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Partial Differential Equations — *Multiplicity of solutions of nonlinear scalar field equations*, by Riccardo Molle and Donato Passaseo, communicated on 14 November 2014.

Abstract. — In this Note we present new multiplicity results for the solutions of nonlinear elliptic problems of the form $-\Delta u + a(x)u = |u|^{p-1}u$ in $\mathbb{R}^N$, $u \in H^1(\mathbb{R}^N)$, where $N \geq 2$, $p > 1$, $p < \frac{N+2}{N-2}$ if $N \geq 3$, $a \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} a > 0$. In particular, we have infinitely many positive solutions when there exists $a_\infty > 0$ such that $\lim_{|x|\to\infty} a(x) = a_\infty$ and $\lim_{|x|\to\infty}[a(x) - a_\infty]e^{\eta|x|} = +\infty\ \forall \eta > 0$.

Key words: Nonlinear scalar field equations, infinitely many solutions, variational methods.

Mathematics Subject Classification: 35J10, 35J20, 35J61.

1. Introduction and statement of the main result

In this Note we are concerned with existence and multiplicity of nontrivial solutions for nonlinear elliptic problems of the form

$$(1.1) \qquad -\Delta u + a(x)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

where $N \geq 2$, $p > 1$, $p < \frac{N+2}{N-2}$ if $N \geq 3$, $a \in L^{N/2}_{\rm loc}(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} a > 0$.

Because of the unboundedness of the domain, problem (1.1) lacks of compactness, the corresponding energy functional does not satisfy the well known Palais-Smale compactness condition and the classical variational methods cannot be applied in the usual way. A nonexistence result is proved in [4]: problem (1.1) has only the trivial solution $u \equiv 0$ when the potential $a(x)$ is increasing along a direction (see Theorem 1.1 in [4]).

If $a(x)$ has radial symmetry, the compactness is restored when we look for solutions in the subspace consisting of the functions having radial symmetry (or some other symmetric configuration). In particular, if $a(x)$ is a constant function, there exists a positive ground state solution, which is unique (up to translation) and has radial symmetry (see [3]).

If $\lim_{|x|\to\infty} a(x) = a_\infty > 0$, the Palais-Smale sequences may be described using the concentration-compactness principle (see [10]): if a Palais-Smale sequence is not relatively compact, then it differs from its weak limit by sequences of functions which, after translations, converge to a solution of the limit equation $-\Delta u + a_\infty u = |u|^{p-1}u$ in $\mathbb{R}^N$. This description of the Palais-Smale sequences has been used in several papers in order to avoid energy levels where the Palais-Smale condition fails and to obtain ground state solutions or solutions corresponding to higher critical values, under suitable assumptions on the behaviour of the potential $a(x)$ at infinity (see, for example, [1–3]).

More recently, the following result has been proved in [6]: if there exists $\eta < \sqrt{a_\infty}$ such that $\lim_{|x|\to\infty}[a(x) - a_\infty]e^{\eta|x|} = +\infty$, then problem (1.1) has infinitely many solutions provided $\sup_{x\in\mathbb{R}^N}\|a(x) - a_\infty\|_{L^{N/2}(B(x,1))}$ is small enough; more precisely, there exists a positive constant c (depending only on a_∞) such that, if $\sup_{x\in\mathbb{R}^N}\|a(x) - a_\infty\|_{L^{N/2}(B(x,1))} < c$, then $\forall k \in \mathbb{N}$ problem (1.1) has a k-bumps positive solution u_k. In [7] it is proved that, as $k \to \infty$, u_k converges to a positive solution of the equation $-\Delta u + a(x)u = |u|^{p-1}u$ in $\mathbb{R}^N$, having infinitely many bumps.

Multibump solutions are obtained also in [8, 9] without any smallness assumption on the oscillation $a(x) - a_\infty$: if $a(x)$ has a suitable polinomial decay and satisfies a suitable symmetry assumption, in [8] it is proved by variational methods that there exist infinitely many multibump positive solutions (with a sufficiently large number of bumps); in [9] a similar result is proved in the case $N = 2$ whitout requiring any symmetry assumption on $a(x)$.

