) we associate the *auxiliary system*

$$
(3.5)\qquad \begin{cases} P_r^{-1} \dfrac{\partial \bar{\mathbf{u}}_n}{\partial t} = \Delta \bar{\mathbf{u}}_n + \left(R\bar{\theta}_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \bar{\Phi}_{\alpha n} \right) \mathbf{k} - \nabla \bar{\Pi}_n - \mathbf{u} \cdot \nabla \bar{\mathbf{u}}_n, \\[2ex] \nabla \cdot \bar{\mathbf{u}}_n = 0, \quad \dfrac{\partial}{\partial t} \bar{\theta}_n = HR\bar{w}_n + \Delta \bar{\theta}_n - P_r \mathbf{u} \cdot \nabla \bar{\theta}_n, \\[2ex] P_\alpha \dfrac{\partial}{\partial t} \bar{\Phi}_{\alpha n} = H_\alpha R_\alpha \bar{w}_n + \Delta \bar{\Phi}_{\alpha n} - P_\alpha P_r \mathbf{u} \cdot \nabla \bar{\Phi}_{\alpha n}, \end{cases}
$$

under the i.b.c. ($\alpha = 1, \dots, m$)

$$(3.6)\quad \begin{cases} (\bar{\mathbf{u}}_n)_{t=0} = \mathbf{u}_n^{(0)}, \quad (\bar{\theta}_n)_{t=0} = \theta_n^{(0)}, \quad (\bar{\Phi}_{\alpha n})_{t=0} = \Phi_{\alpha n}^{(0)}, \quad \alpha = 1, \dots, m, \\ \bar{\mathbf{u}}_n \cdot \mathbf{k} = \bar{\theta}_n = \bar{\Phi}_{\alpha n} = 0, \quad z = 0, 1, \quad \alpha = 1, \dots, m. \end{cases}$$

The following basic theorem holds.

THEOREM 3.1. *Let* $(\bar{\mathbf{u}}_n, \bar{\theta}_n, \bar{\Phi}_{1n}, \dots, \bar{\Phi}_{\alpha n})$ *be,* $\forall n \in \mathbb{N}$, *solution of* (3.5)–(3.6). *Then the series* $\sum_{n=1}^{\infty} \bar{\mathbf{u}}_n$, $\sum_{n=1}^{\infty} \bar{\theta}_n$, $\sum_{n=1}^{\infty} \bar{\Phi}_{\alpha n}$, ($\alpha = 1, \dots, m$) *are convergent and it follows that*

$$(3.7)\quad \sum_{n=1}^{\infty} \bar{\mathbf{u}}_n = \mathbf{u}, \qquad \sum_{n=1}^{\infty} \bar{\theta}_n = \theta, \qquad \sum_{n=1}^{\infty} \bar{\Phi}_{\alpha n} = \Phi_\alpha, \quad (\alpha = 1, \dots, m).$$

PROOF. Setting

$$(3.8)\quad \bar{P}_q = \sum_{n=1}^{q} \bar{\Pi}_n, \qquad \bar{\mathbf{U}}_q = \sum_{n=1}^{q} \bar{\mathbf{u}}_n, \qquad \bar{S}_q = \sum_{n=1}^{q} \bar{\theta}_n, \qquad \bar{S}_{\alpha q} = \sum_{n=1}^{q} \bar{\Phi}_{\alpha n},$$

one obtains that the following i.b.v.p. holds

$$(3.9)\quad \begin{cases} P_r^{-1} \dfrac{\partial \bar{\mathbf{U}}_q}{\partial t} = \Delta \bar{\mathbf{U}}_q + \Big(R\bar{S}_q - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \bar{S}_{\alpha q} \Big) \mathbf{k} - \nabla \bar{P}_q - \mathbf{u} \cdot \nabla \bar{\mathbf{U}}_q, \\ \nabla \cdot \bar{\mathbf{U}}_q = 0, \quad \dfrac{\partial}{\partial t} \bar{\theta}_q = HR\bar{\mathbf{U}}_q \cdot \mathbf{k} + \Delta \bar{\theta}_q - P_r \mathbf{u} \cdot \nabla \bar{\theta}_q, \\ P_\alpha \dfrac{\partial}{\partial t} \bar{S}_{\alpha q} = H_\alpha R_\alpha \bar{\mathbf{U}}_q \cdot \mathbf{k} + \Delta \bar{S}_{\alpha q} - P_\alpha P_r \mathbf{u} \cdot \nabla \bar{S}_{\alpha q}, \end{cases}$$

$$(3.10)\quad \begin{cases} (\bar{\mathbf{U}}_q)_{t=0} = \displaystyle\sum_{n=1}^{q} \mathbf{u}_n^{(0)}, \quad (\bar{S}_q)_{t=0} = \displaystyle\sum_{n=1}^{q} \theta_n^{(0)}, \\ (\bar{S}_{\alpha q})_{t=0} = \displaystyle\sum_{n=1}^{q} \Phi_{\alpha n}^{(0)}, \quad \alpha = 1, \dots, m, \\ (\bar{\mathbf{U}})_q \cdot \mathbf{k} = \bar{S}_q = \bar{S}_{\alpha q} = 0, \quad z = 0, 1, \quad \alpha = 1, \dots, m. \end{cases}$$

Setting

$$(3.11)\quad (\mathbf{U})_n^* = \begin{cases} \mathbf{u}_n - \bar{\mathbf{u}}_n, & \text{for } n = 1, 2, \dots, q \\ \mathbf{u}_n, & \text{for } n > q \end{cases}, \qquad \mathbf{U}^* = \sum_{n=1}^{\infty} (\mathbf{U})_n^* = \mathbf{u} - \bar{\mathbf{U}}_q,$$

$$(3.12)\quad \theta_n^* = \begin{cases} \theta_n - \bar{\theta}_n, & \text{for } n = 1, 2, \dots, q \\ \theta_n, & \text{for } n > q \end{cases}, \qquad \theta^* = \sum_{n=1}^{\infty} \theta_n^*,$$

$$\Psi^*_{\alpha n} = \begin{cases} \Phi_{\alpha n} - \bar{\Phi}_{\alpha n}, & \text{for } n = 1, 2, \ldots, q \\ \Phi_{\alpha n}, & \text{for } n > q \end{cases}, \qquad \Psi^*_\alpha = \sum_{n=1}^{\infty} \Psi^*_{\alpha n}, \tag{3.13}$$

$$P^*_n = \begin{cases} \Pi_n - \bar{\Pi}_n, & \text{for } n = 1, 2, \ldots, q \\ \Pi_n, & \text{for } n > q \end{cases}, \qquad P^* = \sum_{n=1}^{\infty} \Pi^*_n, \tag{3.14}$$

by virtue of (3.5)–(3.6) and (3.8)–(3.10) one obtains

$$\begin{cases} P_r^{-1} \dfrac{\partial}{\partial t} \mathbf{U}^* = \Delta \mathbf{U}^* + \left(R\theta^*_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \Psi^*_\alpha \right) \mathbf{k} - \nabla P^* - \mathbf{u} \cdot \nabla \mathbf{U}^*, \\ \nabla \cdot \mathbf{U}^* = 0, \quad \dfrac{\partial}{\partial t} \theta^* = HR\mathbf{U}^* \cdot \mathbf{k} + \Delta\theta^* - P_r \mathbf{u} \cdot \nabla\theta^*, \\ P_\alpha \dfrac{\partial}{\partial t} \Psi^*_\alpha = H_\alpha R_\alpha \mathbf{U}^* \cdot \mathbf{k} + \Delta \Psi^*_\alpha - P_\alpha P_r \mathbf{u} \cdot \nabla \Psi^*_\alpha, \end{cases} \tag{3.15}$$

$$\begin{cases} (\mathbf{U}^*)_{t=0} = \displaystyle\sum_{n=1}^{\infty} \mathbf{u}_n^{(0)}, \quad (\theta^*)_{t=0} = \sum_{n=1}^{q} \theta_n^{(0)}, \\ (\Psi^*_\alpha)_{t=0} = \displaystyle\sum_{n=1}^{q} \Psi^{(0)}_{\alpha n}, \quad \alpha = 1, \ldots, m, \\ \mathbf{U}^* \cdot \mathbf{k} = \theta^* = \Psi^*_\alpha = 0, \quad z = 0, 1, \quad \alpha = 1, \ldots, m. \end{cases} \tag{3.16}$$

Since

$$\lim_{q\to\infty} \sum_{n=q+1}^{\infty} \mathbf{u}_n^{(0)} = \lim_{q\to\infty} \sum_{n=q+1}^{\infty} \theta_n^{(0)} = \lim_{q\to\infty} \sum_{n=q+1}^{\infty} \Psi^{(0)}_{\alpha n} = 0 \tag{3.17}$$

and (3.15) under the zero i.b.c. admits only the null solution, it follows that

$$\lim_{q\to\infty} (\mathbf{u} - \bar{\mathbf{U}}_q) = \lim_{q\to\infty} (\theta - \bar{S}_q) = \lim_{q\to\infty} (\Phi_\alpha - \bar{S}_{\alpha q}) = 0 \tag{3.18}$$

and (3.7) holds.

Remark 3.1. i) Since in $L^*_2(\Omega)$ the Fourier components are a.e. uniquely determined, it follows that in $L^*_2(\Omega)$

$$\mathbf{u}_n = \bar{\mathbf{u}}_n, \quad \theta_n = \bar{\theta}_n, \quad \Phi_{\alpha n} = \bar{\Phi}_{\alpha n}, \quad \Pi_n = \bar{\Pi}_n, \quad \textit{a.e. in } \Omega \times \mathbb{R}^+$$

and the system (3.5)–(3.6) can be written

$$(3.19)\qquad \begin{cases} P_r^{-1}\dfrac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}_n = -\nabla\Pi_n + \Big(R\theta_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha\Phi_{\alpha n}\Big)\mathbf{k} + \Delta\mathbf{u}_n, \\ \nabla\cdot\mathbf{u}_n = 0, \\ \dfrac{\partial\theta_n}{\partial t} + P_r\mathbf{u}\cdot\nabla\theta_n = HR\mathbf{u}_n\cdot\mathbf{k} + \Delta\theta_n, \\ P_\alpha\Big(\dfrac{\partial}{\partial t}\Phi_{\alpha n} + P_r\mathbf{u}\cdot\nabla\Phi_{\alpha n}\Big) = H_\alpha R_\alpha\mathbf{u}_n\cdot\mathbf{k} + \Delta\Phi_{\alpha n}, \end{cases}$$

$$(3.20)\qquad \begin{cases} (\mathbf{u}_n)_{t=0} = \mathbf{u}_n^{(0)}, \quad (\theta_n)_{t=0} = \theta_n^{(0)}, \quad (\Phi_{\alpha n})_{t=0} = \Phi_{\alpha n}^{(0)}, \quad \alpha = 1,\dots,m, \\ w_n = \mathbf{u}_n\cdot\mathbf{k} = \theta_n = \Phi_{\alpha n} = 0, \quad z = 0,1, \quad \alpha = 1,\dots,m; \end{cases}$$

ii) the global asymptotic stability of the null solution of (3.19)–(3.20) is guaranteed by the global asymptotic stability, $\forall n \in \mathbb{N}$, of the null solution of (2.7)–(2.8);
iii) the instability of the null solution of (3.19)–(3.20), for at least one $n \in \mathbb{N}$, implies the instability of the null solution of (2.7)–(2.8);
iv) the nonlinear "auxiliary system" (3.19)–(3.20) of the Navier-Stokes-Boussinesq fluid mixtures, as far as we know, has never been introduced before in the existing literature and appears to be the system governing the evolution of the nth-Fourier component of the solution $(\mathbf{u}, \theta, \Phi_1, \dots, \Phi_m)$.

