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Partial Differential Equations — *On a Dirichlet problem with p-Laplacian and asymmetric nonlinearity*, by SALVATORE A. MARANO and NIKOLAOS S. PAPAGEORGIOU, communicated on 14 November 2014.

ABSTRACT. — The existence of at least two nonnegative smooth solutions to a homogeneous Dirichlet problem with p-Laplacian and reaction $(p-1)$-linear, but asymmetric, at $\pm\infty$ is investigated through variational and truncation techniques. The case $p = 2$ is separately examined, obtaining a third nontrivial smooth solution via Morse's theory.

KEY WORDS: p-Laplacian, asymmetric nonlinearity, critical groups, Morse identity.

MATHEMATICS SUBJECT CLASSIFICATION: 35J20, 35J60, 35J92.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, let $1 < p < +\infty$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) \equiv 0$. Consider the homogeneous Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplace differential operator, namely $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. As usual, a function $u \in W_0^{1,p}(\Omega)$ is called a (weak) solution to (1.1) provided

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)\,dx = \int_\Omega f(x,u(x))v(x)\,dx \quad \forall v \in W_0^{1,p}(\Omega).$$

The literature concerning (1.1) is by now very wide and many existence or multiplicity results are already available. Quite popular hypotheses are the following:

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} = \alpha \quad \text{uniformly in } x \in \Omega, \tag{1.2}$$

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = \beta \neq 0 \quad \text{uniformly in } x \in \Omega. \tag{1.3}$$

If $\alpha \in \mathbb{R}\setminus\{0\}$ then one usually says that $t \mapsto f(x,t)$ exhibits a symmetric $(p-1)$-linear growth at infinity; see [3, 17] and the references therein. The recent paper

[8] treats the case $\alpha \leq 0$ and $f(x,t) := \lambda g(x,t)$ with $\lambda > 0$ large enough, while $\alpha = +\infty$ in [9, 10, 11].

Let λ_1 (respectively, λ_2) be the first (respectively, second) eigenvalue of the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. Roughly speaking, in this paper, we shall consider a reaction term f whose behavior is $(p-1)$-linear, but asymmetric, near $-\infty$ and $+\infty$, in the sense that the graph of the function $t \mapsto \frac{f(x,t)}{|t|^{p-2}t}$ crosses λ_1 as t moves from $-\infty$ to $+\infty$. Such an f is usually called crossing or jumping nonlinearity. The existence of two solutions to (1.1) lying in $C_0^1(\overline{\Omega})\backslash\{0\}$ is established via variational and truncation methods; see Theorem 3.3. Section 4 investigates the case $p = 2$. A third nontrivial $C_0^1(\overline{\Omega})$-solution is obtained through Morse's theory.

Equations with p-Laplacian and $(p-1)$-linear asymmetric reactions have previously been studied by mainly using the so-called Fučik spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$; see [16], besides the seminal work [1]. This approach depends on the knowledge of the Fučik spectrum and requires that the limit (1.2) exists.

Our arguments are patterned after those of [6] (cf. also [15]) where, however, a further sign condition on f is taken on and the semi-linear case is not separately treated. Accordingly, (1.2)–(1.3) become here

$$\limsup_{t\to+\infty}\frac{f(x,t)}{t^{p-1}} \leq a_1 < \lambda_1 < a_2 \leq \liminf_{t\to-\infty}\frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{t\to-\infty}\frac{f(x,t)}{|t|^{p-2}t} \leq b_2 \tag{1.4}$$

and

$$\lambda_2 < a_3 \leq \liminf_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{t\to 0}\frac{f(x,t)}{|t|^{p-2}t} \leq b_3 \tag{1.5}$$

uniformly in $x \in \Omega$, with a_i, b_j being nonnegative constants. It should be noted that none of limits (1.2)–(1.3) needs to exist. Moreover, (1.2) and (1.4) are mutually independent, whereas (1.3) forces (1.5) as soon as $\lambda_2 < \beta < +\infty$.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Given a set $V \subseteq X$, write $\overline{V}$ for the closure of V, ∂V for the boundary of V, and $\mathrm{int}(V)$ for the interior of V. If $x \in X$ and $\delta > 0$ then

$$B_\delta(x) := \{z \in X : \|z - x\| < \delta\}.$$

The symbol $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of X, $\langle\cdot,\cdot\rangle$ indicates the duality pairing between X and X^*, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'.

Let T be a topological space and let L be a multifunction from T into X (briefly, $L : T \to 2^X$), namely a function which assigns to each $t \in T$ a nonempty subset $L(t)$ of X. We say that L is lower semi-continuous when $\{t \in T : L(t) \cap V \neq \emptyset\}$ turns out to be open in T for every open set $V \subseteq X$. A function $l : T \to X$ is called a selection of L provided $l(t) \in L(t)$ for all $t \in T$.

We say that $\Phi : X \to \mathbb{R}$ is coercive when

$$\lim_{\|x\| \to +\infty} \Phi(x) = +\infty.$$

The function Φ is called weakly sequentially lower semi-continuous if $x_n \rightharpoonup x$ in X implies $\Phi(x) \leq \liminf_{n\to\infty} \Phi(x_n)$. Let $\Phi \in C^1(X)$. The classical Cerami compactness condition for Φ reads as follows.

(C) *Every sequence* $\{x_n\} \subseteq X$ *such that* $\{\Phi(x_n)\}$ *is bounded and*

$$\lim_{n\to+\infty} (1 + \|x_n\|)\|\Phi'(x_n)\|_{X^*} = 0$$

possesses a convergent subsequence.