Notice that the smallness condition on $a(x) - a_\infty$ used in [6, 7], the symmetry condition on $a(x)$ exploited in [8] and the assumption $N = 2$ in [9] play all the same role to localize the bumps in regions where $a(x) - a_\infty$ is small. In [5] we obtain infinitely many positive and nodal multibump solutions using (in place of these conditions) suitable arbitrarily small perturbations of the potential $a(x)$, which have the double role to localize all the bumps in far regions (where $a(x) - a_\infty$ is small) and to control the interactions between positive and negative bumps (which would tend to collapse).

We refer to [5–9] for a more detailed description of these problems, of their interest in Mathematical Physics and for more complete bibliographical references, concerning also some singularly perturbed problems and other related results obtained by different techniques (as Lyapunov-Schmidt reductions) under more restrictive assumptions on the behaviour of $a(x)$ as $|x| \to \infty$.

The main result presented in this Note allows us to remove the restriction on the dimension N, the symmetry assumption on $a(x)$ and the smallness condition on $a(x) - a_\infty$, still obtaining infinitely many positive solutions. In fact, using a variational method developed in [11–14] for the study of elliptic problems with jumping nonlinearities (and already applied in [5–7]), we can prove the following theorem.

THEOREM 1.1. *Let $N \geq 2$ and assume that there exists $a_\infty > 0$ such that $\lim_{|x|\to\infty} a(x) = a_\infty$ and $\lim_{|x|\to\infty}[a(x) - a_\infty]e^{\eta|x|} = +\infty$ $\forall \eta > 0$. Then problem* (1.1) *has infinitely many positive solutions. More precisely, there exists $\bar{k} \in \mathbb{N}$ such that for all $k \geq \bar{k}$ there exists a positive k-bumps solution u_k of* (1.1)*; moreover, for all $k \geq \bar{k}$ there exist k points $x_{k,1}, \dots, x_{k,k}$ in $\mathbb{R}^N$ such that*

$$\lim_{k\to\infty} \min\{|x_{k,i}| : i = 1, \dots, k\} = \infty, \tag{1.2}$$

$$\lim_{k\to\infty} \min\{|x_{k,i} - x_{k,j}| : i, j = 1, \dots, k, i \neq j\} = \infty, \tag{1.3}$$

$$\lim_{k\to\infty} \sup\{|u_k(x + x_{k,i}) - w(x)| : |x| \le R, i = 1, \dots, k\} = 0 \quad \forall R > 0, \tag{1.4}$$

where w is the positive radial solution of the equation $-\Delta u + a_\infty u = |u|^{p-1}u$ in $\mathbb{R}^N$.

Furthermore, there exists $\bar{R} > 0$ such that, for all $R \ge \bar{R}$ and $k \ge \bar{k}$,

$$\begin{aligned} \sup\left\{ u_k(x) : x \in \mathbb{R}^N \Big\backslash \bigcup_{i=1}^{k} B(x_{k,i}, R) \right\} \\ = \sup\left\{ u_k(x) : x \in \partial\Big(\bigcup_{i=1}^{k} B(x_{k,i}, R) \Big) \right\} \end{aligned} \tag{1.5}$$

(*so $u_k \to 0$, as $k \to \infty$, uniformly on the compact subsets of* $\mathbb{R}^N$).

In next session we describe the main steps of the proof of this theorem, which will appear in a paper in preparation, presented and proved in a more complete and detailed way.