4. Preliminaries to the linearization principle

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the scalar product and the norm of $L_2^*(\Omega)$. Further we introduce the energy $E_n^{(m)}$ of the n-th Fourier component of the perturbation fields on setting

$$(4.1)\qquad E_n^{(m)} = \frac{1}{2}\Big(P_r^{-1}\|\mathbf{u}_n\|^2 + \|\theta_n\|^2 + \sum_{\alpha=1}^{m} P_\alpha\|\Phi_{\alpha n}\|^2\Big),$$

with

$$(4.2)\qquad \|\mathbf{u}_n\|^2 = \|u_n\|^2 + \|v_n\|^2 + \|w_n\|^2$$

and denote by $Q_n^{(m)}$ the quadratic form

$$(4.3)\qquad Q_n^{(m)} = -\xi_n\Big(u_n^2 + v_n^2 + w_n^2 + \theta_n^2 + \sum_{\alpha=1}^{m}\Phi_{\alpha n}^2\Big) + \Big[(1+H)R\theta_n - \sum_{\alpha=1}^{m}(1-H_\alpha)R_\alpha\Phi_{\alpha n}\Big]w_n.$$

To (3.19)–(3.20) we associate the linear system

$$
(4.4)\qquad \begin{cases} P_r^{-1}\dfrac{\partial \bar{\mathbf{u}}_n}{\partial t} = \Big(R\bar{\theta}_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \bar{\Phi}_{\alpha n}\Big)\mathbf{k} + \Delta\bar{\mathbf{u}}_n, \\ \nabla \cdot \bar{\mathbf{u}}_n = 0, \\ \dfrac{\partial \bar{\theta}_n}{\partial t} = HR\bar{\mathbf{u}}_n \cdot \mathbf{k} + \Delta\bar{\theta}_n, \\ P_\alpha \dfrac{\partial}{\partial t}\bar{\Phi}_{\alpha n} = H_\alpha R_\alpha \bar{\mathbf{u}}_n \cdot \mathbf{k} + \bar{\Delta}\Phi_{\alpha n}, \quad \alpha = 1, \dots, m, \end{cases}
$$

under the i.b.c. of (3.19)–(3.20) i.e.

$$
(4.5)\qquad \begin{cases} (\bar{\mathbf{u}}_n)_{t=0} = \bar{\mathbf{u}}_n^{(0)}, \quad (\bar{\theta}_n)_{t=0} = \bar{\theta}_n^{(0)}, \quad (\bar{\Phi}_{\alpha n})_{t=0} = \bar{\Phi}_{\alpha n}^{(0)}, \quad \alpha = 1, \dots, m, \\ \bar{w}_n = \bar{\mathbf{u}}_n \cdot \mathbf{k} = \bar{\theta}_n = \bar{\Phi}_{\alpha n} = 0, \quad z = 0, 1, \quad \alpha = 1, \dots, m. \end{cases}
$$

Denoting by $\left(\dfrac{dE_n^{(m)}}{dt}\right)_{NL}$ and $\left(\dfrac{dE_n^{(m)}}{dt}\right)_L$ the time derivative of $E_n^{(m)}$ evaluated respectively along the solutions of (3.19)–(3.20) and (4.4)–(4.5), it easily follows that

$$
(4.6)\qquad \left(\frac{dE_n^{(m)}}{dt}\right)_{NL} = \int_\Omega Q_n^{(m)}(u_n, v_n, w_n, \theta_n, \Phi_{1n}, \dots, \Phi_{mn})\, d\Omega,
$$

$$
(4.7)\qquad \left(\frac{dE_n^{(m)}}{dt}\right)_{L} = \int_\Omega Q_n^{(m)}(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{\theta}_n, \bar{\Phi}_{1n}, \dots, \bar{\Phi}_{mn})\, d\Omega.
$$

The linearization principle is based on the following two theorems.

THEOREM 4.1. *Let*

$$
(4.8)\qquad \left(\frac{dE_n^{(m)}}{dt}\right)_{L} < 0, \quad \forall t \in [0, \infty[,
$$

for arbitrary initial data. Then

$$
(4.9)\qquad \left(\frac{dE_n^{(m)}}{dt}\right)_{NL} < 0, \quad \forall t \in [0, \infty[.
$$

PROOF. In fact (4.8), at $t = 0$, gives

$$
(4.10)\qquad \int_\Omega Q_n^{(m)}(\bar{u}_n^{(0)}, \bar{v}_n^{(0)}, \bar{w}_n^{(0)}, \bar{\theta}_n^{(0)}, \bar{\Phi}_{1n}^{(0)}, \dots, \bar{\Phi}_{mn}^{(0)})\, d\Omega < 0,
$$

for arbitrary initial values. On choosing for any fixed $t \in [0, \infty[$

$$\bar{u}_n^{(0)} = u_n(t), \quad \bar{v}_n^{(0)} = v_n(t), \quad \bar{\theta}_n^{(0)} = \theta_n(t), \quad \bar{\Phi}_{\alpha n}^{(0)} = \Phi_{\alpha n}(t), \quad \alpha = 1, \dots, m, \tag{4.11}$$

(4.10) becomes

$$\int_\Omega Q_n^{(m)}(u_n, v_n, w_n, \theta_n, \Phi_{1n}, \dots, \Phi_{mn})\, d\Omega < 0 \tag{4.12}$$

and, in view of (4.6), (4.9) immediately follows.

By virtue of theorem 4.1, the following linearization principle holds.

THEOREM 4.2. *The conditions guaranteeing*

$$\left(\frac{dE_n^{(m)}}{dt}\right)_L < 0, \quad \forall t \in [0, \infty[, \ \forall n \in \mathbb{N}, \tag{4.13}$$

for arbitrary initial data, guarantee

1) *the linear asymptotic stability of the thermal conduction solution;*
2) *the absence of subcritical instability and the nonlinear asymptotic stability in the $L^2(\Omega)$-norm.*

PROOF. Let (4.13) hold. Then by virtue of the previous theorem, $E_n^{(m)}$ $\forall n \in \mathbb{N}$—along the solution of (3.3)–(3.4)—is a decreasing function of time. Therefore, by virtue of ii) of remark 3.1, either 1) or 2) are immediately obtained.

5. LINEARIZATION PRINCIPLE

It remains to obtain the conditions able to satisfy (4.12). One easily obtains

$$\begin{cases} \nabla \times \mathbf{u}_n = \left(\dfrac{\partial w_n}{\partial y} - \dfrac{\partial v_n}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial u_n}{\partial z} - \dfrac{\partial w_n}{\partial x}\right)\mathbf{j} + \zeta_n \mathbf{k}, \\ \nabla \times \left[\left(R\theta_n - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_{\alpha n}\right)\mathbf{k}\right] = \left(R\theta_n - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_{\alpha n}\right)_y \mathbf{i} \\ \qquad - \left(R\theta_n - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_{\alpha n}\right)_x \mathbf{j}, \\ \mathbf{k} \cdot \nabla \times (\nabla \times \mathbf{u}_n) = -\Delta w_n, \\ \mathbf{k} \cdot \nabla \times \left[\nabla \times \left(R\theta_n - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_{\alpha n}\right)\mathbf{k}\right] = -\Delta_1 \left(R\theta_n - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_{\alpha n}\right). \end{cases} \tag{5.1}$$

Therefore—in view of (4.4)–(4.5)—omitting the bars and taking the vertical component of the double curl of $\mathbf{u}_n$, one obtains ($\alpha = 1, \dots, m$)

$$
(5.2)\qquad \begin{cases} P_r^{-1}\dfrac{\partial u_n}{\partial t}=\Delta u_n, \quad P_r^{-1}\dfrac{\partial v_n}{\partial t}=\Delta v_n,\\ P_r^{-1}\dfrac{\partial \Delta w_n}{\partial t}=\Delta_1\Big(R\theta_n-\displaystyle\sum_{\alpha=1}^{m}R_\alpha\Phi_{\alpha n}\Big)+\Delta\Delta w_n,\\ \dfrac{\partial\theta_n}{\partial t}=HRw_n+\Delta\theta_n, \quad P_\alpha\dfrac{\partial\Phi_{\alpha n}}{\partial t}=H_\alpha R_\alpha w_n+\Delta\Phi_{\alpha n} \end{cases}
$$

and hence, by virtue of (2.9)–(2.10), it follows that

$$
(5.3)\qquad \begin{cases} P_r^{-1}\dfrac{\partial u_n}{\partial t}=-\xi_n u_n, \quad P_r^{-1}\dfrac{\partial v_n}{\partial t}=-\xi_n v_n,\\ P_r^{-1}\dfrac{\partial w_n}{\partial t}=\dfrac{a^2}{\xi_n}\Big(R\theta_n-\displaystyle\sum_{\alpha=1}^{m}R_\alpha\Phi_{\alpha n}\Big)-\xi_n w_n,\\ \dfrac{\partial\theta_n}{\partial t}=HRw_n-\xi_n\theta_n, \quad P_\alpha\dfrac{\partial\Phi_{\alpha n}}{\partial t}=H_\alpha R_\alpha w_n-\xi_n\Phi_{\alpha n}, \end{cases}
$$

under the b.c.

$$
(5.4)\qquad w_n=\theta_n=\Phi_{\alpha n}=0, \quad \text{on } z=0,1, \quad \alpha=1,\dots,m.
$$

In view of $(5.3)_1$–$(5.3)_2$, one obtains

$$
(5.5)\qquad \frac{1}{2}P_r^{-1}\frac{d}{dt}(\|u_n\|^2+\|v_n\|^2)=-\xi_n(\|u_n\|^2+\|v_n\|^2)<0, \quad \forall n\in\mathbb{N}.
$$

Therefore, (4.12) is verified when

$$
(5.6)\qquad \frac{d\mathscr{E}_n^{(m)}}{dt}<0, \quad \forall t\in[0,\infty[, \quad \forall n\in\mathbb{N},
$$

with

$$
(5.7)\qquad \mathscr{E}_n^{(m)}=\frac{1}{2}\Big(\|w_n\|^2+\|\theta_n\|^2+\sum_{\alpha=1}^{m}P_\alpha\|\Phi_{\alpha n}\|^2\Big).
$$

Setting

$$
(5.8)\qquad \eta_n=\frac{a^2}{\xi_n},
$$

in view of (5.3), one obtains

(5.9)
$$\begin{cases} \dfrac{\partial w_n}{\partial t} = \eta_n P_r \Big(R\theta_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \Phi_{\alpha n} \Big) - \xi_n P_r w_n, \\ \dfrac{\partial \theta_n}{\partial t} = HRw_n - \xi_n \theta_n, \\ \dfrac{\partial \Phi_{\alpha n}}{\partial t} = \dfrac{H_\alpha R_\alpha}{P_\alpha} w_n - \dfrac{\xi_n}{P_\alpha} \Phi_{\alpha n}, \quad \alpha = 1, 2, \ldots, m, \end{cases}$$

under the boundary conditions (5.4).

On setting

(5.10)
$$p = m + 2, \quad \Psi_1 = w, \quad \Psi_2 = \theta, \quad \Psi_3 = \Phi_1, \ldots, \Psi_p = \Phi_m,$$

(5.9) can be written

(5.11)
$$\frac{\partial}{\partial t} \begin{pmatrix} \Psi_{1n} \\ \Psi_{2n} \\ \vdots \\ \Psi_{pm} \end{pmatrix} = \mathscr{L}_n^{(p)} \begin{pmatrix} \Psi_{1n} \\ \Psi_{2n} \\ \vdots \\ \Psi_{pm} \end{pmatrix},$$

with

(5.12)
$$\mathscr{L}_n^{(p)} = \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \cdots & \cdots & \cdots \\ a_{p1} & \cdots & a_{pp} \end{pmatrix},$$

(5.13)
$$\begin{cases} a_{11} = -P_r \xi_n, \quad a_{12} = P_r \eta_n R, \quad a_{13} = -P_r \eta_n R_1, \ldots, \quad a_{1p} = -P_r \eta_n R_m, \\ a_{21} = HR, \quad a_{22} = -\xi_n, \quad a_{23} = a_{24} = \cdots = a_{2p} = 0, \\ a_{31} = \dfrac{H_1}{P_1} R_1, \quad a_{32} = 0, \quad a_{33} = -\dfrac{\xi_n}{P_1}, \quad a_{34} = a_{35} = \cdots = a_{3p} = 0, \\ a_{41} = \dfrac{H_2}{P_2} R_2, \quad a_{42} = a_{43} = 0, \quad a_{44} = -\dfrac{\xi_n}{P_2}, \quad a_{45} = a_{46} = \cdots = a_{4p} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{p1} = \dfrac{H_m}{P_m} R_m, \quad a_{p2} = a_{p3} = \cdots = a_{p(p-1)} = 0, \quad a_{pp} = -\dfrac{\xi_n}{P_m}. \end{cases}$$

Let

(5.14)
$$\lambda_{nr} = A_{nr} + iB_{nr}, \quad r = 1, 2, \ldots, p,$$

be the eigenvalues of $\mathscr{L}_n^{(p)}$ and let

(5.15)
$$\mathscr{E}_n^{(p)} = \frac{1}{2} \sum_{r=1}^{p} \|\Psi_n\|^2 = \frac{1}{2} \sum_{r=1}^{p} \langle \Psi_{nr}, \bar{\Psi}_{nr} \rangle,$$

where now the over bar designates the complex conjugate. Then the following theorem holds.