Define, provided $c \in \mathbb{R}$,

$$\Phi^c := \{x \in X : \Phi(x) \leq c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual, $K(\Phi)$ denotes the critical set of Φ, i.e., $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$. Given a topological pair (A, B) fulfilling $B \subset A \subseteq X$, the symbol $H_q(A, B)$, $q \in \mathbb{N}_0$, indicates the q$^{\text{th}}$-relative singular homology group of (A, B) with integer coefficients. Let $x_0 \in K_c(\Phi)$ be an isolated point of $K(\Phi)$. Then

$$C_q(\Phi, x_0) := H_q(\Phi^c \cap V, \Phi^c \cap V \setminus \{x_0\}), \quad q \in \mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, V stands for any neighborhood of x_0 such that $K(\Phi) \cap \Phi^c \cap V = \{x_0\}$. By excision, this definition does not depend on the choice of V. Suppose Φ satisfies condition (C). When $\Phi|_{K(\Phi)}$ is bounded below and $c < \inf_{x \in K(\Phi)} \Phi(x)$, we define

$$C_q(\Phi, \infty) := H_q(X, \Phi^c), \quad q \in \mathbb{N}_0.$$

The second deformation lemma [4, Theorem 5.1.33] implies that this definition does not depend on the choice of c. If $K(\Phi)$ is finite then, setting

$$\left.\begin{aligned} M(t,x) &:= \sum_{q=0}^{+\infty} \operatorname{rank} C_q(\Phi, x) t^q \\ P(t,\infty) &:= \sum_{q=0}^{+\infty} \operatorname{rank} C_q(\Phi, \infty) t^q \end{aligned}\right\} \quad \forall (t,x) \in \mathbb{R} \times K(\Phi),$$

the following Morse relation holds:

$$\sum_{x \in K(\Phi)} M(t,x) = P(t,\infty) + (1+t)Q(t), \tag{2.1}$$

where $Q(t)$ denotes a formal series with nonnegative integer coefficients; see for instance [14, Theorem 6.62].

Now, let X be a Hilbert space, let $x \in K(\Phi)$, and let Φ be C^2 in a neighborhood of x. If $\Phi''(x)$ turns out to be invertible then x is called non-degenerate. The Morse index d of x is the supremum of the dimensions of the vector subspaces of X on which $\Phi''(x)$ turns out to be negative definite. When x is non-degenerate and with Morse index d one has

$$C_q(\Phi, x) = \delta_{q,d}\mathbb{Z}, \quad q \in \mathbb{N}_0. \tag{2.2}$$

The monographs [12, 14] represent general references on the subject.

Throughout the paper, Ω is a bounded domain of the real euclidean N-space $(\mathbb{R}^N, |\cdot|)$ with a smooth boundary $\partial\Omega$, m stands for the Lebesgue measure, $p \in (1, +\infty)$, $p' := p/(p-1)$, $\|\cdot\|_{L^q(\Omega)}$ with $q \geq 1$ indicates the usual norm of $L^q(\Omega)$, and $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\|_{1,p} := \left(\int_\Omega |\nabla u(x)|^p \, dx\right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

Write p^* for the critical exponent of the Sobolev embedding $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding is compact whenever $1 \leq q < p^*$.

Define $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$. Obviously, $C_0^1(\overline{\Omega})$ turns out to be an ordered Banach space with positive cone

$$C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \ \forall x \in \overline{\Omega}\}.$$

Moreover, one has

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega, \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega\right\},$$

where $n(x)$ is the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$; see, for example, [4, Remark 6.2.10].

Let $W^{-1,p'}(\Omega)$ be the dual space of $W_0^{1,p}(\Omega)$ and let $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative p-Laplacian, i.e.,

$$\langle A_p(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) \, dx \quad \forall u, v \in W_0^{1,p}(\Omega). \tag{2.3}$$

The Liusternik-Schnirelman theory gives a strictly increasing sequence $\{\lambda_n\}$ of eigenvalues for the operator $-\Delta_p$ in $W_0^{1,p}(\Omega)$. The following assertions involving λ_1, λ_2, and A_p can be found in [4, Section 6.2]; see also [14, Sections 9.1–9.2].

(p_1) $0 < \lambda_1 < \lambda_2$.

(p_2) $\|u\|^p_{L^p(\Omega)} \le \dfrac{1}{\lambda_1}\|u\|^p_{1,p}$ *for all* $u \in W^{1,p}_0(\Omega)$.

(p_3) *There exists an eigenfunction* ϕ_1 *corresponding to* λ_1 *such that* $\phi_1 \in \mathrm{int}(C^1_0(\overline{\Omega})_+)$ *as well as* $\|\phi_1\|_{L^p(\Omega)} = 1$.

(p_4) *If* $U := \{u \in W^{1,p}_0(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ *and*

$$\Gamma_0 := \{\gamma \in C^0([-1,1],U) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\},$$

then

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \|u\|^p_{1,p}.$$

(p_5) $u_n \rightharpoonup u$ *in* $W^{1,p}_0(\Omega)$ *and* $\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \le 0$ *imply* $u_n \to u$ *in* $W^{1,p}_0(\Omega)$.

Let $\alpha \in L^\infty(\Omega)\setminus\{0\}$ satisfy $\alpha \ge 0$. Consider the weighted eigenvalue problem

$$-\Delta_p u = \lambda\alpha(x)|u|^{p-2}u \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{2.4}$$

As before, there exists a strictly increasing sequence $\{\lambda_n(\alpha)\}$ of eigenvalues for (2.4) enjoying the properties [4, Section 6.2]:

(p_6) $0 < \lambda_1(\alpha) < \lambda_2(\alpha)$.

(p_7) *If* $\alpha, \beta \in L^\infty(\Omega)\setminus\{0\}$, $0 \le \alpha \le \beta$, *and* $\alpha \ne \beta$ *then* $\lambda_1(\beta) < \lambda_1(\alpha)$. *If* $0 \le \alpha < \beta$ *then* $\lambda_2(\beta) < \lambda_2(\alpha)$.

Obviously, $\lambda_n = \lambda_n(1)$, $n \in \mathbb{N}$. Now, suppose $p = 2$ and denote by $E(\lambda_n)$ the eigenspace associated with λ_n. It is known (see e.g. [4, Section 6.2]) that:

(p_8) $E(\lambda_n) \subseteq C^1_0(\overline{\Omega})$ *for all* $n \in \mathbb{N}$.

(p_9) *If* u *lies in* $E(\lambda_n)$ *and vanishes on a set of positive Lebesgue measure then* $u = 0$.