2. Sketch of the proof of Theorem 1.1 and final remarks

The solutions are obtained as critical points of the energy functional $\mathscr{E} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\mathscr{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|Du|^2 + a(x)u^2)\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\, dx. \tag{2.1}$$

Let us consider a positive number δ, choose $R_\delta > 0$ (large enough) so that $w(x) < \delta\ \forall x \in \mathbb{R}^N \backslash B(0, R_\delta/2)$ and for all $k \ge 2$ consider the set

$$D_k = \{(x_1, \dots, x_k) \in (\mathbb{R}^N)^k : |x_i - x_j| \ge 3R_\delta \text{ for } i \neq j, i, j = 1, \dots, k\}. \tag{2.2}$$

For all $(x_1, \dots, x_k) \in D_k$, let us consider the set $S^\delta_{x_1,\dots,x_k}$ consisting of all the functions $u \in H^1(\mathbb{R}^N)$ satisfying the following conditions: $u \ge 0$ in $\mathbb{R}^N$, $u - u \wedge \delta = \sum_{i=1}^k v_i$ where, for all $i \in \{1, \dots, k\}$, $v_i \in H^1(\mathbb{R}^N)$, $v_i \not\equiv 0$, $v_i(x) = 0\ \forall x \notin B(x_i, R_\delta)$, $\mathscr{E}'(u)[v_i] = 0$ and $\left(\int_{\mathbb{R}^N} v_i^2\, dx\right)^{-1} \int_{\mathbb{R}^N} x v_i^2(x)\, dx = x_i$.

In [6] it is proved that (since $\inf_{\mathbb{R}^N} a > 0$ and $p > 1$) there exists $\delta > 0$, small enough, such that $S^\delta_{x_1,\dots,x_k} \neq \emptyset\ \forall (x_1, \dots, x_k) \in D_k$, and $\forall k \ge 2$; moreover, $\inf\{\mathscr{E}(u) : u \in S^\delta_{x_1,\dots,x_k}\} > 0$ and the infimum is achieved. If $\bar{u}$ is a minimizing function, then $\bar{u} > 0$ in $\mathbb{R}^N$, $\bar{u} < \delta$ in $\mathbb{R}^N \backslash \bigcup_{i=1}^k \bar{B}(x_i, R_\delta)$, $-\Delta\bar{u}(x) + a(x)\bar{u}(x) = \bar{u}^p(x)\ \forall x \in \mathbb{R}^N$ such that $\bar{u}(x) < \delta$ and there exist Lagrange multipliers $\lambda_i \in \mathbb{R}^N$, for $i \in \{1, \dots, k\}$, such that

$$\mathscr{E}'(\bar{u})[\psi] = \int_{B(x_i, R_\delta)} (\bar{u} - \bar{u} \wedge \delta)(x)\psi(x)[\lambda_i \cdot (x - x_i)]\, dx \quad \forall \psi \in H^1_0(B(x_i, R_\delta)), \tag{2.3}$$

namely, $\bar{u}$ is a weak solution of the equation

$$\begin{aligned}(2.4)\qquad -\Delta u(x) + a(x)u(x) = |u(x)|^{p-1}u(x) + v_i(x)[\lambda_i \cdot (x - x_i)] \\ \forall x \in B(x_i, R_\delta), \quad \forall i \in \{1, \dots, k\}.\end{aligned}$$

Thus, we can fix $\delta > 0$, $R_\delta > 0$ and then define $f : D_k \to \mathbb{R}^+$ by

$$(2.5)\qquad f(x_1, \dots, x_k) = \min\{\mathscr{E}(u) : u \in S^\delta_{x_1,\dots,x_k}\} \quad \forall (x_1, \dots, x_k) \in D_k,\ \forall k \geq 2.$$

One can verify that f is a continuous function and that the maximum

$$\begin{aligned}(2.6)\qquad g(\rho_1, \dots, \rho_k) = \max\{f(x_1, \dots, x_k) : (x_1, \dots, x_k) \in D_k, \\ |x_i| = \rho_i \text{ for } i = 1, \dots, k\}\end{aligned}$$

is achieved for all the k-tuples $(\rho_1, \dots, \rho_k)$ in $\mathbb{R}^k$ such that the set $\{(x_1, \dots, x_k) \in D_k : |x_i| = \rho_i \text{ for } i = 1, \dots, k\}$ (which is a bounded closed subset of $(\mathbb{R}^N)^k$) is non empty. Thus, if we denote by C_k the set of all these k-tuples $(\rho_1, \dots, \rho_k)$, we can consider the function $g : C_k \to \mathbb{R}$ defined by the maximum (2.6) $\forall (\rho_1, \dots, \rho_k) \in C_k$ (notice that C_k is a closed unbounded subset of $\mathbb{R}^N$, as one can easily verify).