THEOREM 5.1. *Let the eigenvalues λ_{nr}, $r \in \{1, 2, \dots, p\}$, have all negative real parts, i.e.*

$$A_{nr} < 0, \quad \forall r \in \{1, 2, \dots, p\}. \tag{5.16}$$

Then there exists a positive constant K_{np} such that, along the solutions of (5.11), *it follows that*

$$\frac{d}{dt}\mathscr{E}_{np} \leq -K_{np}\lambda^*_{np}\mathscr{E}_{np}, \tag{5.17}$$

with

$$\lambda^*_{np} = \min|A_{nr}|, \quad r \in \{1, 2, \dots, p\}. \tag{5.18}$$

PROOF. In the case $R = R_1 = \cdots = R_m = 0$ (formally in the case $p = 1$ and hence $m = -1$), (5.11) reduces to

$$\frac{\partial \Psi_1}{\partial t} = -P_r\xi_n\Psi_1 \tag{5.19}$$

and immediately it follows that

$$\frac{1}{2}\frac{d}{dt}\|\Psi_1\|^2 \leq -P_r\xi_n\|\Psi_1\|^2 < -P_r\pi^2\|\Psi_1\|, \tag{5.20}$$

i.e. (5.17), for $p = 1$, holds with

$$K_{n1} = -P_r, \quad \lambda^*_{n1} = \pi^2 = \min(a^2 + n^2\pi^2). \tag{5.21}$$

Therefore—since (5.17) holds for $p = 1$—it is enough to show that if it holds for $p = s - 1$, then it holds also for $p = s$. For $p = s$, (5.11) becomes

$$\begin{cases} \dfrac{\partial}{\partial t}\Psi_{n1} = a_{11}\Psi_{n1} + a_{12}\Psi_{n2} + \cdots + a_{1s}\Psi_{ns}, \\ \dfrac{\partial}{\partial t}\Psi_{n2} = a_{21}\Psi_{n1} + a_{22}\Psi_{n2} + \cdots + a_{2s}\Psi_{ns}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial}{\partial t}\Psi_{ns} = a_{s1}\Psi_{n1} + a_{s2}\Psi_{n2} + \cdots + a_{ss}\Psi_{ns}. \end{cases} \tag{5.22}$$

Let

$$\lambda^{(n)}_{s1} = A^{(n)}_{s1} + iB^{(n)}_{s1}, \quad \lambda_{n1} = A_{n1} + iB_{n1}, \tag{5.23}$$

be an eigenvalue of $\mathscr{L}_n^{(1)}$ and $\mathbf{U}_n = (1, U_{2n}, \ldots, U_{sn})$ an associate eigenvector i.e. let $(1, U_{2n}, \ldots, U_{sn})$ (generally complex) verifies the system

$$(5.24) \qquad \begin{cases} a_{11} + a_{12}U_{2n} + a_{13}U_{3n} + \cdots + a_{1s}U_{sn} = \lambda_{n1}, \\ a_{21} + a_{22}U_{2n} + a_{23}U_{3n} + \cdots + a_{2s}U_{sn} = \lambda_{n1}U_{2n}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{s1} + a_{s2}U_{2n} + a_{s3}U_{3n} + \cdots + a_{ss}U_{sn} = \lambda_{n1}U_{sn}. \end{cases}$$

Setting

$$(5.25) \qquad Z_{n1} = \Psi_{n1}, \qquad Z_{nj} = \Psi_{nj} - U_{jn}\Psi_{n1}, \quad j = 2, \ldots, s,$$

i.e.

$$(5.26) \qquad \Psi_{n1} = Z_{n1}, \qquad \Psi_{nj} = Z_{nj} + U_{jn}Z_{n1}, \quad j = 2, \ldots, s,$$

it follows that

$$\begin{aligned} \frac{\partial}{\partial t}Z_{n1} &= a_{11}Z_{n1} + \sum_{j=2}^{s}(Z_{nj} + U_{jn}Z_{n1}) \\ &= \Big(a_{11} + \sum_{j=2}^{s} a_{ij}U_{jn}\Big)Z_{n1} + \sum_{j=2}^{s} a_{1j}Z_{nj} \end{aligned}$$

and in view of $(5.24)_1$, it follows that

$$(5.27) \qquad \frac{\partial}{\partial t}Z_{n1} = \lambda_{s1}^{(n)}Z_{n1} + \sum_{j=2}^{s} a_{1j}Z_{nj}.$$

Analogously, in view of $(5.25)_1$ one obtains

$$\begin{aligned} \frac{\partial}{\partial t}Z_{n2} = \frac{\partial}{\partial t}\Psi_{n2} - U_{2n}\frac{\partial}{\partial t}Z_{n1} &= a_{21}Z_{n1} + \sum_{j=2}^{s} a_{2j}(Z_{nj} + U_{jn}Z_{n1}) \\ &\quad - U_{2n}\Big(\lambda_{s1}^{(n)}Z_{n1} + \sum_{j=2}^{s} a_{1j}Z_{nj}\Big) \\ &= \left[\Big(a_{21} + \sum_{j=2}^{s} a_{2j}U_{jn}\Big) - U_{2n}\lambda_{s1}^{(n)}\right]Z_{n1} + \sum_{j=2}^{s}(a_{2j} - a_{1j}U_{2n})Z_{nj} \end{aligned}$$

and—in view of $(5.24)_2$—it follows that

$$(5.28) \qquad \frac{\partial}{\partial t}Z_{n2} = \sum_{j=2}^{s}(a_{2j} - a_{1j}U_{2n})Z_{nj}.$$

Since in general it follows that for $\alpha \in \{2, \dots, s\}$

$$\frac{\partial Z_{n\alpha}}{\partial t} = \sum_{j=2}^{s} \Big(a_{\alpha j} - a_{1j} U_{\alpha n} \Big) Z_{nj}, \tag{5.29}$$

setting

$$b_{\alpha j} = a_{\alpha j} - a_{1j} U_{\alpha n}, \quad \alpha, j \in \{2, 3, \dots\}, \tag{5.30}$$

one obtains

$$\begin{cases} \dfrac{\partial Z_{n1}}{\partial t} = \lambda_{s1}^{(n)} Z_{n1} + \displaystyle\sum_{j=2}^{s} a_{1j} Z_{nj}, \\ \dfrac{\partial Z_{n\alpha}}{\partial t} = \displaystyle\sum_{j=2}^{s} b_{\alpha j} Z_{nj}, \quad \alpha = 2, 3, \dots, s. \end{cases} \tag{5.31}$$

Setting

$$Z_{n1} = \frac{1}{\mu_n} Y_{n1}, \tag{5.32}$$

one obtains

$$\frac{\partial Y_{n1}}{\partial t} = \lambda_{s1}^{(n)} Y_{n1} + \mu_n \sum_{j=2}^{s} a_{1j} Z_{nj} \tag{5.33}$$

and the system

$$\frac{\partial}{\partial t} \begin{pmatrix} Z_{n2} \\ \vdots \\ Z_{ns} \end{pmatrix} = \mathscr{L}_{n(s-1)}^{(s-1)} \begin{pmatrix} Z_{n2} \\ \vdots \\ Z_{ns} \end{pmatrix}, \tag{5.34}$$

with

$$\mathscr{L}_{n}^{(s-1)} = \begin{pmatrix} b_{22} & b_{23} & \cdots & b_{2s} \\ b_{32} & b_{33} & \cdots & b_{3s} \\ \cdots & \cdots & \cdots & \cdots \\ b_{s2} & b_{s3} & \cdots & b_{ss} \end{pmatrix}. \tag{5.35}$$

To (5.33)–(5.34) one has to add the boundary conditions

$$Z_{n\alpha} = 0, \quad \text{on } z = 0, 1, \quad \forall \alpha \in (2, 3, \dots, s). \tag{5.36}$$

Since the eigenvalues are invariant with respect to the linear transformations, the eigenvalues of $\mathscr{L}_n^{(s-1)}$ are given by the eigenvalues

$$\lambda_{n\alpha} = A_{n\alpha} + iB_{n\alpha}, \quad \alpha \in \{2, \dots, s\}, \tag{5.37}$$

of $\mathscr{L}_n^{(s)}$ and hence

$$A_{n\alpha} < 0, \quad \forall \alpha \in \{2, \dots, s\}. \tag{5.38}$$

Therefore, by assumption, one has

$$\frac{d}{dt}\mathscr{E}_{n(s-1)} < -K_{n(s-1)}\lambda^*_{n(s-1)}\mathscr{E}_{n(s-1)}, \tag{5.39}$$

with $K_{n(s-1)}$ positive constant and

$$\lambda^*_{n(s-1)} = \min|A_{nr}|, \quad r = 2, 3, \dots, p, \tag{5.40}$$

$$\mathscr{E}_{n(s-1)} = \frac{1}{2}\sum_{\alpha=2}^{s}\langle Z_{n\alpha}, \bar{Z}_{n\alpha}\rangle. \tag{5.41}$$

Setting

$$Z_{n\alpha} = P_{n\alpha} + iQ_{n\alpha}, \quad \alpha = 1, 2, \dots, s \tag{5.42}$$

and denoting by *rp* the real part, it follows that

$$\begin{cases} rp\langle \lambda_{s1}^{(n)} Z_{n1}, \bar{Z}_{n1}\rangle = A_{s1}\langle Z_{n1}, \bar{Z}_{n1}\rangle, \\ rp\left\langle \bar{Z}_{n1}, \dfrac{\partial Z_{n1}}{\partial t}\right\rangle = \dfrac{1}{2}\dfrac{d}{dt}\langle \bar{Z}_{n1}, Z_{n1}\rangle, \\ rp\langle Z_{nj}, \bar{Z}_{n1}\rangle = \langle P_{n1}, P_{nj}\rangle + \langle Q_{n1}, Q_{nj}\rangle, \quad j = 2, \dots, s \end{cases} \tag{5.43}$$

and, in view of (5.33), one obtains

$$\frac{1}{2}\frac{d}{dt}\langle Y_{n1}, \bar{Y}_{n1}\rangle = A_{11}^{(n)}\langle Y_{n1}, \bar{Y}_{n1}\rangle + \mu_n\sum_{\alpha=2}^{s} a_{1\alpha}(\langle P_{1n}, P_{\alpha n}\rangle + \langle Q_{1n}, Q_{\alpha n}\rangle). \tag{5.44}$$

Since

$$\begin{aligned}\langle P_{1n}, P_{\alpha n}\rangle + \langle Q_{1n}, Q_{\alpha n}\rangle &\le \frac{1}{2}(\|P_{1n}\|^2 + \|P_{\alpha n}\|^2 + \|Q_{1n}\|^2 + \|Q_{\alpha n}\|^2) \\ &= \frac{1}{2}(\langle Z_{1n}, \bar{Z}_{1n}\rangle + \langle Z_{\alpha n}, \bar{Z}_{\alpha n}\rangle),\end{aligned}$$

setting

$$m^* = \frac{1}{2} \max|a_{1\alpha}|, \quad \alpha = 2, \dots, p, \tag{5.45}$$

one obtains

$$\frac{1}{2}\frac{d}{dt}\langle Z_{1n}, \bar{Z}_{1n}\rangle \leq -(|A_{1n}| - m^* p\mu_n)\langle Z_{1n}, \bar{Z}_{1n}\rangle + \mu_n m^* \mathscr{E}_{n(p-1)} \tag{5.46}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\sum_{\alpha=1}^{s}\langle Z_{\alpha n}, \bar{Z}_{\alpha n}\rangle \leq -(|A_{1n}| - m^* s\mu_n)\langle Z_{1n}, \bar{Z}_{1n}\rangle + \\ - (-\mu_n m^* + K_{n(s-1)\lambda^*_{n(s-1)}})\mathscr{E}_{n(s-1)}. \tag{5.47}$$

Choosing

$$\mu_n < \min\left(\frac{|A_{1n}|}{m^* s}, \frac{K_{n(s-1)}\lambda^*_{n(s-1)}}{m^*}\right), \tag{5.48}$$

it follows that there exist positive constants K_{ns} such that

$$\frac{d\mathscr{E}_{ns}}{dt} \leq -K_{ns}\mathscr{E}_{n(s-1)}. \tag{5.49}$$

Remark 5.1. We remark that

i) condition (5.16) is also necessary for the asymptotic stability;
ii) (5.17) imply the exponential asymptotic stability;
iii) theorem 5.1 shows that the linear instability captures completely the physics of the onset of convection since the absence of subcritical instability, together with the property of the linear stability to guarantee also the global nonlinear stability, has been obtained.