Setting, for every integer $m \ge 1$, $\overline{H}_m := \bigoplus_{n=1}^m E(\lambda_n)$ and $\hat{H}_m := \overline{H}_m^\perp$, we get

$$H^1_0(\Omega) = \overline{H}_m \oplus \hat{H}_m.$$

Consequently, each $u \in H^1_0(\Omega)$ can uniquely be written as $u = \bar{u} + \hat{u}$, where $\bar{u} \in \overline{H}_m$, $\hat{u} \in \hat{H}_m$. A simple argument, based on orthogonality and (p_9), yields the next result.

Lemma 2.1. *Let* $m \in \mathbb{N}$ *and let* $\theta \in L^\infty(\Omega)\setminus\{\lambda_m\}$ *satisfy* $\theta \ge \lambda_m$. *Then there exists a constant* $\bar{c} > 0$ *such that*

$$\|\bar{u}\|^2_{1,2} - \int_\Omega \theta(x)\bar{u}(x)^2\,dx \le -\bar{c}\|\bar{u}\|^2_{1,2} \quad \forall \bar{u} \in \overline{H}_m.$$

Let $m \in \mathbb{N}_0$ *and let* $\theta \in L^\infty(\Omega)\setminus\{\lambda_{m+1}\}$ *satisfy* $\theta \le \lambda_{m+1}$. *Then there exists a constant* $\hat{c} > 0$ *such that*

$$\|\hat{u}\|_{1,2}^2 - \int_\Omega \theta(x)\hat{u}(x)^2\,dx \ge \hat{c}\|\hat{u}\|_{1,2}^2 \quad \forall \hat{u} \in \hat{H}_m.$$

Define $U_C := \{u \in C_0^1(\overline{\Omega}) : \|u\|_{L^p(\Omega)} = 1\}$. Evidently, U_C turns out to be dense in the set U given by (p_4). Moreover, if

$$\Gamma_C := \{\gamma \in C^0([-1,1], U_C) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\}$$

then the following result holds.

LEMMA 2.2. *The set* Γ_C *is dense in* Γ_0.

PROOF. Pick any $\gamma_0 \in \Gamma_0$. We shall prove that there exists a sequence $\{\gamma_n\} \subseteq \Gamma_C$ fulfilling

$$\lim_{n\to+\infty} \max_{t\in[-1,1]} \|\gamma_n(t) - \gamma_0(t)\| = 0. \tag{2.5}$$

The multifunction $L_n : [-1,1] \to 2^{C_0^1(\overline{\Omega})}$ defined by

$$L_n(t) := \begin{cases} \{-\phi_1\} & \text{if } t = -1, \\ \{u \in C_0^1(\overline{\Omega}) : \|u - \gamma_0(t)\| < 1/n\} & \text{if } t \in (-1,1), \\ \{\phi_1\} & \text{if } t = 1 \end{cases}$$

takes nonempty convex values and is lower semi-continuous. So, Theorem 3.1‴ in [13] provides a continuous selection $l_n : [-1,1] \to C_0^1(\overline{\Omega})$ of L_n. This entails

$$\|l_n(t) - \gamma_0(t)\| < \frac{1}{n} \quad \forall t \in (-1,1), \quad l_n(-1) = -\phi_1, \quad l_n(1) = \phi_1. \tag{2.6}$$

Consequently,

$$\lim_{n\to+\infty} \|l_n(t)\|_{L^p(\Omega)} = \|\gamma_0(t)\|_{L^p(\Omega)} = 1 \tag{2.7}$$

uniformly with respect to $t \in [-1,1]$. For any n large enough we can thus set

$$\gamma_n(t) := \frac{l_n(t)}{\|l_n(t)\|_{L^p(\Omega)}}, \quad t \in [-1,1].$$

On account of (2.6) and (p_3) one has $\gamma_n \in \Gamma_C$. Moreover, thanks to (2.6),

$$\begin{aligned} \|\gamma_n(t) - \gamma_0(t)\| &\le \|\gamma_n(t) - l_n(t)\| + \|l_n(t) - \gamma_0(t)\| \\ &< |1 - \|l_n(t)\|_{L^p(\Omega)}| \frac{\|l_n(t)\|}{\|l_n(t)\|_{L^p(\Omega)}} + \frac{1}{n} \quad \forall t \in [-1,1]. \end{aligned} \tag{2.8}$$

Recall that $\gamma_0 \in \Gamma_0$. Since, by (2.6) again,

$$\begin{aligned}\max_{t\in[-1,1]} |1 - \|l_n(t)\|_{L^p(\Omega)}| &= \max_{t\in[-1,1]} |\|\gamma_0(t)\|_{L^p(\Omega)} - \|l_n(t)\|_{L^p(\Omega)}| \\ &\le \max_{t\in[-1,1]} \|\gamma_0(t) - l_n(t)\|_{L^p(\Omega)} \\ &\le c \max_{t\in[-1,1]} \|\gamma_0(t) - l_n(t)\| \le \frac{c}{n}\end{aligned}$$

for some $c > 0$, (2.5) immediately follows from (2.6)–(2.8). □

Finally, put, provided $t \in \mathbb{R}$, $u : \Omega \to \mathbb{R}$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$t^- := \max\{-t, 0\}, \quad t^+ := \max\{t, 0\},$$
$$u^-(x) := u(x)^-, \quad u^+(x) := u(x)^+, \quad N_g(u)(x) := g(x, u(x)).$$

3. Existence results

To avoid unnecessary technicalities, 'for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' and the variable x will be omitted when no confusion can arise.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x, 0) \equiv 0$ and let

$$F(x,z) := \int_0^z f(x,t)\,dt, \quad (x,z) \in \Omega \times \mathbb{R}. \tag{3.1}$$

We will posit the following assumptions, where a_i and b_j denote appropriate nonnegative constants.

(f_0) $|f(x,t)| \le a_0(1 + |t|^{p-1})$ *for every* $(x,t) \in \Omega \times \mathbb{R}$.

(f_1) $\displaystyle\limsup_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} \le a_1 < \lambda_1$ *uniformly with respect to* $x \in \Omega$.