Now, for suitable $\sigma > 0$ that we fix later, let us set

$$\begin{aligned}(2.7)\qquad \rho_k^\sigma = \inf\Bigg\{\rho > 0 : \sum_{i=1}^k \rho_i^2 = k\rho^2 \text{ for some } k\text{-tuple } (\rho_1, \dots, \rho_k) \in C_k \\ \text{and } \frac{\rho}{1 + 2\sigma} \leq \rho_i \leq (1 + 2\sigma)\rho \text{ for } i = 1, \dots, k\Bigg\}.\end{aligned}$$

Then, one can verify that $\rho_k^\sigma > 0$, that for every $\rho \geq \rho_k^\sigma$ the set

$$\begin{aligned}(2.8)\qquad \Bigg\{(\rho_1, \dots, \rho_k) \in C_k : \sum_{i=1}^k \rho_i^2 = k\rho^2 \text{ and } \frac{\rho}{1 + 2\sigma} \leq \rho_i \leq (1 + 2\sigma)\rho \\ \text{for } i = 1, \dots, k\Bigg\}\end{aligned}$$

(which is a bounded closed subset of $\mathbb{R}^N$) is non empty and that the minimum

$$\begin{aligned}(2.9)\qquad h_k^\sigma(\rho) = \min\Bigg\{g(\rho_1, \dots, \rho_k) : (\rho_1, \dots, \rho_k) \in C_k, \sum_{i=1}^k \rho_i^2 = k\rho^2, \\ \frac{\rho}{1 + 2\sigma} \leq \rho_i \leq (1 + 2\sigma)\rho\Bigg\}\end{aligned}$$

is achieved $\forall \rho \geq \rho_k^\sigma$. Thus, we can consider the function $h_k^\sigma : [\rho_k^\sigma, +\infty) \to \mathbb{R}$ defined by the minimum (2.9) $\forall \rho \geq \rho_k^\sigma$.

Taking into account the behaviour of $a(x)$ as $|x| \to \infty$, it follows that $h_k^\sigma(r) > \lim_{\rho\to+\infty} h_k^\sigma(\rho)$ for $r > 0$ large enough, so h_k^σ achieves its maximum. Thus, for all $k \geq 2$, there exists $(x_{k,1}, \dots, x_{k,k}) \in D_k$ and $u_k \in S^\delta_{x_{k,1},\dots,x_{k,k}}$ such that $\frac{r_k}{1+2\sigma} \leq \rho_{k,i} \leq (1+2\sigma) r_k$, where $\rho_{k,i} = |x_{k,i}|$ for $i = 1, \dots, k$ and $r_k = \left(\frac{1}{k}\sum_{i=1}^k \rho_{k,i}^2\right)^{\frac{1}{2}}$, $\mathscr{E}(u_k) = f(x_{k,1}, \dots, x_{k,k}) = g(\rho_{k,1}, \dots, \rho_{k,k}) = h_k^\sigma(r_k)$ and $h_k^\sigma(r_k)$ is the maximum of h_k^σ.

We say that there exists $\bar{k} \in \mathbb{N}$ such that $-\Delta u_k + a(x) u_k = u_k^p$ in $\mathbb{R}^N$ $\forall k \geq \bar{k}$. The proof is based on the following steps.

Step 1. Since the balls $B(x_{k,i}, R_\delta)$, for $i = 1, \dots, k$, are pairwise disjoint, we have $\lim_{k\to\infty} r_k = +\infty$ (which obviously implies (1.2)). Moreover, (1.3), (1.4) and (1.5) may be proved arguing as in [6] and [7].