6. Critical Rayleigh numbers

The equation governing the eigenvalues of $\mathscr{L}_n^{(m)}$ can be written

$$\prod_{r=1}^{m+2}(\lambda - \lambda_{nr}) = 0, \tag{6.1}$$

i.e.

$$\lambda^{m+2} - \mathtt{I}_{n1}\lambda^{m+1} + \mathtt{I}_{n2}\lambda^m + \cdots + (-1)^{m+2}\mathtt{I}_{n(m+2)} = 0, \tag{6.2}$$

with $\mathtt{I}_{nr}$, $r \in \{1, 2, \dots, m+2\}$, characteristic values (invariants) of $\mathscr{L}_n^{(m)}$ given by

$$\text{(6.3)} \qquad \mathtt{I}_{1n} = \sum_{r=1}^{m+2} \lambda_{nr}, \quad \mathtt{I}_{2n} = \sum_{r \neq s=1}^{m+2} \lambda_{nr}\lambda_{ns}, \quad \mathtt{I}_{n(m+2)} = \prod_{r=1}^{m+2} \lambda_{nr},$$

where, in terms of the entries of $\mathscr{L}_n^{(m)}$, $\mathtt{I}_{nr}$ is obtained by adding the principal minors of order r. Introducing the Hurwitz matrix {see [35] and [36]}

$$\text{(6.4)} \qquad \begin{pmatrix} -\mathtt{I}_{n1} & -\mathtt{I}_{n3} & -\mathtt{I}_{n5} & \cdots & 0 \\ 1 & \mathtt{I}_{n2} & \mathtt{I}_{nr} & \cdots & 0 \\ 0 & -\mathtt{I}_{n1} & -\mathtt{I}_{n3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathtt{I}_{n(m+2)} \end{pmatrix}$$

and the principal minors

$$\text{(6.5)} \quad \Delta_{n1} = -\mathtt{I}_{n1}, \quad \Delta_{2n} = \begin{vmatrix} -\mathtt{I}_{n1} & -\mathtt{I}_{n3} \\ 1 & \mathtt{I}_{n2} \end{vmatrix}, \dots, \Delta_{n(m+2)} = \mathtt{I}_{n(m+2)} \cdot \Delta_{n(m+1)},$$

the following well known Routh-Hurwitz conditions hold [35]–[36]

i) *in order for all the roots of* (6.2) *to have negative real parts, it is necessary that all the coefficients of* (6.2) *are positive, while the necessary and sufficient condition is that all the principle diagonal minors* (6.5) *be positive*

$$\text{(6.6)} \qquad \Delta_{n1} > 0, \quad \Delta_{n2} > 0, \dots, \Delta_{n(m+2)} > 0;$$

ii) *either if one of the coefficients of* (6.2) *is negative or one of the inequalities* (6.6) *is reversed, then some roots will have positive real parts.*

By virtue of theorems 4.1–5.1 and i)–ii), the following theorem of absence of subcritical instabilities and nonlinear global asymptotic stability (via the Routh-Hurwitz conditions) holds.

THEOREM 6.1. *If and only if* (6.6) *hold for any* $(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$, *the onset of convection is not allowed and the global nonlinear asymptotic stability is guaranteed. If* (6.6) *does not hold for any* $(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$ *and* $(\bar{n}, \bar{a}^2)$ *is the first couple for which one of* (6.6) *holds reversed, then convection arises along the* $\bar{n}$*-th Fourier component of perturbation, for* $a^2 = \bar{a}^2$.

REMARK 6.1. By virtue of the linearization principle (theorems 4.1–6.1) the conditions necessary and sufficient for the linear stability obtained in the past in the case at stake, become necessary and sufficient for the global nonlinear asymptotic energy stability in the $L^2(\Omega)$-norm. In the sequel we will be concentrated to the case $m \geq 2$ and especially in obtaining stability conditions in algebraic closed form. In this section, for the sake of completeness, we confine ourselves to the cases $(m = 0, m = 1)$.

6.1. *Layer heated from below (Bènard problem)*

One has

$$\mathscr{L}_n^{(0)} = \begin{pmatrix} -P_r\xi_n & P_r\eta_n R \\ R & -\xi_n \end{pmatrix}, \tag{6.7}$$

the Hurwitz matrix reduces to

$$\begin{pmatrix} -\mathtt{I}_{n1} & 0 \\ 1 & \mathtt{I}_{n2} \end{pmatrix} \tag{6.8}$$

and the spectral equation is given by

$$\lambda_n^2 - \mathtt{I}_{n1}\lambda_n + \mathtt{I}_{n2} = 0. \tag{6.9}$$

In view of

$$\mathtt{I}_{n1} = -(1+P_r)\xi_n < 0, \qquad \mathtt{I}_{n2} = P_r\eta_n\left(\frac{\xi_n^2}{\eta_n} - R^2\right), \tag{6.10}$$

(6.6) reduces to

$$-\mathtt{I}_{n1}\mathtt{I}_{n2} > 0 \quad \Leftrightarrow \quad R^2 < \frac{\xi_n^2}{\eta_n}. \tag{6.11}$$

On the other hand

$$\min_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+} \frac{\xi_n^2}{\eta_n} = \left[\frac{(a^2+n^2\pi^2)^3}{a^2}\right]_{\left(a^2=\frac{\pi^2}{2}\right)}^{(n=1)} = \frac{27}{4}\pi^4 \tag{6.12}$$

and one recovers immediately the celebrated condition of linear and global non linear stability {[1], [38]}*:*

$$R^2 < \frac{27}{4}\pi^4 \simeq 675.5. \tag{6.13}$$

We remark that (6.13) *can be obtained immediately also by each one of the following observations*

i) *in view of*

$$\mathtt{I}_{n1}^2 - 4\mathtt{I}_{n2} = \xi_n^2(1-P_r)^2 + 4P_r\eta_n R^2 > 0, \quad \forall(n,a^2) \tag{6.14}$$

the roots of (6.9) *are real numbers and hence the "strong principle of exchange of stability" holds* (*i.e. convection arises via a stationary state*)*;*

ii) *since* $\mathscr{L}_n^{(0)}$ *is symmetrizable then the eigenvalues are real numbers, the strong principle of exchange of stability holds and the stability condition is given by* $\mathtt{I}_2 > 0$.

6.2. Double diffusive-convection in a layer heated and salted from below

In the sub-case $(m = 1, H_1 = 1)$ one obtains

$$\mathscr{L}_n^{(1)} = \begin{pmatrix} -P_r\xi_n & P_r\eta_n R & -P_r\eta_n R_1 \\ R & -\xi_n & 0 \\ \dfrac{R_1}{P_1} & 0 & -\dfrac{\xi_n}{P_1} \end{pmatrix}, \tag{6.15}$$

the spectral equation and the Hurwitz matrix are respectively given by

$$\lambda^3 - \mathtt{I}_{n1}\lambda^2 + \mathtt{I}_{n2}\lambda - \mathtt{I}_{n3} = 0, \tag{6.16}$$

$$\begin{pmatrix} -\mathtt{I}_{n1} & -\mathtt{I}_{n3} & 0 \\ 1 & \mathtt{I}_{n2} & 0 \\ 0 & -\mathtt{I}_{n1} & -\mathtt{I}_{n3} \end{pmatrix} \tag{6.17}$$

with

$$\begin{cases} \mathtt{I}_{n1} = -\left(1 + P_r + \dfrac{1}{P_1}\right)\xi_n < 0, \\ \mathtt{I}_{n2} = P_r\eta_n\left[\left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_1 P_r}\right)\dfrac{\xi_n^2}{\eta_n} + \dfrac{R_1^2}{P_1} - R^2\right], \\ \mathtt{I}_{n3} = \dfrac{P_r}{P_1}\left(R^2 - \dfrac{R_1^2}{P_1} - \dfrac{\xi_n^2}{\eta_n}\right)\eta_n\xi_n, \end{cases} \tag{6.18}$$

Setting

$$\begin{cases} R_{C_1} = \dfrac{R_1^2}{P_1} + \left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_1 P_r}\right)\dfrac{27}{4}\pi^4 \\ R_{C_2} = \dfrac{R_1^2}{P_1} + \dfrac{27}{4}\pi^4 < R_{C_1} \end{cases} \tag{6.19}$$

it follows that the Hurwitz conditions reduce to

$$\begin{cases} [\Delta_{n2}]^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)} = (\mathtt{I}_{n1}\mathtt{I}_{n2} - \mathtt{I}_{n3})^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)} < 0, \\ [\Delta_{n3}]^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)} = [-\mathtt{I}_{n3}(-\mathtt{I}_{n1}\mathtt{I}_{n2} + \mathtt{I}_{n3})]^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)} > 0. \end{cases} \tag{6.20}$$

Since

$$R^2 < R_{C_2} \quad \Rightarrow \quad (\mathtt{I}_{n2} > 0, \mathtt{I}_{n3} < 0) \tag{6.21}$$

one has to require $(6.20)_1$ which is equivalent to

$$R^2 < R_{C_3} = R_{C_1} + \frac{R_{C_1} - R_{C_2}}{(1 + P_r)P_1}. \tag{6.22}$$

Therefore, in view of (6.21)–(6.22), it follows that, *if and only if*

$$R^2 < R_C^2 = R_{C_2} = \frac{R_1^2}{P_1} + \frac{27}{4}\pi^4 \tag{6.23}$$

the thermal conduction solution is nonlinearly globally asymptotically stable when L is heated and salted from below.

We remark that also in the case at stake *the principle of exchange of stability holds*. In fact $R^2 = R_C^2$ gives $\mathtt{I}_{n3} = 0$ at $\left(n = 1, a^2 = \frac{\pi^2}{2}\right)$, to which is associated the zero solution of (6.16) at $\left(n = 1, a^2 = \frac{\pi^2}{2}\right)$.

6.3. *Double diffusive-convection in a layer salted above and heated below*

In the subcase $(m = 1, H_1 = -1)$, one obtains

$$\mathscr{L}_n^{(1)} = \begin{pmatrix} -P_r\xi_n & P_r\eta_n R & -P_r\eta_n R_1 \\ R & -\xi_n & 0 \\ -\dfrac{R_1}{P_1} & 0 & -\dfrac{\xi_n}{P_1} \end{pmatrix}, \tag{6.24}$$

while the spectral equation, the Hurwitz matrix and $\mathtt{I}_{n1}$ are still given by (6.16)–(6.17) and $(6.18)_1$, but with

$$\begin{cases} \mathtt{I}_{n2} = P_r\eta_n\left[\left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_1P_r}\right)\dfrac{\xi_n^2}{\eta_n} - \dfrac{R_1^2}{P_1} - R^2\right] \\ \mathtt{I}_{n3} = \dfrac{P_r}{P_1}\left(R^2 + \dfrac{R_1^2}{P_1} - \dfrac{\xi_n^2}{\eta_n}\right)\eta_n\xi_n. \end{cases} \tag{6.25}$$

Setting

$$\begin{cases} R_{C_1} = \left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_1P_r}\right)\dfrac{27}{4}\pi^4 - \dfrac{R_1^2}{P_1} \\ R_{C_2} = \dfrac{27}{4}\pi^4 - \dfrac{R_1^2}{P_1} < R_{C_1} \end{cases} \tag{6.26}$$

the Hurwitz conditions require

$$(6.27)\qquad \begin{cases} R^2 < \min(R_{C_1}, R_{C_2}) = R_{C_2} \\ R^2 < R_{C_1} + \dfrac{R_{C_1} - R_{C_2}}{(1+P_r)P_1}. \end{cases}$$

Therefore, in view of (6.26)–(6.27), it follows that

i) *if and only if*

$$(6.28)\qquad R^2 < R_C^2 = \frac{27}{4}\pi^4 - \frac{R_1^2}{P_1}$$

the thermal conduction solution is nonlinearly globally asymptotically stable when L is heated from below and salted from above;

ii) *since* $R^2 = R_C^2 \Leftrightarrow (\mathtt{I}_{n3})^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)} = 0$, *the marginal state is stationary.*

iii) *the Rayleigh critical value of the salt for the onset of the* cold convection [37], *is*

$$(6.29)\qquad R_{1C}^{(2)} = \frac{27}{4}\pi^4 P_1,$$

i.e.

$$(6.30)\qquad R_1^2 > R_{1C},$$

implies the instability of the thermal conduction solution irrespective of the temperature gradient (instability named *cold convection* [37]).

