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Partial Differential Equations — *On the critical polynomial of functionals related to p-area* ($1 < p < \infty$) *and p-Laplace* ($1 < p \leq 2$) *type operators*, by SILVIA CINGOLANI, MARCO DEGIOVANNI and GIUSEPPINA VANNELLA, communicated on 14 November 2014.

ABSTRACT. — We consider a class of quasilinear elliptic equations whose principal part includes the p-area (for $1 < p < \infty$) and the p-Laplace (for $1 < p \leq 2$) operator. For the critical points of the associated functional, we provide estimates of the corresponding critical polynomial.

KEY WORDS: p-area operator, p-Laplace operator, functionals with lack of smoothness, critical polynomial, Morse index.

MATHEMATICS SUBJECT CLASSIFICATION: 35J62, 35J92, 58E05.

1. INTRODUCTION

In this note we outline some results that are discussed and proved in a more complete form in [9].

Consider the quasilinear elliptic problem

$$(1.1) \qquad \begin{cases} -\mathrm{div}[(\kappa^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with $\partial\Omega$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1]$, while $p > 1$ and $\kappa \geq 0$ are real numbers.

Under suitable assumptions on g, weak solutions u of (1.1) correspond to critical points of the C^1-functional $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$(1.2) \qquad f(u) = \int_\Omega \Psi_{p,\kappa}(\nabla u)\, dx + \int_\Omega G(x,u)\, dx,$$

where

$$\Psi_{p,\kappa}(\xi) = \frac{1}{p}[(\kappa^2 + |\xi|^2)^{\frac{p}{2}} - \kappa^p], \quad G(x,s) = \int_0^s g(x,t)\, dt.$$

About the principal part of the equation, the reference cases are $\kappa = 1$, which yields the *p-area operator*, and $\kappa = 0$, which yields the *p-Laplace operator*. In the case $p = 2$ the value of κ is irrelevant.

In this work we aim to describe the behavior of the functional f near a critical point u_0, checking the critical polynomial of f at u_0 (see [4, 6, 21]) via Hessian-type notions.

For functionals defined on Banach spaces, serious difficulties arise in extending Morse theory (see [26, 25, 5, 6, 7]). More precisely, by standard deformation results, which hold also in general Banach spaces, one can prove the so-called Morse relations, which can be written as

$$\sum_{\substack{f'(u)=0\\ a\le f(u)\le b}} i(f,u)(t) = \sum_{m=0}^{\infty} \beta_m t^m + (1+t)Q(t),$$

where (β_m) is the sequence of the Betti numbers of a pair of sublevels $(\{f \le b\}, \{f < a\})$ and $i(f,u)(t)$ is the (generalized) critical polynomial of f at u (see e.g. [6, Theorem I.4.3]). The problem, in the extension from Hilbert to Banach spaces, concerns the estimate of the critical polynomials, by the Hessian of f or some related concept. In a Hilbert setting, the classical Morse lemma and the generalized Morse lemma [17] provide a satisfactory answer. For Banach spaces, a similar general result is so far not known.

More recently, for $p > 2$ and $\kappa > 0$, the first and the last author have proved an extension of the Morse Lemma and established a connection between the critical polynomial and the Morse index (see [10, 11, 12]), taking advantage of the fact that, under suitable assumptions on g, the functional f is actually of class C^2 on $W_0^{1,p}(\Omega)$ and that

$$\Psi''_{p,\kappa}(\eta)[\xi]^2 \ge \nu_{p,\kappa}|\xi|^2 \quad \text{with } \nu_{p,\kappa} > 0.$$

Moreover, an approximation result of Marino-Prodi type is proved in [13].

On the contrary, in the case $1 < p < 2$ the functional f is not of class C^2 on $W_0^{1,p}(\Omega)$. For $\kappa = 0$, even the function $\Psi_{p,\kappa}$ is not of class C^2 on $\mathbb{R}^N$. This adds new difficulties to the problem.

For any $p > 1$ estimates of the critical polynomial associated to p-Laplacian equations on a ball are obtained by Aftalion and Pacella [2] at the positive radial solutions u such that $|\nabla u(x)| \ne 0$ for $x \ne 0$. Moreover, estimates in the line of the Morse lemma and of the generalized Morse lemma for quasilinear elliptic equations with natural growth conditions have been proved in [14, 19].

Our purpose is to consider a class of functionals including (1.2) in the two cases:

- $1 < p < \infty$ and $\kappa > 0$;
- $1 < p \le 2$ and $\kappa = 0$.

Actually, we are mainly interested in the case $1 < p \le 2$ also when $\kappa > 0$, but our results are new also for $p > 2$, as our assumptions are less restrictive than in previous papers. On the contrary, our results do not cover the case $p > 2$ with $\kappa = 0$.