(f_2) $\displaystyle\lambda_1 < a_2 \le \liminf_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \le b_2$ *uniformly in* $x \in \Omega$.

(f_3) $\displaystyle\lambda_2 < a_3 \le \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le b_3$ *uniformly with respect to* $x \in \Omega$.

(f_4) *There exists* $a_4 > \lambda_1$ *such that* $\displaystyle\frac{a_4}{p}|z|^p \le F(x,z)$ *for all* $(x,z) \in \Omega \times \mathbb{R}_0^-$.

On account of (f_0) and (f_3), to every $\rho > 0$ there corresponds $\mu_\rho > 0$ satisfying

$$f(x,t) + \mu_\rho t^{p-1} \ge 0, \quad (x,t) \in \Omega \times [0,\rho]. \tag{3.2}$$

Remark 3.1. The constants that appear in (f_0)–(f_4) can evidently be replaced by suitable functions belonging to $L^\infty(\Omega)$. In particular, we might have $a_1, a_2, a_4 \in L^\infty(\Omega)\setminus\{\lambda_1\}$ with $0 \le a_1 \le \lambda_1 \le \min\{a_2, a_4\}$.

Write $X := W_0^{1,p}(\Omega)$ and $C_+ := C_0^1(\overline{\Omega})_+$. The energy functional $\varphi : X \to \mathbb{R}$ stemming from Problem (1.1) is

$$\varphi(u) := \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x, u(x))\, dx \quad \forall u \in X, \tag{3.3}$$

with F as in (3.1). Obviously, $\varphi \in C^1(X)$. Moreover, if

$$f_+(x,t) := f(x,t^+), \quad F_+(x,z) := \int_0^z f_+(x,t)\, dt$$

then $F_+(x,z) = F(x,z^+)$ and the corresponding truncated function

$$\varphi_+(u) := \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F_+(x,u(x))\, dx, \quad u \in X,$$

turns out to be C^1 as well.

Lemma 3.1. *Under hypotheses* (f_0)–(f_1), *the functional* φ_+ *is weakly sequentially lower semi-continuous and coercive.*

Proof. The space X compactly embeds in $L^p(\Omega)$ while the Nemitskii operator N_{f_+} turns out to be continuous on $L^p(\Omega)$. Thus, a standard argument ensures that φ_+ is weakly sequentially lower semi-continuous.

Pick $\varepsilon \in (0, \lambda_1 - a_1)$. By (f_0)–(f_1) there exists $c_0 > 0$ fulfilling

$$F(x,z) < \frac{a_1 + \varepsilon}{p} z^p + c_0 \quad \forall (x,z) \in \Omega \times \mathbb{R}_0^+.$$

Consequently, on account of (p_2),

$$\begin{aligned}
\varphi_+(u) &\geq \frac{1}{p}[\|u\|_{1,p}^p - (a_1+\varepsilon)\|u^+\|_{L^p(\Omega)}^p] - c_0 m(\Omega) \\
&\geq \frac{1}{p}[\|u\|_{1,p}^p - (a_1+\varepsilon)\|u\|_{L^p(\Omega)}^p] - c_0 m(\Omega) \\
&\geq \frac{1}{p}\left(1 - \frac{a_1+\varepsilon}{\lambda_1}\right)\|u\|_{1,p}^p - c_0 m(\Omega)
\end{aligned}$$

for any $u \in X$. Since $a_1 + \varepsilon < \lambda_1$, the conclusion follows. □

Theorem 3.1. *Let* (f_0), (f_1), *and* (f_3) *be satisfied. Then Problem* (1.1) *admits a solution* $u_0 \in \operatorname{int}(C_+)$, *which is a local minimizer of* φ.

Proof. Thanks to Lemma 3.1 we can find $u_0 \in X$ such that

$$\varphi_+(u_0) = \inf_{u \in X} \varphi_+(u). \tag{3.4}$$

Bearing in mind (p_1), fix $\varepsilon \in (0, a_3 - \lambda_1)$. By ($f_3$) one has

$$F(x,z) \geq \frac{a_3 - \varepsilon}{p}|z|^p \quad \text{in } \Omega \times [-\delta, \delta] \tag{3.5}$$

for appropriate $\delta > 0$. If $t > 0$ is so small that

$$0 \leq t\phi_1(x) \leq \delta \quad \forall x \in \overline{\Omega},$$

where ϕ_1 comes from (p_3), then (3.5) yields

$$\varphi_+(t\phi_1) \leq \frac{t^p}{p}[\varepsilon - (a_3 - \lambda_1)]\|\phi_1\|^p_{L^p(\Omega)} < 0. \tag{3.6}$$

Hence,

$$\varphi_+(u_0) < 0 = \varphi_+(0), \tag{3.7}$$

which clearly means $u_0 \neq 0$. Now, through (3.4) we get $\varphi'_+(u_0) = 0$, namely

$$\langle A_p(u_0), v\rangle = \int_\Omega f_+(x, u_0(x))v(x)\,dx, \quad v \in X.$$

Choosing $v := -u_0^-$ leads to $\|u_0^-\|^p_{1,p} = 0$. Thus, $u_0 \geq 0$ and, a fortiori, the function u_0 solves (1.1). Standard regularity results [5, Theorems 1.5.5–1.5.6] ensure that $u_0 \in C_+ \setminus \{0\}$. Let $\rho := \|u_0\|_{L^\infty(\Omega)}$. Due to (3.2) one has

$$-\Delta_p u_0(x) + \mu_\rho u_0(x)^{p-1} = f(x, u_0(x)) + \mu_\rho u_0(x)^{p-1} \geq 0 \quad \text{a.e. in } \Omega.$$

Therefore, by Theorem 5 in [18], $u_0 \in \operatorname{int}(C_+)$. This also implies that u_0 is a local $C^1_0(\overline{\Omega})$-minimizer of φ, because $\varphi|_{C_+} = \varphi_+|_{C_+}$. Finally, owing to [2, Theorem 1.1], the same holds true with $C^1_0(\overline{\Omega})$ replaced by X. □

Lemma 3.2. *Under hypotheses* (f_0)–(f_2), *the functional φ fulfills condition* (C).

