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Partial Differential Equations — *Estimates for p-Laplace type equation in a limiting case*, by Fernando Farroni, Luigi Greco and Gioconda Moscariello, communicated on 26 June 2014.

Abstract. — We study the Dirichlet problem for a p-Laplacian type operator in the setting of the Orlicz–Zygmund space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N)$, $q > 1$ and $\alpha > 0$. More precisely, our aim is to establish under which assuptions on $\alpha > 0$ existence and uniqueness of the solution are assured.

Key words: Dirichlet problem, p-Laplace operators, Orlicz–Sobolev spaces.

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1. Introduction

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$. We consider the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field satisfying the following conditions for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$

$$\mathcal{A}(x, 0) = 0 \tag{1.2}$$

$$\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \geq a|\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} \tag{1.3}$$

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq b|\xi - \eta|(|\xi| + |\eta|)^{p-2} \tag{1.4}$$

where $p > 1$, $0 < a \leq b$.

Let $f = (f^1, f^2, \dots, f^N)$ be a vector field of class $\mathscr{L}^s(\Omega, \mathbb{R}^N)$, $1 \leq s \leq q$ where q is the conjugate exponent to p, i.e. $pq = p + q$.

Definition 1.1. A function $u \in W_0^{1,r}(\Omega)$, $\max\{1, p-1\} \leq r \leq p$, is a solution of (1.1) if

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla \varphi \rangle \, dx = \int_\Omega \langle f, \nabla \varphi \rangle \, dx, \tag{1.5}$$

for every $\varphi \in C_0^\infty(\Omega)$.

By a routine argument, if $s \geq r/(p-1)$, it can be seen that the identity (1.5) still holds for functions $\varphi \in \mathscr{W}^{1,\frac{r}{r-p+1}}(\Omega)$ with compact support. We shall refer to a solution in the sense of Definition 1.1 as a distributional solution or (as some people say) as a very weak solution [15, 17].

We point out that, if $r < p$, such a solution may have infinite energy, i.e. $|\nabla u| \notin \mathscr{L}^p(\Omega)$. The existence of a solution $u \in \mathscr{W}_0^{1,1}(\Omega)$ to problem (1.1) is obtained in [5] when $\operatorname{div} f$ belongs to $\mathscr{L}^1(\Omega)$. It is well known that the uniqueness of solutions to (1.1) in the sense of Definition 1.1 generally fails [20, 1]. At the present time the problem remains unclear, unless for $p = 2$ [4, 11]. In this case the range of exponents r allowing for a comprehensive theory is known, see [2, 16]. In the general case, uniqueness is proved in the setting of the grand Sobolev space (see [12] and also [10]). See also [8] for the case $p = N$.

We present existence and uniqueness results for problem (1.1) assuming that the datum f lies in the Orlicz–Zygmund space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N)$, $\alpha > 0$. More precisely, we establish under which assuptions on the parameter $\alpha > 0$ we can define a continuous operator

$$\mathcal{H} : \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N) \to \mathscr{L}^p \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N) \tag{1.6}$$

which carries a given vector field f into the gradient field ∇u. For embedding theorems for functions with gradient in Zygmund spaces, see [13].

In the case $\alpha \leq 0$, in the literature there are several results on the continuity of the operator defined in (1.6) [18, 6, 14]. Moreover, as a consequence of the results in [11] and [4] (see also [7]) and the interpolation theorem of [3], when $p = 2$ the operator $\mathcal{H}$ is Lipschitz continuous for any $-\infty < \alpha < \infty$. Actually, for $p = 2$ and suitable $\alpha > 0$, the existence for problem (1.1) is also ensured for not uniformly elliptic equations [19].

Here we consider the case p different from 2. Our main results are the following.

THEOREM 1.1. *Let $1 < p < \infty$, $p \neq 2$. For each $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N)$, with $pq = p + q$ and $0 < \alpha \leq \frac{p}{|p-2|}$, the problem (1.1) admits a unique solution $u : \Omega \to \mathbb{R}$, such that $\nabla u \in \mathscr{L}^p \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N)$. There exists a constant $C > 0$, $C = C(N, p, \alpha, a, b)$, such that the following estimate holds true*

$$\|\nabla u\|^p_{\mathscr{L}^p \log^{-\alpha} \mathscr{L}} \leq C \|f\|^q_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}} \tag{1.7}$$

Moreover the operator $\mathcal{H}$ is continuous.

THEOREM 1.2. *Let $1 < p < \infty$, $p \neq 2$. There exists a constant $C > 0$, $C = C(N, p, \alpha, a, b)$, such that, if f and g belong to $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega, \mathbb{R}^N)$, with $pq = p + q$ and $0 < \alpha < \frac{p}{|p-2|}$, then*

$$\|\mathcal{H}f - \mathcal{H}g\|^p_{\mathscr{L}^p \log^{-\alpha} \mathscr{L}} \leq C(\|f - g\|^\gamma_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}} \| \, |f| + |g| \, \|^{1-\gamma}_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}})^q, \tag{1.8}$$

where

$$\gamma = 1 - \alpha \frac{p-2}{p} \quad \text{if } p > 2 \tag{1.9}$$

$$\gamma = \frac{p}{q}\left(1 - \alpha \frac{2-p}{p}\right) \quad \text{if } 1 < p < 2 \tag{1.10}$$

We point out that Theorem 1.1 improves the result of [12] in two different directions. First of all, when $0 < \alpha < \frac{p}{|p-2|}$, it gives higher integrability of the solutions found in [12]. On the other hand, the case $\alpha = \frac{p}{|p-2|}$ is not covered by [12].

Since for $\alpha > 0$ the solutions of our problem could have infinite energy, we cannot use in the equations test functions whose gradient is proportional to the gradient of the solution. In order to prove our results we construct suitable test functions and we develop fine properties related to the norm in the Orlicz–Zygmund spaces. Tipically, these spaces are equipped with the Luxemburg norm that is not convenient in our setting. Then we introduce the quantity

$$\|f\|_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}} = \left\{ \int_0^{\varepsilon_0} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q \, d\varepsilon \right\}^{1/q}$$

which is a norm equivalent to the Luxemburg one.

For the proofs of our results see [9].

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