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**Probability Theory** — *Sets of admissible shifts of convex measures*, by Lavrentin M. Arutyunyan and Egor D. Kosov, communicated on 14 November 2014.

Abstract. — We prove that for any convex probability measure on a linear space the set of its non-singular shifts is convex and the set of its equivalent shifts is a linear subspace.

Key words: Convex measure, logarithmically concave measure, admissible shift.

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We study the sets of non-singular and equivalent shifts for convex measures on locally convex spaces. It is well known that for a Radon Gaussian measure (see [1]) these sets are linear and coincide with the Cameron–Martin space, for example, for the countable power of the standard Gaussian measure on the real line this is the usual $l^2$. For a general measure, these sets need not be convex. We prove that for a Radon convex measure, the set of its equivalent shifts is always a linear space and the set of its non-singular shifts is always convex.

Let us recall some concepts and notation. A Borel probability measure $\mu$ on a locally convex space $X$ is called Radon if for every Borel set $A$ and for every $\varepsilon > 0$ there is a compact set $K \subset A$ such that $\mu(A \backslash K) < \varepsilon$. In addition to the Borel $\sigma$-field $\mathscr{B}(X)$ we shall also need the cylindric $\sigma$-field $\sigma(X)$ of $X$, i.e., the smallest $\sigma$-field with respect to which all continuous linear functionals on $X$ are measurable. In the case of a separable Banach space the two $\sigma$-fields coincide.

Let $\mu$ and $\nu$ be two probability measures absolutely continuous with respect to a positive measure $\lambda$, i.e. $\mu = \varphi \cdot \lambda$, $\nu = \psi \cdot \lambda$. The number

$$H(\mu, \nu) = \int \sqrt{\varphi\psi}\, d\lambda$$

is called the Hellinger integral of this pair of measures. It is independent of our choice of a measure $\lambda$ and the following estimate is true (see [2, Theorem 4.7.37]):

$$2(1 - H(\mu, \nu)) \leq \int |\varphi - \psi|\, d\lambda = \|\mu - \nu\| \leq 2\sqrt{1 - H^2(\mu, \nu)},$$

where $\|\cdot\|$ denotes the total variation norm. The latter estimate shows that $\mu \perp \nu$ is equivalent to $H(\mu, \nu) = 0$, where $\perp$ denotes the mutual singularity relation between measures. The equivalence of $\mu$ and $\nu$ is denoted by $\mu \sim \nu$.

A probability measure $\mu$ on the $\sigma$-field $\sigma(X)$ is called convex (or logarithmically concave) if

$$\mu(tA + (1-t)B) \geq (\mu(A))^t(\mu(B))^{1-t}$$

for all $t \in [0,1]$ and $A, B \in \sigma(X)$. The convexity of a Radon probability measure is defined as the convexity of its restriction to the $\sigma$-field $\sigma(X)$. It is easily seen from the definition that the topological support of a Radon convex measure, i.e., the smallest closed set of full measure, is convex.

Another equivalent definition can be formulated in terms of finite-dimensional projections (see [5]). A probability measure on $\mathbb{R}^n$ is called convex if it is defined by a density of the form $e^{-V}$ with respect to Lebesgue measure on some affine subspace $L$, where $V : L \to (-\infty, \infty]$ is a convex function on this subspace. A Radon probability measure $\mu$ on a locally convex space $X$ is convex if and only if for every continuous linear operator $P : X \to \mathbb{R}^n$ the measure $\mu \circ P^{-1}$ is convex on $\mathbb{R}^n$, where $\mu \circ P^{-1}(A) = \mu(P^{-1}(A))$.