Proof. Since X compactly embeds in $L^p(\Omega)$, the Nemitskii operator N_f is continuous on $L^p(\Omega)$, and A_p enjoys property (p_5), it suffices to show that every sequence $\{u_n\} \subseteq X$ satisfying

$$|\varphi(u_n)| \leq c_1 \quad \forall n \in \mathbb{N}, \tag{3.8}$$

$$\lim_{n\to+\infty}(1 + \|u_n\|_{1,p})\varphi'(u_n) = 0 \tag{3.9}$$

turns out to be bounded. Obviously, this happens once the same holds for both $\{u_n^+\}$ and $\{u_n^-\}$. We are thus reduced to verifying two claims.

Claim 1. *The sequence $\{u_n^+\}$ is bounded in X.*

If the assertion were false then, up to subsequences, $\|u_n^+\|_{1,p} \to +\infty$. Write $v_n := u_n^+/\|u_n^+\|_{1,p}$. From $\|v_n\|_{1,p} \equiv 1$ it follows, along a subsequence when necessary,

$$v_n \rightharpoonup v \quad \text{in } X, \qquad v_n \to v \quad \text{in } L^p(\Omega), \qquad v_n \to v \geq 0 \quad \text{a.e. in } \Omega. \tag{3.10}$$

Through (3.9) one has $\langle \varphi'(u_n), u_n^+ \rangle \to 0$, which, dividing by $\|u_n^+\|_{1,p}^p$, easily entails

$$\|v_n\|_{1,p}^p \leq \varepsilon_n + \int_\Omega \frac{f(x, u_n^+(x))}{\|u_n^+\|_{1,p}^{p-1}} v_n(x)\, dx \quad \forall n \in \mathbb{N}, \tag{3.11}$$

where $\varepsilon_n \to 0^+$. Because of (f_0) the sequence $\{\|u_n^+\|_{1,p}^{-p+1} N_f(u_n^+)\} \subseteq L^{p'}(\Omega)$ is bounded. Via the same reasoning made in [14, pp. 302–303] we thus get a function $\alpha \in L^\infty(\Omega)$ such that $0 \leq \alpha \leq a_1$ and

$$\frac{1}{\|u_n^+\|_{1,p}^{p-1}} N_f(u_n^+) \rightharpoonup \alpha v^{p-1} \quad \text{in } L^{p'}(\Omega).$$

Thanks to (3.10)–(3.11) this produces, as $n \to +\infty$,

$$\|v\|_{1,p}^p \leq \int_\Omega \alpha(x) v(x)^p\, dx \leq \lambda_1 \|v\|_{L^p(\Omega)}^p. \tag{3.12}$$

Consequently, $v = t\phi_1$ for some $t \geq 0$. If $t = 0$ then, by (3.10)–(3.11) again, $v_n \to 0$ in X, which contradicts $\|v_n\|_{1,p} = 1$ for all $n \in \mathbb{N}$. Otherwise, on account of (3.12) and (f_1),

$$\|\phi_1\|_{1,p}^p = \frac{1}{t^p}\|v\|_{1,p}^p \leq \frac{1}{t^p}\int_\Omega \alpha(x) v(x)^p\, dx < \int_\Omega \lambda_1 \phi_1(x)^p\, dx = \lambda_1 \|\phi_1\|_{L^p(\Omega)}^p,$$

but this is impossible; cf. (p_3).

Claim 2. *The sequence $\{u_n^-\}$ is bounded in X.*

If the assertion were false then, up to subsequences, $\|u_n^-\|_{1,p} \to +\infty$. Write, like before, $w_n := u_n^-/\|u_n^-\|_{1,p}$. From $\|w_n\|_{1,p} \equiv 1$ it follows, along a subsequence when necessary,

$$w_n \rightharpoonup w \quad \text{in } X, \qquad w_n \to w \quad \text{in } L^p(\Omega), \qquad w_n \to w \geq 0 \quad \text{a.e. in } \Omega. \tag{3.13}$$

Through (3.9) one has

$$\left| \langle A_p(u_n), v \rangle - \int_\Omega f(x, u_n(x)) v(x)\, dx \right| \leq \varepsilon_n \|v\|_{1,p} \quad \forall v \in X, \tag{3.14}$$

where $\varepsilon_n \to 0^+$. Assumption (f_0) and the boundedness of $\{u_n^+\}$ readily lead to

$$\left| \langle A_p(u_n^+), v \rangle - \int_\Omega f(x, u_n^+(x)) v(x)\, dx \right| \leq c_2 \|v\|_{1,p} \tag{3.15}$$

for appropriate $c_2 > 0$. Since $u_n = u_n^+ - u_n^-$, inequalities (3.14)–(3.15) produce, after dividing by $\|u_n^-\|_{1,p}^{p-1}$,

$$\left| \langle A_p(-w_n), v\rangle - \frac{1}{\|u_n^-\|_{1,p}^{p-1}} \int_\Omega f(x, -u_n^-(x))v(x)\, dx \right| \le \varepsilon_n' \|v\|_{1,p}, \quad v \in X, \tag{3.16}$$

with $\varepsilon_n' \to 0^+$. Observe next that, by (f_0) besides (3.13),

$$\lim_{n\to+\infty} \frac{1}{\|u_n^-\|_{1,p}^{p-1}} \int_\Omega f(x, -u_n^-(x))(w_n(x) - w(x))\, dx = 0.$$

So, (3.16) written for $v := w_n - w$ and (3.13) again provide

$$\lim_{n\to+\infty} \langle A_p(w_n), w_n - w\rangle = 0,$$

namely, because of (p_5),

$$\lim_{n\to+\infty} w_n = w \quad \text{in } X, \tag{3.17}$$

whence $\|w\|_{1,p} = 1$. Thanks to (f_0) the sequence $\{\|u_n^-\|_{1,p}^{-p+1} N_f(-u_n^-)\} \subseteq L^{p'}(\Omega)$ is bounded. Using the arguments made in [14, pp. 302–303] we thus obtain a function $\alpha \in L^\infty(\Omega)$ such that $a_2 \le \alpha \le b_2$ and

$$\frac{1}{\|u_n^-\|_{1,p}^{p-1}} N_f(-u_n^-) \rightharpoonup -\alpha w^{p-1} \quad \text{in } L^{p'}(\Omega).$$