We recall that both the cases (6.21)–(6.22) have been deeply studied (with different procedures) and a good account of the results obtained can be found in {[3], [6]–[18], [38] and the references therein}.

7. Ternary diffusion-convection in a layer heated from below and salted from above and below

The case $(m = 2, H_1 = 1, H_2 = -1)$ is a prototype case of diffusion-convection in layers heated from below and salted from above and below. Since $\mathscr{L}_n^{(2)}$ in this case is given by

$$(7.1)\qquad \mathscr{L}_n^{(2)} = \begin{pmatrix} -P_r\xi_n & P_r\eta_n R & -P_r\eta_n R_1 & -P_r\eta_n R_2 \\ R & -\xi_n & 0 & 0 \\ \dfrac{R_1}{P_1} & 0 & -\dfrac{\xi_n}{P_1} & 0 \\ -\dfrac{R_2}{P_2} & 0 & 0 & -\dfrac{\xi_n}{P_2} \end{pmatrix},$$

the spectral equation and the Hurwitz matrix are respectively given by

$$\lambda^4 - \mathtt{I}_{n1}\lambda^3 + \mathtt{I}_{n2}\lambda^2 - \mathtt{I}_{n3}\lambda + \mathtt{I}_{n4} = 0, \tag{7.2}$$

$$\begin{pmatrix} -\mathtt{I}_{n1} & -\mathtt{I}_{n3} & 0 & 0 \\ 1 & \mathtt{I}_{n2} & \mathtt{I}_{n4} & 0 \\ 0 & -\mathtt{I}_{n1} & -\mathtt{I}_{n3} & 0 \\ 0 & 1 & \mathtt{I}_{n2} & \mathtt{I}_{n4} \end{pmatrix}, \tag{7.3}$$

with

$$\begin{cases} \mathtt{I}_{n1} = -\Big(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2} + P_r\Big)\xi_n, \\ \mathtt{I}_{n2} = \eta_n P_r \Big\{ \dfrac{R_1^2}{P_1} - \dfrac{R_2^2}{P_2} + \Big[1 + \dfrac{1}{P_1} + \dfrac{1}{P_2} \\ \qquad\qquad + \dfrac{1}{P_r}\Big(\dfrac{1}{P_1} + \dfrac{1}{P_2} + \dfrac{1}{P_1 P_2}\Big)\Big] \dfrac{\xi_n^2}{\eta_n} - R^2 \Big\}, \\ \mathtt{I}_{n3} = \dfrac{P_r \eta_n \xi_n}{P_1 P_2} \Big\{ -(1+P_2)R_1^2 + R_2^2(1+P_1) + (P_1+P_2)R^2 + \\ \qquad\qquad - \Big[1 + P_1 + P_2 + \dfrac{(P_1P_2 + P_1 + P_2)}{P_r}\Big] \dfrac{\xi_n^2}{\eta_n} \Big\}, \\ \mathtt{I}_{n4} = \dfrac{P_r}{P_1 P_2} \xi_n^2 \eta_n \Big(R_1^2 - R_2^2 + \dfrac{\xi_n^2}{\eta_n} - R^2 \Big). \end{cases} \tag{7.4}$$

We set

$$\begin{cases} R_{C_1} = \dfrac{R_1^2}{P_1} - \dfrac{R_2^2}{P_2} + \Big[1 + \dfrac{1}{P_1} + \dfrac{1}{P_2} + \dfrac{1}{P_r}\Big(\dfrac{1}{P_1} + \dfrac{1}{P_2} + \dfrac{1}{P_1P_2}\Big)\Big] \dfrac{27}{4}\pi^4, \\ R_{C_2} = \dfrac{(1+P_2)}{P_1+P_2} R_1^2 - \dfrac{(1+P_1)}{P_1+P_2} R_2^2 \\ \qquad + \Big[\Big(1 + \dfrac{1}{P_1+P_2}\Big) + P_r^{-1}\Big(1 + \dfrac{P_1P_2}{P_1+P_2}\Big)\Big] \dfrac{27}{4}\pi^4, \\ R_{C_3} = R_1^2 - R_2^2 + \dfrac{27}{4}\pi^4 \end{cases} \tag{7.5}$$

and notice that the Hurwitz determinants are given by

$$\begin{cases} \Delta_{n1} = -\mathtt{I}_{n1}, \quad \Delta_{n2} = \mathtt{I}_{n3} - \mathtt{I}_{n1}\mathtt{I}_{n2}, \\ \Delta_{n3} = -\mathtt{I}_{n3}\Delta_{n2} - \mathtt{I}_{n1}^2\mathtt{I}_{n4} = -\mathtt{I}_{n3}^2 + \mathtt{I}_{n1}\mathtt{I}_{n2}\mathtt{I}_{n3} - \mathtt{I}_{n1}^2\mathtt{I}_{n4}, \\ \Delta_{n4} = \mathtt{I}_{n4}\Delta_{n3}. \end{cases} \tag{7.6}$$

Since

$$\mathtt{I}_{n1} < 0, \quad \forall (n, a^2) \in \mathbb{N} \times \mathbb{R}^+, \tag{7.7}$$

it follows that

LEMMA 7.1. *If and only if*

$$\begin{cases} \mathtt{I}_{n2} > 0, \quad \mathtt{I}_{n3} < 0, \quad \mathtt{I}_{n4} > 0, \\ \Delta_{n3} > 0, \end{cases} \tag{7.8}$$

all the roots of (7.2) *have negative real parts.*

PROOF. In fact $(\Delta_{n3} > 0, \mathtt{I}_{n4} > 0) \Rightarrow \Delta_{n4} > 0$; $(\Delta_{n3} > 0, \mathtt{I}_{n3} < 0, \mathtt{I}_{n4} > 0) \Rightarrow \Delta_{n2} > 0$.

REMARK 7.1. Since (7.8) have to be verified $\forall (n, a^2) \in \mathbb{N} \times \mathbb{R}^+$, in view of (7.4)–(7.5), it is necessary and sufficient that are verified for $\left(n = 1, a^2 = \frac{\pi^2}{2}\right)$.

Setting

$$\Delta_\alpha = [\Delta_{n\alpha}]^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)}, \qquad \mathtt{I}_\alpha = [\mathtt{I}_{n\alpha}]^{(n=1)}_{\left(a^2=\frac{\pi^2}{2}\right)}, \quad \alpha = 1, 2, 3, 4, \tag{7.9}$$

then the following theorem is immediately implied by Lemma 7.1.

THEOREM 7.1. *If and only if*

$$R^2 < \min(R_{C_1}, R_{C_2}, R_{C_3}), \quad \Delta_3 > 0, \tag{7.10}$$

the thermal conduction solution is globally nonlinearly asymptotically stable.

LEMMA 7.2. *The spectral equation*

$$\lambda^4 - \mathtt{I}_1\lambda^3 + \mathtt{I}_2\lambda^2 - \mathtt{I}_3\lambda + \mathtt{I}_4 = 0, \tag{7.11}$$

admits the root

$$\lambda = iY, \qquad Y \in \mathbb{R}, \tag{7.12}$$

if and only if

$$\Delta_3 = 0 \quad \Leftrightarrow \quad Y^4 - \mathtt{I}_2 Y^2 + \mathtt{I}_4 = 0, \tag{7.13}$$

with Y given by

$$Y^2 = \frac{\mathtt{I}_3}{\mathtt{I}_1}. \tag{7.14}$$

PROOF. In fact, inserting $\lambda = iY$ in (7.11), one obtains

$$(Y^4 - \mathtt{I}_2 Y^2 + \mathtt{I}_4) + i(\mathtt{I}_1 Y^2 - \mathtt{I}_3) Y = 0. \tag{7.15}$$

Obviously the consistency of (7.13)–(7.14) requires

$$\mathtt{I}_2^2 \geq 4\mathtt{I}_4, \qquad \frac{\mathtt{I}_3}{\mathtt{I}_1} > 0. \tag{7.16}$$

Since the roots of (7.12) are continuous functions of R^2, convection occurs or via a null root (stationary convection) or via an imaginary root (oscillatory convection, Hopf bifurcation). The following theorem holds.

Theorem 7.2. *If and only if*

$$\begin{cases} \min(R_{C_1}, R_{C_2}, R_{C_3}) > 0, \\ R_{C_3} < \dfrac{R_{C_2}}{\mathtt{I}_1^2}(|\mathtt{I}_1|R_{C_1} - R_{C_2}), \end{cases} \tag{7.17}$$

exists a critical Rayleigh number R_C^2 such that

$$R^2 < R_C^2, \tag{7.18}$$

inhibits the onset of convection and guarantees the global nonlinear asymptotic stability of the thermal conduction. Then, denoting by $\bar{R}^2$ the lowest positive root of (7.13), *it follows that*

$$R_C^2 = \min(R_{C_3}, \bar{R}^2) \tag{7.19}$$

and convection occurs:

i) *via a stationary state at $R_C^2 = R_{C_3}$;*
ii) *via an oscillatory state at $R_C^2 = \bar{R}^2$.*

Proof. In fact (7.17) are obtained requiring the validity of (7.8) at $R = 0$. Therefore—for continuity—(7.17) guarantee the existence of R_C^2 such that (7.18) implies (7.8). As concerns (7.19) and i)–ii) we underline that for $R^2 = R_{C_3}$, (7.11) reduces to

$$\lambda(\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3) = 0 \tag{7.20}$$

and admits the solution $\lambda = 0$ to which is associate a stationary state. On the other hand—in view of Lemma 7.2, to (7.13) are associated imaginary solution, hence the proof is completely reached.

Theorem 7.3. *The condition*

$$R^2 < \frac{27}{4}\pi^4 - R_2^2, \tag{7.21}$$

guarantees the global nonlinear asymptotic stability of the thermal conduction solution.

PROOF. In fact the linear evolution system associated to (7.1) is

$$\frac{\partial}{\partial t}\begin{pmatrix} w_n \\ \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix} = \mathscr{L}_n^{(2)} \begin{pmatrix} w_n \\ \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix}. \tag{7.22}$$

Setting

$$\begin{cases} \theta_n = \mu_n \theta_n^*, \quad \Phi_{\alpha n} = \mu_{\alpha n}\Phi_{\alpha n}^*, \\ \mu_n = \dfrac{1}{\sqrt{P_r \eta_n}}, \quad \mu_{\alpha n} = \dfrac{1}{\sqrt{P_\alpha P_r \eta_n}}, \end{cases} \tag{7.23}$$

(7.22)—omitting the stars—becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} w_n \\ \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix} = L_n^{(2)} \begin{pmatrix} w_n \\ \theta_n \\ \Phi_{1n} \\ \Phi_{2n} \end{pmatrix}, \tag{7.24}$$

with

$$L_n^{(2)} = \begin{pmatrix} -P_r\xi_n & R\sqrt{P_r\eta_n} & -R_1\sqrt{\dfrac{P_r\eta_n}{P_1}} & -R_2\sqrt{\dfrac{P_r\eta_n}{P_2}} \\ R\sqrt{P_r\eta_n} & -\xi_n & 0 & 0 \\ R_1\sqrt{\dfrac{P_r\eta_n}{P_1}} & 0 & -\dfrac{\xi_n}{P_1} & 0 \\ -R_2\sqrt{\dfrac{P_r\eta_n}{P_2}} & 0 & 0 & -\dfrac{\xi_n}{P_2} \end{pmatrix}. \tag{7.25}$$

In view of (7.24)–(7.25), one obtains

$$\frac{1}{2}\frac{d}{dt}(\|w_n\|^2 + \|\theta_n\|^2 + \|\Phi_{1n}\|^2 + \|\Phi_{2n}\|^2) = -\frac{\xi_n}{P_1}\|\Phi_{1n}\|^2 + \int_\Omega Q_n\, d\Omega, \tag{7.26}$$

where Q_n is the quadratic form in $(w_n, \theta_n, \Phi_{2n})$ given by

$$Q_n = -\xi_n\left(P_r w_n^2 + \theta_n^2 + \frac{1}{P_2}\Phi_{2n}^2\right) + 2R\sqrt{P_r\eta_n}w_n\theta_n - 2R_2\sqrt{\frac{P_r\eta_n}{P_2}}w_n\Phi_{2n}, \tag{7.27}$$

which is negative definite when (7.21) holds.

REMARK 7.2. We remark that (7.21) does not contain any contribution of the stabilizing effect of the salt salting L from below.