Step 2. We have $\limsup_{k\to\infty} \sup\{u_k(x) : x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k B(x_{k,i}, R_\delta/2)\} < \delta$. Therefore (as the condition $u \geq 0$) the unilateral constraint $u(x) \leq \delta$ $\forall x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k B(x_{k,i}, R_\delta)$, we used to define the set $S^\delta_{x_1,\dots,x_k}$, does not give rise to any variational inequality. A similar argument holds for the unilateral constraint $\frac{\rho}{1+2\sigma} \leq |x_i| \leq (1+2\sigma)\rho$, we used to define $h_k^\sigma(\rho)$. In fact, there exists $\sigma > 0$ (small enough) such that

$$\frac{1}{1+\sigma} \leq \liminf_{k\to\infty} \frac{1}{r_k} \min\{|x_{k,i}| : i = 1, \dots, k\} \tag{2.10}$$

$$\limsup_{k\to\infty} \frac{1}{r_k} \max\{|x_{k,i}| : i = 1, \dots, k\} \leq 1 + \sigma. \tag{2.11}$$

The formulas (2.10) and (2.11) play a crucial role in the proof of Theorem 1.1. Roughly speaking, they are true because the interaction between the bumps of u_k is attractive and the function h_k^σ is defined by the minimum (2.9) (so we have a contradiction if we assume that (2.10) or (2.11) are not true for σ small).

Notice that, exploiting again the attractive interaction between the bumps of u_k, we obtain also $h_k^\sigma(\rho_k^\sigma) < \max\{h_k^\sigma(\rho) : \rho \geq \rho_k^\sigma\}$, for k large enough, which implies $r_k > \rho_k^\sigma$.

Step 3. Let us denote by $\lambda_{k,1}, \dots, \lambda_{k,k}$ the Lagrange multipliers corresponding to the minimizing function u_k in $S^\delta_{x_{k,1},\dots,x_{k,k}}$. It remains to show that, for k large enough, $\lambda_{k,i} = 0$ $\forall i \in \{1, \dots, k\}$. Arguing as in [6], from $f(x_{k,1}, \dots, x_{k,k}) = g(\rho_{k,1}, \dots, \rho_{k,k})$ it follows that, for k large enough, $\lambda_{k,i} - (\lambda_{k,i} \cdot x_{k,i}) \frac{x_{k,i}}{|x_{k,i}|^2} = 0$ $\forall i \in \{1, \dots, k\}$; since $g(\rho_{k,1}, \dots, \rho_{k,k}) = h_k^\sigma(r_k)$, we infer that there exists a Lagrange multiplier $\mu_k \in \mathbb{R}$ such that $\lambda_{k,i} = \mu_k x_{k,i}$ $\forall i \in \{1, \dots, k\}$; finally, we obtain $\mu_k = 0$ because $h_k^\sigma(r_k)$ is the maximum of h_k^σ and $r_k > \rho_k^\sigma$. Thus, we get $\mathscr{E}'(u_k) = 0$ for k large enough and all the other assertions of Theorem 1.1 follow now by standard arguments.

Remark 2.1. If in Theorem 1.1 we assume in addition that $a(x)$ has radial symmetry, it is natural to expect that the k-bumps of the solution u_k are distributed near $(N-1)$-dimensional spheres. However, our method can be adapted to construct also infinitely many positive solutions with bumps distributed on circles. In fact, for all $k \geq 2$ and for $\rho > 0$ large enough, consider the point $(x_{k,1}(\rho), \dots, x_{k,k}(\rho)) \in D_k$ such that $x_{k,i}(\rho) = (\rho \cos \frac{2\pi i}{k}, \rho \sin \frac{2\pi i}{k}, 0, \dots, 0)$ for $i = 1, \dots, k$ and set $\chi_k(\rho) = f(x_{k,1}(\rho), \dots, x_{k,k}(\rho))$. Then, as in the proof of Theorem 1.1, there exists $\bar{r}_k > 0$ such that $\chi_k(\bar{r}_k)$ is the maximum of χ_k and, for k large enough, every minimizing function for the energy functional $\mathscr{E}$ in $S^\delta_{x_{k,1}(\bar{r}_k), \dots, x_{k,k}(\bar{r}_k)}$ is a positive k-bumps solution of problem (1.1).