On account of (3.16)–(3.17) this implies, as $n \to +\infty$,

$$\langle A_p(w), v\rangle = \int_\Omega \alpha(x) w(x)^{p-1}\, dx \quad \forall v \in X,$$

i.e., w turns out to be a weak positive solution of the problem

$$-\Delta_p u = \alpha(x)|u|^{p-2}u \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

Now, recalling (f_1), from (p_7) it follows

$$\lambda_1(\alpha) < \lambda_1(\lambda_1) = 1 = \lambda_2(\lambda_2) < \lambda_2(\alpha).$$

Therefore $w = 0$, which contradicts $\|w\|_{1,p} = 1$. □

A further nontrivial smooth solution to (1.1) can now be found.

THEOREM 3.2. *Let* (f_0)–(f_4) *be satisfied. Then Problem* (1.1) *possesses a nontrivial solution* $u_1 \in C_0^1(\overline{\Omega})\setminus\{u_0\}$.

Proof. We may evidently assume that the local minimizer u_0 of φ given by Theorem 3.1 is proper. Thus, for sufficiently small $\rho > 0$ one has

$$\varphi(u_0) < c_\rho := \inf_{u \in \partial B_\rho(u_0)} \varphi(u). \tag{3.18}$$

Since, due to (f_2),

$$\lim_{t \to -\infty} \varphi(t\phi_1) = -\infty,$$

there exists $t_1 > 0$ such that

$$\|t_1\phi_1 + u_0\|_{1,p} > \rho, \quad \varphi(-t_1\phi_1) < c_\rho.$$

On account of Lemma 3.2, the Mountain-Pass Theorem can be applied, which yields a point $u_1 \in X$ complying with $\varphi'(u_1) = 0$ and

$$c_\rho \le \varphi(u_1) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{3.19}$$

where

$$\Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = -t_1\phi_1, \gamma(1) = u_0\}.$$

Obviously, the function u_1 solves (1.1). Through (3.18)–(3.19) we get $u_1 \ne u_0$, while standard regularity arguments ensure that $u_1 \in C_0^1(\overline{\Omega})$. The proof is thus completed once one verifies that $u_1 \ne 0$. This will follow from the inequality

$$\varphi(u_1) < 0, \tag{3.20}$$

which, in view of (3.19), can be shown by constructing a path $\hat{\gamma} \in \Gamma$ such that

$$\varphi(\hat{\gamma}(t)) < 0 \quad \forall t \in [0,1]. \tag{3.21}$$

By (f_3) to every $\eta > 0$ small there corresponds $\delta > 0$ such that

$$\frac{\lambda_2 + \eta}{p} |z|^p \le F(x,z), \quad (x,z) \in \Omega \times [-\delta, \delta]. \tag{3.22}$$

Combining (p_4) with Lemma 2.2 entails

$$\max_{t \in [-1,1]} \|\gamma_\eta(t)\|_{1,p}^p < \lambda_2 + \eta \tag{3.23}$$

for appropriate $\gamma_\eta \in \Gamma_C$. Since $\gamma_\eta([-1,1])$ is compact in $C_0^1(\overline{\Omega})$ and $t_1\phi_1, u_0 \in \operatorname{int}(C_+)$ we can find $\varepsilon > 0$ so small that

$$-t_1\phi_1(x) \le \varepsilon\gamma_\eta(t)(x) \le u_0(x), \quad |\varepsilon\gamma_\eta(t)(x)| \le \delta$$

whenever $x \in \Omega$, $t \in [-1, 1]$. Thanks to (3.22)–(3.23) one has

$$\begin{aligned}\varphi(\varepsilon\gamma_\eta(t)) &= \frac{\varepsilon^p}{p}\|\gamma_\eta(t)\|_{1,p}^p - \int_\Omega F(x, \varepsilon\gamma_\eta(t)(x))\,dx \\ &< \frac{\varepsilon^p}{p}(\lambda_2 + \eta) - \frac{\varepsilon^p}{p}(\lambda_2 + \eta)\int_\Omega |\gamma_\eta(t)(x)|^p\,dx = 0 \quad \forall t \in [-1, 1],\end{aligned}$$

because $\gamma_\eta(t) \in U_C$. Consequently,

$$\varphi|_{\varepsilon\gamma_\eta([-1,1])} < 0. \tag{3.24}$$

Next, write $a := \varphi_+(u_0)$. From (3.7) it follows $a < 0$. We may suppose

$$K(\varphi_+) = \{0, u_0\},$$

otherwise the conclusion is straightforward. Hence, no critical value of φ_+ lies in $(a, 0)$ while

$$K_a(\varphi_+) = \{u_0\}.$$

Due to the second deformation lemma [4, Theorem 5.1.33], there exists a continuous function $h : [0, 1] \times (\varphi_+^0 \setminus \{0\}) \to \varphi_+^0$ satisfying

$$h(0, u) = u, \quad h(1, u) = u_0, \quad \text{and} \quad \varphi_+(h(t, u)) \le \varphi_+(u)$$

for all $(t, u) \in [0, 1] \times (\varphi_+^0 \setminus \{0\})$. Let $\gamma_+(t) := h(t, \varepsilon\phi_1)^+$, $t \in [0, 1]$. Then $\gamma_+(0) = \varepsilon\phi_1$, $\gamma_+(1) = u_0$, as well as

$$\varphi(\gamma_+(t)) = \varphi_+(\gamma_+(t)) \le \varphi_+(h(t, \varepsilon\phi_1)) \le \varphi_+(\varepsilon\phi_1) = \varphi(\varepsilon\gamma_\eta(1)) < 0; \tag{3.25}$$

cf. (3.24). Finally, define

$$\gamma_-(t) := -(t_1 t + \varepsilon(1 - t))\phi_1, \quad t \in [0, 1].$$

By (f_4) and (p_2)–(p_3) we easily have

$$\varphi(\gamma_-(t)) \le \frac{(t_1 t + \varepsilon(1 - t))^p}{p}(\lambda_1 - a_4)\|\phi_1\|_{L^p(\Omega)}^p < 0. \tag{3.26}$$

Concatenating γ_-, $\varepsilon\gamma_\eta$, and γ_+ one obtains a path $\hat{\gamma} \in \Gamma$ which, in view of (3.24)–(3.26), fulfills (3.21). This shows (3.20), whence $u_1 \neq 0$. □

The next multiplicity result directly stems from Theorems 3.1–3.2.