Let $\mu$ be a Borel measure on a locally convex space. Let $\mu_h$ be the shift of the measure $\mu$ by the vector $h$, i.e., the measure defined by the formula

$$\mu_h(A) = \mu(A-h), \quad A \in \mathscr{B}(X).$$

Let $M(\mu)$ denote the set of all its non-singular shifts and let $\tilde{M}(\mu) \subset M(\mu)$ be the set of all its equivalent shifts, i.e.,

$$M(\mu) = \{h : \mu \not\perp \mu_h\},$$
$$\tilde{M}(\mu) = \{h : \mu \sim \mu_h\}.$$

In relation to these sets let us also mention the subspace of continuity $C(\mu)$ of $\mu$ consisting of all vectors $h$ such that $\lim_{t\to 0}\|\mu - \mu_{th}\| = 0$. The subspace of quasiinvariance $Q(\mu)$ is the set of all vectors $h$ such that $\mu_{th} \sim \mu$ for all $t$. Clearly, $Q(\mu) \subset \tilde{M}(\mu)$, but the inclusion may be strict. An important difference is that $Q(\mu)$ is always a linear subspace, moreover, $Q(\mu) \subset C(\mu)$ and $C(\mu)$ is also linear (see [3, Chapter 5], [4]). Some difference between these sets occurs already in the one-dimensional case: if $\mu = \sum_n 2^{-|n|-1}\delta_n$, where summation is taken over all integer numbers $n$ and $\delta_n$ is Dirac's measure at $n$, then $C(\mu) = Q(\mu) = \{0\}$, $M(\mu) = \tilde{M}(\mu) = \mathbb{Z}$, for Lebesgue measure $\lambda$ on the interval $[0,1]$ regarded as a measure on $\mathbb{R}$ we have $C(\lambda) = \mathbb{R}$, $Q(\lambda) = \tilde{M}(\lambda) = \{0\}$, $M(\lambda) = (-1,1)$.

In infinite dimensions, some more subtle phenomena take place (see [7], [12]).

Another important set is the set $D_C(\mu)$ of all vectors $h$ such that the measure $\mu$ has a Skorohod derivative $d_h\mu$ along $h$. The latter is defined by means of the integration by parts formula

$$\int_X \partial_h f(x)\mu(dx) = -\int_X f(x)\, d_h\mu(dx)$$

for all functions $f$ of the form $f(x) = \varphi(l_1(x), \ldots, l_n(x))$, where $\varphi \in C_b^\infty(\mathbb{R}^n)$ and $l_i \in X^*$. The Skorohod derivative $d_h\mu$ can be equivalently defined as a limit of

$(\mu_{th}-\mu)/t$ in the weak topology as $t\to 0$; its existence is equivalent to the estimate $\|\mu_{th}-\mu\|\le C|t|$ with some constant $C$. The set $D_C(\mu)$ is a linear subspace of $C(\mu)$. For a convex measure $\mu$ given by a density on $\mathbb{R}^n$, the sets $C(\mu)$ and $D_C(\mu)$ coincide with $\mathbb{R}^n$ (for $D_C(\mu)$ this is not trivial and was first shown by Krugova [8]). For any convex measure $\mu$ (in finite or infinite dimensions) there is the following useful inequality due Krugova [9]:

$$\|\mu_{th}-\mu\|\ge 2-2e^{-\frac{1}{2}\|d_h\mu\|},$$

which in the case where $d_h\mu$ does not exist asserts that $\mu$ and $\mu_h$ are mutually singular. Therefore, for any convex measure $\mu$ we have

$$M(\mu)\subset C(\mu)=D_C(\mu).$$

However, it can happen that $M(\mu)$ is not linear: for example, for Lebesgue measure $\lambda$ on $[0,1]$ regarded as a measure on the real line we have $M(\lambda)=(-1,1)$. More generally, if $\mu$ is an absolutely continuous convex measure on $\mathbb{R}^d$, then its topological support is a convex set $U$ with a nonempty interior $W$ and then $M(\mu)=W-W$. Our result says that $M(\mu)$ is a convex subset in $C(\mu)$ and that $\tilde{M}(\mu)=Q(\mu)$.