More in general, we can construct infinitely many solutions with bumps distributed near d-dimensional spheres for every integer d such that $1 \leq d \leq N-1$. In fact, consider the sphere $S^d = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x| = 1, x_i = 0 \ \forall i \geq d+2\}$. Then, arguing as in the proof of Theorem 1.1, we infer that $\forall k \geq 2$ there exist $\bar{\theta}_{k,1}, \dots, \bar{\theta}_{k,k}$ in S^d and $\bar{\rho}_{k_1}, \dots, \bar{\rho}_{k_k}$ in $\mathbb{R}^+$ such that $\lim_{k\to\infty} \min\{\bar{\rho}_{k_i} : i = 1, \dots, k\} = +\infty$, $\lim_{k\to\infty}(\max\{\bar{\rho}_{k_i} : i = 1, \dots, k\}/\min\{\bar{\rho}_{k_i} : i = 1, \dots, k\}) = 1$ and, for k large enough, every minimizing function for $\mathscr{E}$ in $S^\delta_{\bar{\rho}_{k_1}\bar{\theta}_{k_1}, \dots, \bar{\rho}_{k_k}\bar{\theta}_{k_k}}$ is a positive k-bumps solution of problem (1.1).

Remark 2.2. Notice that the method used to prove Theorem 1.1 may be also applied to construct a sequence $(\hat{u}_n)_n$ of positive solutions of problem (1.1) which, unlike the sequence $(u_k)_{k \geq \bar{k}}$ given by Theorem 1.1, converges in $H^1_{\text{loc}}(\mathbb{R}^N)$ to a positive solution $\hat{u}$ of the equation $-\Delta u + a(x)u = |u|^{p-1}u$ in $\mathbb{R}^N$, having infinitely many bumps (while the sequence $(u_k)_{k \geq \bar{k}}$ converges to the trivial solution $u \equiv 0$). The solution $\hat{u}$ obtained in this way presents k_1 bumps localized near a sphere $\partial B(0, r_1)$, k_2 bumps near a sphere $\partial B(0, r_2)$ and so on, where $(k_n)_n$ and $(r_n)_n$ are suitable increasing sequences in $\mathbb{N}$ and $\mathbb{R}^+$ respectively, with $\lim_{n\to+\infty}(k_n - k_{n-1}) = \lim_{n\to+\infty}(r_n - r_{n-1}) = +\infty$.

In fact, arguing as in the proof of Theorem 1.1, one obtain a solution $\hat{u}_1$ with k_1 bumps near a sphere $\partial B(0, r_1)$ then, using similar arguments, one can construct a solution $\hat{u}_2$ with k_1 bumps near $\partial B(0, r_1)$ and k_2 bumps near $\partial B(0, r_2)$, for suitable $k_2 \in \mathbb{N}$, $r_2 > 0$ large enough, and then one can iterate this procedure. Thus we obtain a sequence $(\hat{u}_n)_n$ with the desired properties.

Notice that, at every step, we can choose the positive numbers k_n in a quite arbitrary way (provided large enough). Therefore, we can say also that there exist infinitely many positive solutions of the equation $-\Delta u + a(x)u = |u|^{p-1}u$ in $\mathbb{R}^N$ having infinitely many bumps (while the result obtained in [7] guarantees only the existence of one solution with this property under the additional assumption that the oscillation of $a(x) - a_\infty$ is small enough in $\mathbb{R}^N$).