THEOREM 3.3. *Let* (f_0)–(f_4) *be satisfied. Then Problem* (1.1) *possesses at least two nontrivial solutions* $u_0 \in \operatorname{int}(C_+)$ *and* $u_1 \in C_0^1(\overline{\Omega})$.

4. The case $p = 2$

Suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(x, 0) \equiv 0$ and $f(x, \cdot)$ belongs to $C^1(\mathbb{R})$ for every $x \in \Omega$, while $f(\cdot, t)$ and $f_t'(\cdot, t)$ are measurable for all $t \in \mathbb{R}$. The following assumptions will be made in the sequel, where a_i and b_j denote appropriate nonnegative constants.

(f_5) $|f_t'(x, t)| \le a_0(1 + |t|^{r-2})$ *for every* $(x, t) \in \Omega \times \mathbb{R}$, *being* $2 \le r < 2^*$.

(f_6) $\displaystyle\lim_{t\to+\infty} \frac{f(x, t)}{t} = a_1 < \lambda_1$ *uniformly with respect to* $x \in \Omega$.

(f_7) $\displaystyle\lambda_1 < a_2 \le \liminf_{t\to-\infty} \frac{f(x, t)}{t} \le \limsup_{t\to-\infty} \frac{f(x, t)}{t} \le b_2$ *uniformly in* $x \in \Omega$.

(f_8) $\displaystyle f_t'(x, 0) = \lim_{t\to 0} \frac{f(x, t)}{t}$ *uniformly with respect to* $x \in \Omega$. *Moreover, for some* $m \ge 2$ *one has* $\lambda_m < a_3 \le f_t'(x, 0) \le b_3 < \lambda_{m+1}$ *in* Ω.

(f_9) *There exists* $a_4 > \lambda_1$ *fulfilling* $\frac{a_4}{2} z^2 \le F(x, z)$ *for all* $(x, z) \in \Omega \times \mathbb{R}_0^-$.

A comment analogous to that made in Remark 3.1 is true here.

Consider the semi-linear problem

$$
\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}
$$

If $X := H_0^1(\Omega)$ and, to simplify notation, $\|\cdot\| := \|\cdot\|_{1,2}$ then the energy functional $\varphi : X \to \mathbb{R}$ stemming from (4.1) is

$$
\varphi(u) := \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u(x))\, dx \quad \forall u \in X, \tag{4.2}
$$

with F as in (3.1). Obviously, $\varphi \in C^2(X)$.

Adapting the arguments of Section 3 we see that φ satisfies condition (C) and the following result holds.

Theorem 4.1. *Let* (f_5)–(f_9) *be satisfied. Then* (4.1) *admits at least two nontrivial solutions* $u_0 \in \mathrm{int}(C_+)$ *and* $u_1 \in C_0^1(\overline{\Omega})$.

A further nontrivial smooth solution to (4.1) will be found via Morse's theory.

Lemma 4.1. *Under hypotheses* (f_5)–(f_7) *one has* $C_q(\varphi, \infty) = 0$ *for all* $q \in \mathbb{N}_0$.

Proof. Pick any $\beta \in L^\infty(\Omega)\setminus\{0\}$ such that $\beta \ge 0$. Define, provided $u \in X$, $t \in [0, 1]$,

$$
\psi(u) := \frac{1}{2}\|u\|^2 - \frac{a_2}{2}\|u^-\|^2_{L^2(\Omega)} + \int_\Omega \beta(x) u(x)\, dx,
$$

$$
h(t, u) := t\varphi(u) + (1 - t)\psi(u).
$$

On account of (f_5) the function $h:[0,1]\times X\to\mathbb{R}$ maps bounded sets into bounded sets, while $h(0,\cdot)$ and $h(1,\cdot)$ evidently comply with condition (C). Since $u\mapsto h'_t(t,u)$ and $u\mapsto h'_u(t,u)$ are locally Lipschitz continuous, as a simple computation shows, Proposition 3.2 in [7] can be applied once we prove that there exist $c\in\mathbb{R}$, $\delta>0$ fulfilling

$$h(t,u)\le c \quad\Rightarrow\quad (1+\|u\|)\|h'_u(t,u)\|_{X^*}\ge\delta\|u\|^2.$$

If the assertion were false then one might construct two sequences $\{t_n\}\subseteq[0,1]$, $\{u_n\}\subseteq X$ such that $t_n\to t$, $h(t_n,u_n)\to-\infty$, and

$$(1+\|u_n\|)\|h'_u(t_n,u_n)\|_{X^*}<\frac{1}{n}\|u_n\|^2,\quad n\in\mathbb{N}. \tag{4.3}$$

By the properties of h, from $h(t_n,u_n)\to-\infty$ it follows

$$\lim_{n\to+\infty}\|u_n\|=+\infty. \tag{4.4}$$

Set $w_n:=\dfrac{u_n}{\|u_n\|}$, $n\in\mathbb{N}$. Passing to a subsequence when necessary, we may suppose

$$w_n\rightharpoonup w \quad\text{in } X,\qquad w_n\to w\quad\text{in } L^2(\Omega),\qquad w_n(x)\to w(x)\quad\text{a.e. in }\Omega,$$

because $\|w_n\|=1$ for all $n\in\mathbb{N}$. Inequality (4.3) yields

$$\begin{aligned}\bigg|\langle A_2(w_n),v\rangle - t_n\int_\Omega\frac{f(x,u_n)}{\|u_n\|}v\,dx+(1-t_n)a_2\int_\Omega\frac{u_n^-}{\|u_n\|}v\,dx\\ +(1-t_n)\int_\Omega\frac{\beta}{\|u_n\|}v\,dx\bigg|\le\frac{1}{n}\|v\|\quad\forall v\in X.\end{aligned} \tag{4.5}$$