8. Ternary diffusion-convection in a layer heated from below and salted from above

THEOREM 8.1. *In the case* $(m = 2, H_1 = H_2 = -1)$, *if and only if*

$$R^2 < R_C^2 = \frac{27}{4}\pi^4 - R_1^2 - R_2^2, \tag{8.1}$$

it follows that

i) *convection cannot occur;*
ii) *the thermal conduction solution is nonlinearly globally asymptotically stable;*
iii) *convection arises via a stationary state at* $R^2 = R_C^2$.

PROOF. The linear evolution system is given by (7.22) with

$$\mathscr{L}_n^{(2)} = \begin{pmatrix} -P_r\xi_n & P_r\eta_n R & -P_r\eta_n R_1 & -P_r\eta_n R_2 \\ R & -\xi_n & 0 & 0 \\ -\frac{R_1}{P_1} & 0 & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2}{P_2} & 0 & 0 & -\frac{\xi_n}{P_2} \end{pmatrix} \tag{8.2}$$

and, via (7.23), is given by (7.24) with $L_n^{(2)}$ given by the *symmetric operator*

$$L_n^{(2)} = \begin{pmatrix} -P_r\xi_n & R\sqrt{P_r\eta_n} & -R_1\sqrt{\frac{P_r\eta_n}{P_1}} & -R_2\sqrt{\frac{P_r\eta_n}{P_2}} \\ R\sqrt{P_r\eta_n} & -\xi_n & 0 & 0 \\ -R_1\sqrt{\frac{P_r\eta_n}{P_1}} & 0 & -\frac{\xi_n}{P_1} & 0 \\ -R_2\sqrt{\frac{P_r\eta_n}{P_2}} & 0 & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}. \tag{8.3}$$

Since the eigenvalues of $L_n^{(2)}$ are real, convection occurs via a stationary state, i.e., in view of (7.2), for

$$\mathtt{I}_{n4} = \det L_n^{(2)} = 0. \tag{8.4}$$

One easily obtains

$$\mathtt{I}_{n4} = \frac{P_r}{P_1P_2}\xi_n^2\eta_n\Big(- R_1^2 - R_2^2 + \frac{\xi_n^2}{\eta_n}\Big) \tag{8.5}$$

and (8.1) immediately follows.

9. Ternary diffusion-convection in a layer heated and salted from below

In this case $(m = 2, H_1 = H_2 = 1)$, $\mathscr{L}_n^{(2)}$ is given by (7.1) with R_2 at the place of $-R_2$. Further $\mathtt{I}_{n\alpha}$, $(\alpha = 1, 2, 3, 4)$, is given by $\mathtt{I}_{n\alpha}$ of (7.4) with R_2^2 at the place of $(-R_2^2)$ and R_{C_α}, $(\alpha = 1, 2, 3)$, is given analogously by R_{C_α} of (7.5) with R_2^2 at the place of $(-R_2^2)$. Therefore, taking into account these substitutions, the stability conditions becomes

$$R^2 < \frac{27}{4}\pi^4. \tag{9.1}$$

Remark 9.1. We remark that (9.1) does not contain any stabilizing effect of the two salts salting L from below.

10. Difficulties of handling the stability conditions for large number of salts dissolved in

Relevant can become the difficulties of handling (6.6) for large m. As concerns (6.6) and the analogous of (7.19) (when L is salted from below by $S_1, S_2, \ldots, S_r$ $(r \leq m)$ and from above by $S_{r+1}, \ldots, S_m$), the following general remarks can be done:

1) *one verifies that the necessary stability condition*

$$(-1)^{m+2}\mathtt{I}_{n(m+2)} > 0, \quad \forall(a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \tag{10.1}$$

is equivalent to

$$R^2 < \sum_{\alpha=1}^{r} R_\alpha^2 - \sum_{\alpha=r+1}^{m} R_\alpha^2 + \frac{\xi_n^2}{\eta_n}, \quad \forall(a^2, n) \in \mathbb{R}^+ \times \mathbb{N} \tag{10.2}$$

and gives, for $\left(a^2 = \frac{\pi^2}{2}, n = 1\right)$, *the necessary stability condition*

$$R^2 < R_{c(m+1)} = \sum_{\alpha=1}^{r} R_\alpha^2 - \sum_{\alpha=r+1}^{m} R_\alpha^2 + \frac{27}{4}\pi^4; \tag{10.3}$$

2) *for the stability of the thermal conduction solution it is necessary and sufficient to satisfy* (6.6) *only for* $\left(n = 1, a^2 = \frac{\pi^2}{2}\right)$;

3) (7.19)—*with* $R_{c(m+1)}$ *at the place of* R_{C_3} *and* $\overline{R}^2$ *lowest value of* R^2 *for which* (6.2) *admits a pure imaginary root, continues to give the global non linear asymptotic stability condition and also the properties* i)–ii) *of Section* 7 *continue to hold;*
4) *in the case* $(m = 3, r = 2)$, (10.3) *gives*

$$R^2 = R_{C_4} = R_1^2 + R_2^2 - R_3^2 + \frac{27}{4}\pi^4, \tag{10.4}$$

(6.2) *reduces* $\left(\textit{for } n = 1, a^2 = \frac{\pi^2}{2}\right)$ *to*

$$\lambda^5 - \mathtt{I}_1\lambda^4 + \mathtt{I}_2\lambda^3 - \mathtt{I}_3\lambda^2 + \mathtt{I}_4\lambda - \mathtt{I}_5 = 0, \tag{10.5}$$

with

$$\mathtt{I}_5 = R^2 - R_{C_4} \tag{10.6}$$

and, the looking for imaginary roots, is equivalent to solve the system

$$\begin{cases} Y(Y^4 - \mathtt{I}_2 Y^2 + \mathtt{I}_4) = 0, \\ \mathtt{I}_1 Y^4 - \mathtt{I}_3 Y^2 + \mathtt{I}_5 = 0. \end{cases} \tag{10.7}$$

Since $(Y = 0, R^2 = R_{C_4})$ *is a solution of* (10.7), *when* R_{C_4} *is the lowest positive root of* (10.7) *then* $R^2 < R_{C_4}$ *is necessary and sufficient for the global stability and convection occurs at* $R^2 = R_{C_4}$ *via a stationary state. This happens, for instance, when*

$$\mathtt{I}_2^2 < 4\mathtt{I}_4, \quad \mathtt{I}_3^2 < 4\mathtt{I}_1\mathtt{I}_5. \tag{10.8}$$

11. Global nonlinear asymptotic stability via hidden symmetries and skew-symmetries

In view of the difficulties remarked previously, it appears of notable interest to guarantee the global nonlinear asymptotic stability of the thermal conduction solution via simple algebraic conditions, in closed form. To this scope we introduce—at the place of θ_n, $\Phi_{\alpha n}$—new fields. In fact, the diffusion-convection in L has symmetries and skew-symmetries which are hidden in (5.9) and can be seen only introducing—at the place of θ_n, $\Phi_{\alpha n}$—new fields. Let

$$H_\alpha = \begin{cases} 1, & \alpha = 1, 2, \ldots, r, \\ -1, & \alpha = r + 1, \ldots, m. \end{cases} \tag{11.1}$$

Then (5.9) becomes

$$(11.2)\qquad \begin{cases} \dfrac{\partial w_n}{\partial t} = -\xi_n P_r w_n + \eta_n P_r\Big(R\theta_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \Phi_{\alpha n}\Big), \\ \dfrac{\partial \theta_n}{\partial t} = R w_n - \xi_n \theta_n, \\ \dfrac{\partial \Phi_{\alpha n}}{\partial t} = \begin{cases} \dfrac{R_\alpha}{P_\alpha} w_n - \dfrac{\xi_n}{P_\alpha}\Phi_{\alpha n}, & \alpha = 1, 2, \ldots, r, \\ \dfrac{-R_\alpha}{P_\alpha} w_n - \dfrac{\xi_n}{P_\alpha}\Phi_{\alpha n}, & \alpha = r+1, \ldots m. \end{cases} \end{cases}$$

Setting

$$(11.3)\qquad \varphi_{\alpha n} = \begin{cases} R_\alpha \theta_n - P_\alpha R \Phi_{\alpha n}, & \alpha = 1, \ldots, r, \\ R_\alpha \theta_n + P_\alpha R \Phi_{\alpha n}, & \alpha = r+1, \ldots, m, \end{cases}$$

(11.2) imply

$$(11.4)\qquad \frac{\partial \varphi_{\alpha n}}{\partial t} = \begin{cases} -\xi_n R_\alpha \theta_n + R\xi_n \Phi_{\alpha n}, & \alpha = 1, 2, \ldots, r, \\ -\xi_n R_\alpha \theta_n - R\xi_n \Phi_{\alpha n}, & \alpha = r+1, \ldots, m \end{cases}$$

and, by virtue of (11.3), one obtains

$$(11.5)\qquad P_\alpha \frac{\partial \varphi_{\alpha n}}{\partial t} = -\xi_n \varphi_{\alpha n} + (1 - P_\alpha) R_\alpha \xi_n \theta_n, \quad \alpha = 1, 2, \ldots, m.$$

Since

$$(11.6)\qquad \begin{aligned} R\theta_n - \sum_{\alpha=1}^{m} R_\alpha \Phi_{\alpha n} &= R\theta_n + \sum_{\alpha=1}^{r} (\varphi_{\alpha n} - R_\alpha \theta_n) \frac{R_\alpha}{P_\alpha R} \\ &\quad - \sum_{\alpha=r+1}^{m} \frac{R_\alpha}{P_\alpha R} (\varphi_{\alpha n} - R_\alpha \theta_n) \\ &= \frac{1}{R}\Bigg[\Big(R^2 - \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} + \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha}\Big)\theta_n \\ &\quad + \sum_{\alpha=1}^{r} \frac{R_\alpha}{P_\alpha}\varphi_{\alpha n} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha}{P_\alpha}\varphi_{\alpha n}\Bigg], \end{aligned}$$

(11.2) become

$$(11.7)\qquad \begin{cases} \dfrac{\partial w_n}{\partial t} = -\xi_n P_r w_n + \dfrac{P_r \eta_n}{R}\Big(R^* \theta_n + \displaystyle\sum_{\alpha=1}^{r} \frac{R_\alpha}{P_\alpha}\varphi_{\alpha n} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha}{P_\alpha}\varphi_{\alpha n}\Big), \\ \dfrac{\partial \theta_n}{\partial t} = R w_n - \xi_n \theta_n, \\ \dfrac{\partial \varphi_{\alpha n}}{\partial t} = -\dfrac{\xi_n}{P_\alpha}\varphi_{\alpha n} - \dfrac{\xi_n R_\alpha}{P_\alpha}(P_\alpha - 1)\theta_n, \quad \alpha = 1, 2, \ldots, m, \end{cases}$$

under the boundary conditions

$$w_n = \theta_n = \varphi_{\alpha n} = 0, \qquad z = 0, 1, \quad \alpha = 1, 2, \ldots, m, \tag{11.8}$$

with

$$R^* = R^2 - \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} + \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha}. \tag{11.9}$$

For the sake of simplicity and concreteness, in this paper, we confine ourselves to the prototype case $(m = 2, r = 1)$. Setting

$$\varphi_n = R w_n + P_r \xi_n \theta_n, \tag{11.10}$$

it follows that

$$\frac{\partial \varphi_n}{\partial t} = P_r \eta_n \left(R^* - \frac{\xi_n^2}{\eta_n} \right) \theta_n + P_r \eta_n \frac{R_1}{P_1} \varphi_1 - P_r \eta_n \frac{R_2}{P_2} \varphi_2. \tag{11.11}$$

In view of

$$\theta_n = \frac{\varphi_n - R w_n}{P_r \xi_n} \tag{11.12}$$

and (11.7), (11.11), one obtains

$$\frac{\partial}{\partial t} \begin{pmatrix} \varphi_n \\ w_n \\ \varphi_{1n} \\ \varphi_{2n} \end{pmatrix} = L_n \begin{pmatrix} \varphi_n \\ w_n \\ \varphi_{1n} \\ \varphi_{2n} \end{pmatrix}, \tag{11.13}$$

with

$$L_n = \begin{pmatrix} \frac{\eta_n}{\xi_n} \left(R^* - \frac{\xi_n^2}{\eta_n} \right) & -\frac{R\eta_n}{\xi_n} \left(R^* - \frac{\xi_n^2}{\eta_n} \right) & P_r \eta_n \frac{R_1}{P_1} & -P_r \eta_n \frac{R_2}{P_2} \\ \frac{\eta_n}{\xi_n} \frac{R^*}{R} & -\left(P_r \frac{\xi_n^2}{\eta_n} + R^* \right) \frac{\eta_n}{\xi_n} & P_r \eta_n \frac{R_1}{P_1} & -P_r \eta_n \frac{R_2}{P_2} \\ \frac{(1 - P_1)}{P_1} \frac{R_1}{P_r} & -\frac{(1 - P_1)}{P_1} \frac{R_1 R}{P_r} & -\frac{\xi_n}{P_1} & 0 \\ \frac{(1 - P_2)}{P_2} \frac{R_2}{P_r} & -\frac{(1 - P_2)}{P_2} \frac{R_2 R}{P_r} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix}. \tag{11.14}$$

REMARK 11.1. Setting

$$A^* = \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2}, \qquad \gamma = \frac{27}{4} \pi^4, \tag{11.15}$$

it follows that

$$R^* = R^2 - A^* \tag{11.16}$$

and the consistency of

$$R^2 \geq 0; \qquad -P_r \frac{\xi_n^2}{\eta_n} < R^* < \frac{\xi_n^2}{\eta_n}, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \tag{11.17}$$

requires

$$A^* - P_r \frac{\xi_n^2}{\eta_n} < R^2 < A^* + \frac{\xi_n^2}{\eta_n}, \quad \forall R^2 \geq 0,\ \forall (n, a^2) \in \mathbb{N} \times \mathbb{R}^+ \tag{11.18}$$

and hence

$$-\gamma < A^* < P_r \gamma, \qquad R^2 < A^* + \gamma. \tag{11.19}$$

THEOREM 11.1. *Let*

$$P_1 \leq 1, \qquad P_2 \geq 1. \tag{11.20}$$

Then for

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + \frac{P_r}{1 + P_r}\gamma, \tag{11.21}$$

convection cannot occur and the thermal conduction solution is globally nonlinearly asymptotically stable.