More precisely, define

$$f(u) = \int_\Omega \Psi(\nabla u)\,dx + \int_\Omega G(x,u)\,dx. \tag{1.3}$$

We will assume that:

(Ψ_1) the function $\Psi : \mathbb{R}^N \to \mathbb{R}$ is of class C^1 on $\mathbb{R}^N$ with $\Psi(0) = 0$ and $\nabla\Psi(0) = 0$; moreover, there exist $p > 1$, $\kappa \ge 0$ and $0 < \nu \le C$ such that the functions $(\Psi - \nu\Psi_{p,\kappa})$ and $(C\Psi_{p,\kappa} - \Psi)$ are both convex;

(Ψ_2) if $\kappa = 0$ and $1 < p < 2$, then Ψ is of class C^2 on $\mathbb{R}^N\backslash\{0\}$; otherwise, Ψ is of class C^2 on $\mathbb{R}^N$;

(g_1) the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that $g(\cdot, s)$ is measurable for every $s \in \mathbb{R}$ and $g(x,\cdot)$ is of class C^1 for a.e. $x \in \Omega$; if $p \le N$, we also assume there exist $C, q > 0$ such that

$$|g(x,s)| \le C(1 + |s|^q) \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R},$$

where $q \le p^* - 1 = \frac{Np}{N-p} - 1$ if $p < N$, while no restriction on q is required if $p = N$;

(g_2) for every $S > 0$ there exists $C_S > 0$ such that

$$|D_s g(x,s)| \le C_S \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \le S.$$

Under these assumptions, it is easily seen that $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ is of class C^1, while it is of class C^2 if $p > \max\{N, 2\}$. Moreover, even in the case $g = 0$, f is never of class C^2 for $1 < p < 2$ and is of class C^2 in the case $p = 2$ iff Ψ is a quadratic form on $\mathbb{R}^N$ (see [1, Proposition 3.2]).

Now, let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f, namely a weak solution of

$$\begin{cases} -\mathrm{div}[\nabla\Psi(\nabla u)] + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to [18, 16, 20, 23, 24], $u_0 \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in]0,1]$.

Let us recall the first ingredient we need from [6, 15, 21].

Definition 1.1. Let $\mathbb{K}$ be a field, $c = f(u_0)$ and $f^c = \{u \in W_0^{1,p}(\Omega) : f(u) \le c\}$. The *generalized critical polynomial of f at u_0 with coefficients in* $\mathbb{K}$ is defined by

$$i(f, u_0; \mathbb{K})(t) = \sum_{m=0}^{\infty} [\dim_{\mathbb{K}} H^m(f^c, f^c\backslash\{u_0\}; \mathbb{K})]t^m,$$

where H^* stands for Alexander-Spanier cohomology [22].

We will simply write $i(f, u_0)(t)$, if no confusion can arise. In general, $i(f, u_0)(t)$ is a formal power series with coefficients in $\mathbb{N} \cup \{\infty\}$. If however u_0 is an isolated critical point, under assumptions (Ψ_1) and (g_1) it follows from [8, Theorem 1.1] and [3, Theorem 3.4] that $H^*(f^c, f^c \setminus \{u_0\})$ is of finite type, so that $i(f, u_0)(t)$ is a true polynomial with coefficients in $\mathbb{N}$.

The other ingredient is a notion of Morse index, which is not standard, as the functional f is not of class C^2.

In the case $\kappa > 0$ and $1 < p < \infty$, observe that

$$(p-1)\nu(\kappa^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi|^2 \le \Psi''(\eta)[\xi]^2 \le C(\kappa^2 + |\eta|^2)^{\frac{p-2}{2}}|\xi|^2 \quad \text{for any } \eta, \xi \in \mathbb{R}^N,$$

as $(\Psi - \nu\Psi_{p,\kappa})$ and $(C\Psi_{p,\kappa} - \Psi)$ are both convex. Therefore, there exists $\tilde{\nu} > 0$ such that

$$\tilde{\nu}|\xi|^2 \le \Psi''(\nabla u_0(x))[\xi]^2 \le \frac{1}{\tilde{\nu}}|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^N,$$

as ∇u_0 is bounded. Moreover, $D_s g(x, u_0) \in L^\infty(\Omega)$, as u_0 is bounded. Thus, we can define a smooth quadratic form $Q_{u_0} : W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$Q_{u_0}(v) = \int_\Omega \Psi''(\nabla u_0)[\nabla v]^2 \, dx + \int_\Omega D_s g(x, u_0) v^2 \, dx$$

and define the *Morse index of f at u_0* (denoted by $m(f, u_0)$) as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where Q_{u_0} is negative definite and the *large Morse index of f at u_0* (denoted by $m^*(f, u_0)$) as the supremum of the dimensions of the linear subspaces of $W_0^{1,2}(\Omega)$ where Q_{u_0} is negative semidefinite. We clearly have $m(f, u_0) \le m^*(f, u_0) < +\infty$.