Now observe that, on account of (f_5)–(f_7), the sequence $\{\|u_n\|^{-1}N_f(u_n)\}$ is bounded in $L^2(\Omega)$. Choosing $v:=w_n-w$ and letting $n\to+\infty$ in (4.5) easily leads to

$$\lim_{n\to+\infty}\langle A_2(w_n),w_n-w\rangle=0,$$

whence $w_n\to w$ in X by (p_5). Through (f_6)–(f_7) we get

$$\frac{N_f(u_n)}{\|u_n\|}\rightharpoonup a_1w^+-\alpha w^-\quad\text{in } L^2(\Omega)$$

for appropriate $\alpha\in L^2(\Omega)$ such that $a_2\le\alpha\le b_2$; see [6, pp. 1377–1378] or [14, pp. 302–303]. By (4.5) this implies, as $n\to+\infty$,

$$\langle A_2(w),v\rangle=\int_\Omega\{ta_1w^+(x)-[t\alpha(x)+(1-t)a_2]w^-(x)\}v(x)\,dx,\quad v\in X,$$

namely w turns out to be a weak solution of the problem

$$-\Delta u = ta_1 u^+ - \alpha_t(x) u^- \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $\alpha_t(x) := t\alpha(x) + (1-t)a_2$. Since $ta_1 < \lambda_1$ while

$$\lambda_m < a_2 \le \alpha_t(x) \le b_2 < \lambda_{m+1},$$

one has $w = 0$, which however contradicts $\|w\| = 1$. Hence, Proposition 3.2 in [7] provides

$$C_q(\varphi, \infty) = C_q(\psi, \infty) \quad \forall q \in \mathbb{N}_0. \tag{4.6}$$

The conclusion is achieved once we show that $C_q(\psi, \infty) = 0$. If $u \in K(\psi)$ then

$$\langle A_2(u), v\rangle = -\int_\Omega [a_2 u^-(x) + \beta(x)] v(x)\, dx, \quad v \in X.$$

Letting $v := u^+$ immediately leads to $u \le 0$. So, u solves the problem

$$-\Delta u = a_2 u - \beta(x) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{4.7}$$

Since $\beta \in L^\infty(\Omega)\setminus\{0\}$ and $\beta \ge 0$, standard regularity results [5, Theorems 1.5.5–1.5.6], besides [18, Theorem 5], yield $-u \in \operatorname{int}(C_+)$. Define, for every $v \in \operatorname{int}(C_+)$,

$$R(v, -u) := |\nabla v|^2 - \nabla(-u) \cdot \nabla\Big(\frac{v^2}{-u}\Big).$$

From the classical Picone identity (see, e.g., [14, Proposition 9.60]), (4.7), the sign properties of u and β, as well as (f_7) it follows

$$\begin{aligned} 0 &\le \int_\Omega R(v, -u)(x)\, dx = \|v\|^2 - \int_\Omega (-\Delta u) \frac{v^2}{u}\, dx \\ &= \|v\|^2 - a_2 \|v\|^2_{L^2(\Omega)} + \int_\Omega \frac{v^2}{u} \beta\, dx \\ &\le \|v\|^2 - a_2\|v\|^2_{L^2(\Omega)} < \|v\|^2 - \lambda_m \|v\|^2_{L^2(\Omega)}. \end{aligned}$$

Bearing in mind (p_3) this entails, for $v := \phi_1$,

$$0 < \lambda_1 - \lambda_m \le 0,$$

which is clearly impossible. So, $K(\psi) = \emptyset$ and, a fortiori, $C_q(\psi, \infty) = 0$. □

LEMMA 4.2. *Suppose* (f_5) *and* (f_8) *hold true. Then* $C_q(\varphi, 0) = \delta_{q,d_m}\mathbb{Z}$ *for all* $q \in \mathbb{N}_0$, *where* $d_m := \dim \bigoplus_{i=1}^m E(\lambda_i)$.

PROOF. Recall that $\varphi \in C^2(X)$ and one has

$$\langle \varphi''(u)(v), w\rangle = \int_\Omega \nabla v(x)\cdot\nabla w(x)\,dx - \int_\Omega f_t'(x,u(x))v(x)w(x)\,dx \quad \forall u,v,w \in X. \tag{4.8}$$

Thanks to (f_8), Lemma 2.1 can be applied. Thus, $u = 0$ is a non-degenerate critical point of φ with Morse index d_m. Now, the conclusion follows from (2.2). □

THEOREM 4.2. *Let* (f_5)–(f_9) *be satisfied. Then Problem* (4.1) *possesses at least three nontrivial solutions* $u_0 \in \mathrm{int}(C_+)$ *and* $u_1, u_2 \in C_0^1(\overline{\Omega})$.

PROOF. Theorem 4.1 directly gives the solutions $u_0 \in \mathrm{int}(C_+)$, $u_1 \in C_0^1(\overline{\Omega})\setminus\{0\}$. Through Theorem 3.1 we next infer

$$C_q(\varphi, u_0) = \delta_{q,0}\mathbb{Z}, \quad q \in \mathbb{N}_0; \tag{4.9}$$

see [14, Example 6.45]. The proof of Theorem 3.2 ensures that u_1 is a Mountain-Pass type critical point for φ. Hence, taking into account (4.8), Corollary 6.102 in [14] yields

$$C_q(\varphi, u_1) = \delta_{q,1}\mathbb{Z}, \quad q \in \mathbb{N}_0. \tag{4.10}$$

If the assertion were false then $K(\varphi) = \{0, u_0, u_1\}$. Lemmas 4.1–4.2, (4.9), (4.10), and Morse's relation (2.1) written for $t = -1$ would imply

$$(-1)^{d_m} + (-1)^0 + (-1)^1 = 0,$$

which is absurd. Therefore, there exists a further point $u_2 \in K(\varphi)\setminus\{0, u_0, u_1\}$. Standard regularity arguments lead to the conclusion. □

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