It is worth noting that in the general case $C(\mu)$ is complete with respect to the distance $d(a,b)=\sup_{|t|\le 1}\|\mu_{ta}-\mu_{tb}\|$ that is consistent with the vector structure, but a similar distance $d_0(a,b)=\|\mu_a-\mu_b\|$ may fail to be consistent with the vector structure even on $Q(\mu)$ (see [12]). However, it follows by the Krugova inequality that for a convex measure $\mu$ the distance $d_0$ defines the same topology as the distance $d$. Indeed, if $d_0(h_n,h)\to 0$, then by this inequality $\|d_{h-h_n}\mu\|\to 0$, whence it follows that

$$\|\mu_{th}-\mu_{th_n}\|\le |t|\,\|d_{h-h_n}\mu\|\le \|d_{h-h_n}\mu\|\to 0$$

uniformly in $t\in[-1,1]$.

It is readily verified that for any Radon measure $\mu$ the sets introduced above do not change if we consider $\mu$ on $\sigma(X)$ in place of $\mathscr{B}(X)$.

It is unknown whether $M(\mu)\ne\{0\}$ for any non Dirac convex measure.

THEOREM 1. *Let $\mu$ be a convex Radon measure on a locally convex space $X$. Then the function $H(h)=H(\mu,\mu_h)$ is logarithmically concave, i.e.,*

$$H(th+(1-t)q)\ge H^t(h)H^{1-t}(q)\quad \forall t\in[0,1].$$

*It follows that $M(\mu)$ is a convex set.*

PROOF. We first consider the case of a finite-dimensional space. In this case, the measure $\mu$ is absolutely continuous with respect to the standard Lebesgue measure on some affine subspace $L+v$, where $L$ is a linear subspace and $v$ is a vector. If $h$ is not in $L$, then $H(h)=0$ and the inequality is obvious. So, we assume that $h\in L$ and, passing to the subspace $L$, we can assume that $\mu$ has a density with respect to Lebesgue measure on the whole space, i.e., $\mu = e^{-V}\,dx$, where $V$ is a convex function. In this case we have

$$H(h) = \int e^{-(V(x)+V(x+h))/2}\,dx.$$

Let us recall the following Prékopa–Leindler inequality (see, e.g., [10], [11] or [2, Theorem 3.10.21]): if

$$f(tx + (1-t)y) \geq (\varphi(x))^t (\psi(y))^{1-t},$$

then

$$\int f(x)\,dx \geq \Bigl(\int \varphi(x)\,dx\Bigr)^t \Bigl(\int \psi(x)\,dx\Bigr)^{1-t}.$$

Applying this inequality to the functions

$$f(x) = \varphi(x) = \psi(x) = e^{-(V(x)+V(x+h))/2},$$

we immediately obtain the desired assertion.

Let us proceed to the infinite-dimensional case. Let $\lambda$ be a Radon probability measure such that $\mu = \varrho \cdot \lambda$, $\mu_h = \varrho_h \cdot \lambda$, e.g., $\lambda = (\mu + \mu_h)/2$. Note that the functions $\varrho$ and $\varrho_h$ can be chosen measurable with respect to the $\sigma$-field $\sigma(X)$ (see [1, Corollary A.3.13]). Then, there exists a $\sigma$-field generated by countably many continuous linear functionals $\{\ell_i\}$ such that $\varrho_h$ and $\varrho$ are measurable with respect to it (see [1, Lemma 2.1.2]). Let $\varrho^n$ and $\varrho_h^n$ be the conditional expectations of the functions $\varrho$ and $\varrho_h$, correspondingly, with respect to the measure $\lambda$ and the $\sigma$-field generated by the functionals $\{\ell_i\}_{i=1}^n$. Then the functions $\varrho^n$ and $\varrho_h^n$ converge in $L^1(\lambda)$ to the functions $\varrho$ and $\varrho_h$, respectively. Therefore, we have convergence $\sqrt{\varrho^n} \to \sqrt{\varrho}$, $\sqrt{\varrho_h^n} \to \sqrt{\varrho_h}$ in $L^2(\lambda)$. Let us define continuous linear operators $P_n : X \to \mathbb{R}^n$ by the formula $x \mapsto (l_1(x), \dots, l_n(x))$. Obviously,

$$H(h) = \lim_{n\to\infty} \int \sqrt{\varrho^n \varrho_h^n}\,d\lambda.$$

On the other hand,

$$\int \sqrt{\varrho^n \varrho_h^n}\,d\lambda = H(\mu \circ P_n^{-1}, \mu_h \circ P_n^{-1}).$$