In the case $\kappa = 0$ and $1 < p < 2$, observe that

$$\frac{(p-1)\nu}{|\eta|^{2-p}}|\xi|^2 \le \Psi''(\eta)[\xi]^2 \le \frac{C}{|\eta|^{2-p}}|\xi|^2 \quad \text{for any } \eta, \xi \in \mathbb{R}^N \text{ with } \eta \ne 0.$$

Set

$$Z_{u_0} = \{x \in \Omega : \nabla u_0(x) = 0\},$$

$$X_{u_0} = \left\{ v \in W_0^{1,2}(\Omega) : \nabla v(x) = 0 \text{ a.e. in } Z_{u_0} \text{ and } \frac{|\nabla v|^2}{|\nabla u_0|^{2-p}} \in L^1(\Omega \setminus Z_{u_0}) \right\}.$$

Then

$$(v|w)_{u_0} = \int_{\Omega \setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla v, \nabla w] \, dx$$

is a scalar product on X_{u_0} which makes X_{u_0} a Hilbert space continuously embedded in $W_0^{1,2}(\Omega)$. Moreover, we can define a smooth quadratic form $Q_{u_0} : X_{u_0} \to \mathbb{R}$ by

$$Q_{u_0}(v) = \int_{\Omega \setminus Z_{u_0}} \Psi''(\nabla u_0)[\nabla v]^2 \, dx + \int_\Omega D_s g(x, u_0) v^2 \, dx$$

and denote again by $m(f, u_0)$ the supremum of the dimensions of the linear subspaces of X_{u_0} where Q_{u_0} is negative definite and by $m^*(f, u_0)$ the supremum of the dimensions of the linear subspaces of X_{u_0} where Q_{u_0} is negative semidefinite. Since the derivative of Q_{u_0} is a compact perturbation of the Riesz isomorphism, we still have $m(f, u_0) \leq m^*(f, u_0) < +\infty$.

Now we can state our main results.

THEOREM 1.2. *Let $\kappa > 0$ and $1 < p < \infty$. Let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f defined in* (1.3). *Then we have*

$$i(f, u_0)(t) = \sum_{m=m(f,u_0)}^{m^*(f,u_0)} a_m t^m$$

with $a_m \in \mathbb{N} \cup \{\infty\}$.

When the quadratic form Q_{u_0} has no kernel, we can provide a complete description of the critical polynomial.

THEOREM 1.3. *Let $\kappa > 0$ and $1 < p < \infty$. Let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f defined in* (1.3) *with $m(f, u_0) = m^*(f, u_0)$.*

Then u_0 is an isolated critical point of f and we have

$$i(f, u_0)(t) = t^{m(f,u_0)}.$$

If u_0 is an isolated critical point of f, then a sharper form of Theorem 1.2 can be proved. Taking into account Theorem 1.3, only the case $m(f, u_0) < m^*(f, u_0)$ is interesting.

THEOREM 1.4. *Let $\kappa > 0$ and $1 < p < \infty$. Let $u_0 \in W_0^{1,p}(\Omega)$ be an isolated critical point of the functional f defined in* (1.3) *with $m(f, u_0) < m^*(f, u_0)$.*

Then one and only one of the following facts hold:

(a) *we have*

$$i(f, u_0)(t) = t^{m(f,u_0)};$$

(b) *we have*

$$i(f, u_0)(t) = t^{m^*(f,u_0)};$$

(c) *we have*

$$i(f, u_0)(t) = \sum_{m=m(f,u_0)}^{m^*(f,u_0)} a_m t^m$$

with $a_m \in \mathbb{N}$ and $a_{m(f,u_0)} = a_{m^*(f,u_0)} = 0$.

Remark 1.5. Since the value of κ is irrelevant in the case $p = 2$, Theorems 1.2, 1.3 and 1.4 cover also the case $\kappa = 0$ with $p = 2$.

In the case $\kappa = 0$ and $1 < p < 2$, we can prove that the generalized critical polynomial cannot contain t^m with m large.

Theorem 1.6. *Let $\kappa = 0$ and $1 < p < 2$. Let $u_0 \in W_0^{1,p}(\Omega)$ be a critical point of the functional f defined in* (1.3). *Then we have*

$$i(f, u_0)(t) = \sum_{m=0}^{m^*(f,u_0)} a_m t^m$$

with $a_m \in \mathbb{N} \cup \{\infty\}$.

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