Remark 2.3. Unlike the results proved in [6, 7], Theorem 1.1 does not require $\sup_{x\in\mathbb{R}^N} \|a(x) - a_\infty\|_{L^{N/2}(B(x,1))}$ to be small and, indeed, it may be arbitrarily large. For example, let Ω be a bounded domain of $\mathbb{R}^N$ and $a_n(x) = n\bar{a}(x) + a(x)$ $\forall x \in \mathbb{R}^N$, with $a(x)$ as in Theorem 1.1 and $\bar{a}(x)$ such that $\bar{a}(x) > 0$ $\forall x \in \Omega$, $\bar{a}(x) = 0$ $\forall x \notin \Omega$, $\int_\Omega \bar{a}(x)^{N/2} dx < +\infty$; then, there exists $\bar{k}$, independent of n, such that $\forall k \geq \bar{k}$ and $\forall n \in \mathbb{N}$ there exists a positive k-bumps solution $u_{k,n}$ of the equation $-\Delta u + a_n(x)u = u^p$ in $\mathbb{R}^N$; moreover, as $n \to \infty$, $u_{k,n} \to \tilde{u}_k$ in $H^1(\mathbb{R}^N)$ where $\tilde{u}_k \equiv 0$ in Ω while it is a positive k-bumps solution of the equation $-\Delta u + a(x)u = u^p$ in the exterior domain $\tilde{\Omega} := \mathbb{R}^N \backslash \bar{\Omega}$, with zero Dirichlet boundary condition (on the other hand, the solutions $\tilde{u}_k$ in $\tilde{\Omega}$ may be also obtained directly since our method may be easily adapted to deal with Dirichlet problems in exterior domains).

Remark 2.4. Let us point out that the method we use to prove Theorem 1.1 may be adapted to deal also with the case of potentials $a(x)$ non regular at infinity: for example, when the limit $a_\infty(\theta) = \lim_{\rho \to +\infty} a(\rho\theta)$ exists for all $\theta \in S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ but it depends on θ.

Notice that if $a_\infty(\theta)$ is allowed to be a nonconstant function of θ, the Palais-Smale condition may even fail at every level where there exist Palais-Smale sequences; moreover, we may have $\frac{\partial a}{\partial \theta} > 0$ in $\mathbb{R}^N$ for a suitable $\theta \in S^{N-1}$ and, in this case, $u \equiv 0$ is the unique solution of problem (1.1) because of Theorem 1.1 in [4]. But, if we assume that $\sup\{a_\infty(\theta) : \theta \in S^{N-1}\} < +\infty$, and

$$\text{(2.12)} \qquad \lim_{|x| \to \infty} \left[a(x) - a_\infty\left(\frac{x}{|x|}\right) \right] = 0,$$

$$\lim_{|x| \to \infty} \left[a(x) - a_\infty\left(\frac{x}{|x|}\right) \right] e^{\eta |x|} = +\infty \quad \forall \eta > 0,$$

then our method still works and allows us to obtain infinitely many positive solutions as in Theorem 1.1. More precisely, for every integer k large enough, we obtain a k-bumps positive solution u_k, having the same properties as in Theorem 1.1. In addition, our method gives more information about the asymptotic behaviour, as $k \to +\infty$, of the centers $x_{k,1}, \dots, x_{k,k}$ of the bumps. For example, if $a_\infty(\theta)$ depends continuously on θ, we obtain

$$\text{(2.13)} \quad \lim_{k \to \infty} \min\left\{ a_\infty\left(\frac{x_{k,i}}{|x_{k,i}|}\right) : i = 1, \dots, k \right\} = \max\{a_\infty(\theta) : \theta \in S^{N-1}\};$$

moreover if the set $M_\infty = \{\theta \in S^{N-1} : a_\infty(\theta) = \max_{S^{N-1}} a_\infty\}$ has more than one connected component, then for k large enough we can construct k-bumps solutions u_k, with $\frac{x_{k,1}}{|x_{k,1}|}, \dots, \frac{x_{k,k}}{|x_{k,k}|}$ localized near prescribed connected components of M_∞. Finally, let us point out that, even if the potential $a(x)$ is non regular at infinity, arguing as in Remark 2.2 we can construct sequences $(\hat{u}_n)_n$ of positive solutions of problem (1.1) that converge in $H^1_{\text{loc}}(\mathbb{R}^N)$ to positive solutions of the equation $-\Delta u + a(x)u = |u|^{p-1}u$ in $\mathbb{R}^N$ having infinitely many bumps. Moreover, we can say also that there exists infinitely many positive solutions having infinitely many bumps.

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