It follows from the finite-dimensional case that the function

$$H(\mu \circ P_n^{-1}, \mu_h \circ P_n^{-1}) = H(\mu \circ P_n^{-1}, (\mu \circ P_n^{-1})_{P_n(h)})$$

is logarithmically concave. Hence the same is true for the function $H(h)$ that is a limit of logarithmically concave functions. □

Let $\mu$ be a Radon measure on a locally convex space $X$, $h \in X$, $\ell \in X^*$, $\ell(h) = 1$ and $S = \ker \ell$. Then any vector $x$ can be uniquely represented in the form $\alpha h + z$, $z \in S$. Let us consider the sections of the set $A$ by the lines parallel to $h$:

$$A^z = \{\alpha \in \mathbb{R} : z + \alpha h \in A\}, \quad A \in \mathscr{B}(X).$$

Let $\mu^{\{z\}}$ denote the conditional measures of the measure $\mu$ associated with $\ell$ (see [2, Chapter 10]), i.e.,

$$\mu(A) = \int_S \mu^{\{z\}}(A^z)\mu \circ \pi^{-1}(dz),$$

where $\pi(x) = x - \ell(x)h$.

THEOREM 2. *If $\mu$ is a convex Radon measure on a locally convex space $X$, then $\tilde{M}(\mu)$ is a linear subspace in $X$, hence $\tilde{M}(\mu) = Q(\mu)$.*

PROOF. It is known that for $\mu \circ \pi^{-1}$-a.e. $z$ the measure $\mu^{\{z\}}$ is convex (see [6] or [3, Theorem 4.3.6]). Let $h \in \tilde{M}(\mu)$. Let us show that for $\mu \circ \pi^{-1}$-a.e. point $z$ in the subspace $S$, the measure $\mu^{\{z\}}$ on the real line has full support. Suppose the opposite, i.e., there is a compact set $B$ such that $\mu \circ \pi^{-1}(B) > 0$ and for every $z \in B$ we have

$$\mu^{\{z\}}([a_z, b_z]) = 1,$$

where the interval $[a_z, b_z]$ is the support of the measure $\mu^{\{z\}}$ (possibly unbounded or degenerate, i.e., $a_z = b_z$). Let us note that for every $z \in B$

$$\mu^{\{z\}}((b_z - 1, b_z]) > 0.$$

Let us consider the set

$$A = \{x : x = \alpha h + z, z \in B, b_z < \alpha \le b_z + 1\}.$$

It is easy to see that $\mu(A) = 0$ and $\mu_h(A) > 0$. Indeed, for almost every $z$ we have $(\mu_h)^{\{z\}} = (\mu^{\{z\}})_1$, which is an easy consequence of the a.e. uniqueness of conditional measures. Therefore, this contradicts the equivalence of the measures $\mu$ and $\mu_h$ and shows that the supports $[a_z, b_z]$ are unbounded from above; the unboundedness from below is proved similarly. So, $\mu \sim \mu_{th}$ for every $t$, because

$$(\mu_{th})^{\{z\}} = (\mu^{\{z\}})_t \sim \mu^{\{z\}},$$

i.e., $th \in \tilde{M}(\mu)$ for every $t \in \mathbb{R}$ and every $h \in \tilde{M}(\mu)$. The fact that $h + q \in \tilde{M}(\mu)$ for all $h, q \in \tilde{M}(\mu)$ follows immediately from the definition of the equivalence of measures. □

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