PROOF. We give the proof in the case $(P_1 < 1, P_2 > 1)$. In fact, by continuity, (11.21) continues to hold either for $P_1 = 1$ or $P_2 = 1$.

Setting $(\alpha = 1, 2)$

$$w_n = \frac{w_n^*}{R}\sqrt{\frac{|R^*|}{|R^* - \xi_n^2/\eta_n|}}, \qquad \varphi_{\alpha n} = \frac{\varphi_{\alpha n}^*}{P_r}\sqrt{\frac{|1 - P_\alpha|}{\eta_n}} \tag{11.22}$$

and omitting the stars, one obtains (11.13) with L_n given by

(11.23)

$$L_n = \begin{pmatrix} \frac{\eta_n}{\xi_n}\left(R^* - \frac{\xi_n^2}{\eta_n}\right) & \frac{\eta_n}{\xi_n}\sqrt{|R^*(R^* - \xi_n^2/\eta_n)|} & \frac{R_1}{P_1}\sqrt{(1-P_1)\eta_n} & -\frac{R_2}{P_2}\sqrt{(P_2-1)\eta_n} \\ \varepsilon\frac{\eta_n}{\xi_n}\sqrt{|R^*(R^* - \xi_n/\eta_n)|} & -\left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right)\frac{\eta_n}{\xi_n} & \frac{R_1}{P_1}\sqrt{(1-P_1)\eta_n} & -\frac{R_2}{P_2}\sqrt{(P_2-1)\eta_n} \\ \frac{R_1}{P_1}\sqrt{(1-P_1)\eta_n} & -\frac{R_1}{P_1}\sqrt{(1-P_1)\eta_n} & -\frac{\xi_n}{P_1} & 0 \\ -\frac{R_2}{P_2}\sqrt{(P_2-1)\eta_n} & \frac{R_2}{P_2}\sqrt{(P_2-1)\eta_n} & 0 & -\frac{\xi_n}{P_2} \end{pmatrix},$$

with

$$\varepsilon = 1, \quad \text{for } R^* > 0, \qquad \varepsilon = -1, \quad \text{for } R^* < 0. \tag{11.24}$$

Introducing the quadratic form

$$Q_n = \begin{cases} \left(R^* - \dfrac{\xi_n^2}{\eta_n}\right)\varphi_n^2 - \left(P_r\dfrac{\xi_n^2}{\eta_n} + R^*\right)w_n^2 - \dfrac{\xi_n^2}{\eta_n}\displaystyle\sum_{\alpha=1}^{2}\dfrac{\varphi_{\alpha n}^2}{P_\alpha} \\ \quad + (1+\varepsilon)\varphi_n w_n\sqrt{|R^*(R^* - \xi_n^2/\eta_n)|} \\ \quad + \dfrac{2R_1}{P_1}\xi_n\sqrt{\dfrac{(1-P_1)}{\eta_n}}\varphi_n\varphi_{1n} - 2\dfrac{R_2}{P_2}\xi_n\sqrt{\dfrac{(P_2-1)}{\eta_n}}\varphi_n\varphi_{2n}, \end{cases} \tag{11.25}$$

one obtains

$$\frac{1}{2}\frac{d}{dt}\left(\|\varphi_n\|^2 + \|w_n\|^2 + \sum_{\alpha=1}^{2}\|\varphi_{\alpha n}\|^2\right) = \frac{\eta_n}{\xi_n}\int_\Omega Q_n\, d\Omega. \tag{11.26}$$

In view of

$$\begin{cases} 2\dfrac{R_1}{P_1}\sqrt{(1-P_1)\dfrac{\xi_n^2}{\eta_n}}|\varphi_n\varphi_{1n}| \le \dfrac{R_1^2}{P_1}\left(1-P_1\right)\varphi_n^2 + \dfrac{1}{P_1}\dfrac{\xi_n^2}{\eta_n}\varphi_{1n}^2, \\ 2\dfrac{R_2}{P_2}\sqrt{(P_2-1)\dfrac{\xi_n^2}{\eta_n}}|\varphi_n\varphi_{2n}| \le \dfrac{R_2^2}{P_2}\left(P_2-1\right)\varphi_n^2 + \dfrac{1}{P_2}\dfrac{\xi_n^2}{\eta_n}\varphi_{2n}^2, \end{cases} \tag{11.27}$$

for $R^* < 0 \Leftrightarrow \varepsilon = -1$ one obtains

$$Q_n < \left[R^* + \frac{R_1^2}{P_1}\left(1-P_1\right) + \frac{R_2^2}{P_2}(P_2-1) - \frac{\xi_n^2}{\eta_n}\right]\varphi_n^2 \tag{11.28}$$

and—in view of (11.9)—it follows that

$$Q_n < \left[R^2 - \left(R_1^2 - R_2^2 + \frac{\xi_n^2}{\eta_n}\right)\right]\varphi_n^2. \tag{11.29}$$

Therefore (11.21) guarantees $\{Q_n < 0, \forall(n, a^2) \in \mathbb{N} \times \mathbb{R}^+\}$ and hence the validity of the theorem for $\{A^* > 0, R^2 < A^*\}$. In the case $R^* > 0 \Leftrightarrow \varepsilon = 1$, by virtue of (11.27) one obtains

$$(11.30)\qquad Q_n \le \tilde{Q}_n = -\left[\frac{\xi_n^2}{\eta_n} - (R^* + A)\right]\varphi_n^2 + 2\sqrt{R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right)}\varphi_n w_n$$
$$- \left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right)w_n^2,$$

with

$$(11.31)\qquad A = \frac{R_1^2}{P_1}(1 - P_1) + \frac{R_2^2}{P_2}(P_2 - 1) \ge 0.$$

$\tilde{Q}_n$ is negative definite if and only if

$$(11.32)\qquad R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right) < \left[\frac{\xi_n^2}{\eta_n} - (R^* + A)\right]\left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right),$$

which is equivalent to

$$(11.33)\qquad \left(A + P_r\frac{\xi_n^2}{\eta_n}\right)R^* < \left(\frac{\xi_n^2}{\eta_n} - A\right)P_r\frac{\xi_n^2}{\eta_n}$$

i.e. to

$$(11.34)\qquad R^* < \frac{(A + P_r\xi^2_/\eta_n) - A(1 + P_r)}{A + P_r\xi_n^2/\eta_n}\,\frac{\xi_n^2}{\eta_n}$$

and hence to

$$(11.35)\qquad R^* < \frac{\xi_n^2}{\eta_n} - \frac{A(1 + P_r)\xi_n^2/\eta_n}{A + P_r\xi_n^2/\eta_n}.$$

Since (11.35) has to hold $\forall(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$, one obtains

$$(11.36)\qquad R^* < \min_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+}\frac{\xi_n^2}{\eta_n} - \max_{(n,a^2)\in\mathbb{N}\times\mathbb{R}^+}\frac{A(1 + P_r)}{A + P_r\xi_n^2/\eta_n}\,\frac{\xi_n^2}{\eta_n}$$

and hence

$$(11.37)\qquad R^* + \frac{1 + P_r}{P_r}A < \gamma.$$

We end by remarking that (11.37) is implied by (11.21).

REMARK 11.2. We remark that by virtue of (11.29), (11.20) together with

$$(11.38)\qquad \gamma P_r > A^* = \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} > \frac{R_1^2}{P_1} - R_2^2 + \frac{\gamma P_r}{1 + P_r} > 0,$$

imply that

$$R^2 < \min(A^*, R_{C_3} = R_1^2 - R_2^2 + \gamma), \tag{11.39}$$

guarantees:

i) the global nonlinear asymptotic stability of the thermal conduction solution;
ii) for $R_{C_3} \le A$ the onset of convection at $R^2 = R_{C_3}$, via a stationary state.

THEOREM 11.2. *Let*

$$P_1 \le 1, \qquad P_2 \le 1, \qquad \frac{R_1^2}{P_1} - R_2^2 < P_r\gamma. \tag{11.40}$$

Then for

$$R^2 < R_1^2 - \frac{R_2^2}{P_2} + \frac{P_r}{1+P_r}\gamma, \tag{11.41}$$

convection cannot occur and the thermal conduction solution is globally non linearly asymptotically stable.

PROOF. It is easily verified that (11.26) continues to hold with

$$Q_n = \begin{cases} \left(R^* - \frac{\xi_n^2}{\eta_n}\right)\varphi_n^2 + (1+\varepsilon)\sqrt{\left|R^*\left(R^* - \frac{\xi_n^2}{\eta_n}\right)\right|}\, w_n\varphi_n \\ -\left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right)w_n^2 + -\frac{\xi_n^2}{\eta_n}\sum_{\alpha=1}^{2}\frac{\varphi_{\alpha n}^2}{P_\alpha} + \frac{2R_1}{P_1}\left(\sqrt{\frac{1-P_1}{\eta_n}}\right)\xi_n\varphi_n\varphi_{1n} \\ -\frac{2R_2}{P_2}\left(\sqrt{\frac{1-P_2}{\eta_n}}\right)\xi_n w_n\varphi_{2n}. \end{cases} \tag{11.42}$$

For $\varepsilon = -1$, it follows that

$$\begin{aligned} Q_n &\le \left[R^* - \frac{\xi_n^2}{\eta_n} + \frac{R_1^2}{P_1}(1-P_1)\right]\varphi_n^2 + \tilde{Q}_n, \\ \tilde{Q}_n &= -\left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right)w_n^2 - 2\frac{R_2}{P_2}\left(\sqrt{\frac{1-P_2}{\eta_n}}\right)\xi_n w_n\varphi_{2n} - \frac{\xi_n^2}{\eta_n}\frac{\varphi_{2n}^2}{P_2}. \end{aligned} \tag{11.43}$$

Hence Q_n is negative definite if

$$R^* - \frac{\xi_n^2}{\eta_n} + \frac{R_1^2}{P_1}(1-P_1) < 0 \qquad \Leftrightarrow \qquad R^2 < R_1^2 - \frac{R_2^2}{P_2} + \frac{\xi_n^2}{\eta_n}, \tag{11.44}$$
$$\forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N},$$

together with the negative definiteness of $\tilde{Q}_n$ i.e. if

$$\frac{R_2^2}{P_2^2}(1-P_2)\frac{\xi_n^2}{\eta_n} < \frac{1}{P_2}\left(P_r\frac{\xi_n^2}{\eta_n} + R^*\right)\frac{\xi_n^2}{\eta_n}. \tag{11.45}$$

Since (11.44) is implied by (11.41) and (11.45) is implied by $(11.40)_3$, the theorem is proved in the case $\varepsilon = -1$. In the case $\varepsilon = 1$, (11.42) implies

$$Q_n < \tilde{Q}_n = -\left[\frac{\xi_n^2}{\eta_n} - (R^* + A_1)\right]\varphi_n^2 + 2\sqrt{R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right)}\varphi_n w_n + \\ -\left(P_r\frac{\xi_n^2}{\eta_n} + R^* + A_2\right)w_n^2, \tag{11.46}$$

with

$$A_1 = \frac{R_1^2}{P_1}(1-P_1), \qquad A_2 = \frac{R_2^2}{P_2}(P_2 - 1). \tag{11.47}$$

Since $\frac{\xi_n^2}{\eta_n} - (R^* + A_1) > 0$ and $P_r\frac{\xi_n^2}{\eta_n} + R^* + A_2 > 0$ are implied respectively by (11.41) and $(11.40)_2$, $\tilde{Q}_n$ is negative definite if and only if

$$R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right) < \left(\frac{\xi_n^2}{\eta_n} - R^* - A_1\right)\left(P_r\frac{\xi_n^2}{\eta_n} + R^* + A_2\right), \tag{11.48}$$

i.e. if and only if

$$R^*\frac{\xi_n^2}{\eta_n} < \left(\frac{\xi_n^2}{\eta_n} - A_1\right)\left(P_r\frac{\xi_n^2}{\eta_n} + R^* + A_2\right) - R^*\left(P_r\frac{\xi_n^2}{\eta_n} + A_2\right) \tag{11.49}$$

and hence by

$$R^* < \frac{(\xi_n^2/\eta_n - A_1)(P_r\xi_n^2/\eta_n + A_2)}{A + P_r\xi_n^2/\eta_n} = \frac{\xi_n^2}{\eta_n} - A_1 - \frac{A_1(\xi_n^2/\eta_n - A_1)}{A + P_r\xi_n^2/\eta_n}, \tag{11.50}$$

which is equivalent to

$$R^* + A_1 < \frac{\xi_n^2}{\eta_n} - \frac{A_1}{A + P_r\xi_n^2/\eta_n}\frac{\xi_n^2}{\eta_n} + \frac{A_1^2}{A + P_r\xi_n^2/\eta_n}, \tag{11.51}$$

implied by

$$R^* + A_1 < \frac{\xi_n^2}{\eta_n} - \frac{A_1}{A + P_r\xi_n^2/\eta_n}\frac{\xi_n^2}{\eta_n}. \tag{11.52}$$

Requiring the consistency of (11.52) $\forall(n, a^2) \in \mathbb{N} \times \mathbb{R}^+$ one has

$$R^* + A_1 < \gamma - \frac{A_1}{P_r} \qquad \Leftrightarrow \qquad R^* + \frac{(1+P_r)A_1}{P_r} < \gamma, \tag{11.53}$$

which is implied by

$$\frac{1+P_r}{P_r}(R^* + A_1) < \gamma \tag{11.54}$$

and hence by (11.41).

REMARK 11.3. We remark that, when (11.40) hold together with

$$A^* > R_1^2 - \frac{R_2^2}{P_2} + \frac{P_r}{1+P_r}\gamma > 0, \tag{11.55}$$

then, by virtue of (11.44), the stability condition is given by

$$R^2 < \min\left(A^*, R_1^2 - \frac{R_2^2}{P_2} + \gamma\right). \tag{11.56}$$

THEOREM 11.3. *Let*

$$P_1 \geq 1, \quad P_2 \leq 1, \quad R_1^2 + R_2^2 < P_r \gamma. \tag{11.57}$$

Then convection cannot occur for

$$R^2 < \frac{R_1^2}{P_1} - \frac{R_2^2}{P_2} + \gamma \tag{11.58}$$

and (11.58) *guarantees the global nonlinear asymptotic stability of the thermal conduction solution.*

PROOF. In the case (11.57), Q_n is given by

$$Q_n = \begin{cases} \left(R^* - \dfrac{\xi_n^2}{\eta_n}\right)\varphi_n^2 + (1+\varepsilon)\sqrt{\left|R^*\left(R^* - \dfrac{\xi_n^2}{\eta_n}\right)\right|}\, w_n \varphi_n \\ -\left(P_r \dfrac{\xi_n^2}{\eta_n} + R^*\right) w_n^2 - \dfrac{\xi_n^2}{\eta_n} \sum \dfrac{\varphi_{\alpha n}^2}{P_\alpha} + 2\dfrac{R_1}{P_1}\left(\sqrt{\dfrac{P_1 - 1}{\eta_n}}\right)\xi_n w_n \varphi_{1n} \\ -2\dfrac{R_2}{P_2}\left(\sqrt{\dfrac{1-P_2}{\eta_n}}\right)\xi_n w_n \varphi_{2n}. \end{cases} \tag{11.59}$$

In the case $(A^* > 0, R^2 < A^*, \varepsilon = -1)$, one obtains

$$Q_n \le P_n = \left(R^* - \frac{\xi_n^2}{\eta_n}\right)\varphi_n^2 - \left[P_r \frac{\xi_n^2}{\eta_n} + R^* + A_1 + A_2\right] w_n^2 \tag{11.60}$$

and being

$$P_r\gamma - \frac{R_1^2}{P_1} + \frac{R_2^2}{P_2} + A_1 + A_2 > 0 \quad \Leftrightarrow \quad R_1^2 - R_2^2 < P_r\gamma, \tag{11.61}$$

P_n is negative definite by virtue of (11.57)–(11.58).

In the case $\varepsilon = 1$, one obtains

$$\begin{aligned} Q_n \le \tilde{P}_n = \left(R^* - \frac{\xi_n^2}{\eta_n}\right)\varphi_n^2 + 2\sqrt{R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right)} w_n\varphi_n \\ - \left[P_r \frac{\xi_n^2}{\eta_n} + R^* + A_1 + A_2\right] w_n^2 \end{aligned} \tag{11.62}$$

and $\tilde{P}_n$ is negative definite for

$$R^*\left(\frac{\xi_n^2}{\eta_n} - R^*\right) < \left(\frac{\xi_n^2}{\eta_n} - R^*\right)\left[P_r \frac{\xi_n^2}{\eta_n} + R^* + A_1 + A_2\right], \tag{11.63}$$

i.e. if and only if

$$R_1^2 \frac{(P_1 - 1)}{P_1} + R_2^2 \frac{(1 - P_2)}{P_2} < P_r \frac{\xi_n^2}{\eta_n}, \tag{11.64}$$

which is implied by $(11.57)_3$.

Theorem 11.4. *Let*

$$P_1 \ge 1, \qquad P_2 > 1, \qquad R_1^2 - \frac{R_2^2}{P_2} < P_r\gamma. \tag{11.65}$$

Then convection cannot occur for

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + \gamma \frac{P_r}{1 + P_r} \tag{11.66}$$

and (11.66) *guarantees the global nonlinear asymptotic stability of the thermal conduction solution.*

Proof. In the case at stake for $\varepsilon = -1$, one obtains that (11.27) holds with

$$(11.67)\quad Q_n = \begin{cases} \Big(R^* - \dfrac{\xi_n^2}{\eta_n}\Big)\varphi_n^2 - \Big(P_r\dfrac{\xi_n^2}{\eta_n} + R^*\Big)w_n^2 - \dfrac{\xi_n^2}{\eta_n}\displaystyle\sum_{\alpha=1}^{2}\dfrac{\varphi_{\alpha n}^2}{P_\alpha} \\ -2\dfrac{R_2}{P_2}\xi_n\Big(\sqrt{\dfrac{P_2-1}{\eta_n}}\Big)w_n\varphi_{2n} + 2\dfrac{R_1}{P_1}\xi_n\Big(\sqrt{\dfrac{P_1-1}{\eta_n}}\Big)w_n\varphi_{1n}, \end{cases}$$

which implies

$$(11.68)\quad \begin{cases} Q_n \le \left[R^* - \dfrac{\xi_n^2}{\eta_n} + \dfrac{R_2^2}{P_2}\Big(P_2 - 1\Big)\right]\varphi_n^2 + Q_n^*, \\ Q_n^* = -\Big(P_r\dfrac{\xi_n^2}{\eta_n} + R^*\Big)w_n^2 + 2\dfrac{R_1}{P_1}\xi_n\Big(\sqrt{\dfrac{P_1-1}{\eta_n}}\Big)w_n\varphi_{1n} - \dfrac{\xi_n^2}{\eta_n}\dfrac{\varphi_{1n}^2}{P_1}. \end{cases}$$

Hence Q_n is negative definite if

$$(11.69)\quad R^* - \frac{\xi_n^2}{\eta_n} + \frac{R_2^2}{P_2}\Big(P_2 - 1\Big) < 0 \qquad \Leftrightarrow \qquad R^2 < \frac{R_1^2}{P_1} - R_2^2 + \frac{\xi_n^2}{\eta_n},$$
$$\forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N},$$

together with the negative definiteness of Q_n^* i.e.

$$(11.70)\quad \frac{R_1^2}{P_1^2}(P_1 - 1)\frac{\xi_n^2}{\eta_n} < \frac{1}{P_1}\frac{\xi_n^2}{\eta_n}\Big(P_r\frac{\xi_n^2}{\eta_n} + R^*\Big).$$

Obviously (11.69) is implied by (11.66) and (11.70) is implied by $(11.65)_3$. Passing to the case $\varepsilon = 1$, one has

$$(11.71)\quad Q_n < \tilde{Q}_n = -\left[\frac{\xi_n^2}{\eta_n} - \Big(R^* + A_2\Big)\right]\varphi_n^2 + 2\sqrt{R^*\Big(\frac{\xi_n^2}{\eta_n} - R^*\Big)}\varphi_n w_n$$
$$- \Big(P_r\frac{\xi_n^2}{\eta_n} + R^* + A_1\Big)w_n^2.$$

Since $\dfrac{\xi_n^2}{\eta_n} - (R^* + A_2) > 0$ and $P_r\dfrac{\xi_n^2}{\eta_n} + R^* + A_1 > 0$ are implied respectively by (11.66) and $(11.65)_3$, $\tilde{Q}_n$ is negative definite if and only if

$$(11.72)\quad R^*\Big(\frac{\xi_n^2}{\eta_n} - R^*\Big) < \Big(\frac{\xi_n^2}{\eta_n} - R^* - A_2\Big)\Big(P_r\frac{\xi_n^2}{\eta_n} + R^* + A_1\Big).$$

Since (11.72), via the substitution

$$(11.73)\quad \begin{pmatrix} A_2 & A_1 \\ A_1 & A_2 \end{pmatrix}$$

can be obtained from (11.48), then following step by the step the procedure of theorem 11.2 in the case $\varepsilon = 1$, one obtains

$$\frac{1 + P_r}{P_r}(R^* + A_2) < \gamma, \tag{11.74}$$

which is implied by (11.66).

Remark 11.4. We remark that, when (11.65) hold together with

$$A^* > \frac{R_1^2}{P_1} - R_2^2 + \frac{\gamma P_r}{1 + P_r}, \tag{11.75}$$

then, by virtue of (11.69)–(11.70), the stability condition is given by

$$R^2 < \frac{R_1^2}{P_1} - R_2^2 + \gamma. \tag{11.76}$$

12. Final remarks

The paper is concerned with mass and heat transfer by convection in horizontal layers filled by Navier-Stokes fluid mixtures with any number of chemicals (salts) dissolved in. It is shown that:

i) for each Fourier component of the perturbations to the thermal conduction solution, there exists an own nonlinear admissible system of equations named auxiliary system (Section 3);
ii) subcritical instabilities do not exist and the global nonlinear asymptotic L^2-stability is guaranteed by the condition of linear stability (Sections 4–5);
iii) via the Routh-Hurwitz conditions applied to the spectral equation governing the eigenvalues of the linearized associated problem, rigorous stability conditions are characterized for any number of salts (Sections 5–10);
iv) the symmetries and skew-symmetries hidden in the ordinary Navier-Stokes equations governing the fluid mixtures, are put in evidence by substituting the temperature and salts concentrations via new suitable unknown fields (Section 11);
v) in the case of layers heated from below and salted from above and below by only one salt, via hidden symmetries and skew-symmetries, the global nonlinear asymptotic stability is guaranteed by simple algebraic conditions in closed form which appear to be useful not only for theoreticians but also for experimentalists in the research fields of physics of fluid (Section 11);
vi) the Auxiliary System Method, introduced in [32]–[34] for the Darcy-Boussinesq fluid mixtures, continues to hold also for the Navier-Stokes fluid mixtures[1].

[1] Other applications of the Auxiliary System Method can be found in [39]–[43].

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