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Partial Differential Equations — *The Neumann eigenvalue problem for the ∞-Laplacian*, by L. Esposito, B. Kawohl, C. Nitsch and C. Trombetti, communicated on 12 December 2014.

Abstract. — The first nontrivial eigenfunction of the Neumann eigenvalue problem for the p-Laplacian, suitably normalized, converges to a viscosity solution of an eigenvalue problem for the ∞-Laplacian as $p \to \infty$. We show among other things that the limiting eigenvalue, at least for convex sets, is in fact the first nonzero eigenvalue of the limiting problem. We then derive a number of consequences, which are nonlinear analogues of well-known inequalities for the linear (2-)Laplacian.

Key words: Neumann eigenvalues, viscosity solutions, infinity Laplacian.

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1. Introduction and statements

In this paper we study the ∞-Laplacian eigenvalue problem under Neumann boundary conditions

$$\begin{cases} \min\{|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u > 0\} \cap \Omega \\ \max\{-|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u < 0\} \cap \Omega \\ -\Delta_\infty u = -\sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j} = 0 & \text{in } \{u = 0\} \cap \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

A solution u to this problem has to be understood in the viscosity sense, and the Neumann eigenvalue Λ is some nonnegative real constant. For $\Lambda = 0$ problem (1) has constant solutions. We consider those as trivial. Our main result is

Theorem 1. *Let Ω be a smooth bounded open convex set in $\mathbb{R}^n$ then a necessary condition for the existence of nonconstant continuous solutions u to* (1) *is*

$$\Lambda \geq \Lambda_\infty := \frac{2}{\operatorname{diam}(\Omega)}. \tag{2}$$

Moreover problem (1) *admits a Lipschitz solution when $\Lambda = \frac{2}{\operatorname{diam}(\Omega)}$.*

The results of this paper were presented on Sept. 09, 2014 by the second author in a seminar with the same title at Accademia Nazionale dei Lincei.

If Ω is merely bounded, connected and has Lipschitz boundary, then the notion of diameter can be generalized as in Definition 1. In that case solutions of (1) exist, see Section 2 or [16]. However, it is still unclear whether Λ_∞ is always the first eigenvalue.

Theorem 1 has a number of interesting consequences, one of which we list right here. By the isodiametric inequality we may conclude

COROLLARY 1. *If Ω^* denotes the ball of same volume as Ω, then the Szegö-Weinberger inequality $\Lambda_\infty(\Omega) \leq \Lambda_\infty(\Omega^*)$ holds.*

For the case of the ordinary Laplacian ($p = 2$) this result was shown in [17] and [19]. For the 1-Laplacian case and convex plane Ω we refer to [9]. While the Faber-Krahn inequality $\lambda_p(\Omega^*) \leq \lambda_p(\Omega)$ holds for any p, the Szegö-Weinberger inequality has resisted attempts to be generalized to general p, and for general p we are unaware of any results in this direction. The reason why we call problem (1) ∞-Laplacian eigenvalue problem under Neumann boundary conditions is that (1) can be derived as the limit $p \to \infty$ of Neumann eigenvalue problems for the p-Laplacian

$$\begin{cases} -\Delta_p u = \Lambda_p^p |u|^{p-2} u & \text{in } \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

whenever Ω is a bounded open Lipschitz set of $\mathbb{R}^n$.

For the Dirichlet p-Laplacian eigenvalue problem on open bounded sets $\Omega \subset \mathbb{R}^n$

$$\begin{cases} -\Delta_p v = \lambda_p^p |v|^{p-2} v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

the same limit was studied by Juutinen, Lindqvist and Manfredi in [12, 13]. They formulate and fully investigate the so-called Dirichlet ∞-Laplacian eigenvalue problem employing the notion of viscosity solutions. Recall for instance that, when λ_p denotes for all $p \geq 1$ the first nontrivial eigenvalue of (4), the limit yields

$$\lim_{p \to \infty} \lambda_p = \lambda_\infty := \frac{1}{R(\Omega)},$$

where $R(\Omega)$ denotes inradius, i.e. the radius of the largest ball contained in Ω. Moreover, they identify the limiting eigenvalue problem as

$$\begin{cases} \min\{|\nabla v| - \lambda v, -\Delta_\infty v\} = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

in the sense that nonnegative normalized eigenfunctions of (4) converge, up to a subsequence, to a positive Lipschitz function v_∞ which solves (5) in the viscosity sense with $\lambda(\Omega) = \lambda_\infty(\Omega)$. Finally they also show that the infinity Laplacian eigenvalue problem (5) admits nontrivial solutions if and only if $\lambda \geq \lambda_\infty$ and positive solutions if and only if $\lambda = \lambda_\infty$. Therefore they call λ_∞ the principal eigenvalue of the ∞-Laplacian eigenvalue problem under Dirichlet boundary condition.

In the Neumann case (see [16]) and for any bounded connected Ω with Lipschitz boundary the limiting problem $p \to \infty$ for (3) is given by (1).

In analogy to the Dirichlet case, the first nontrivial eigenvalues of (3) satisfy

$$\lim_{p\to\infty} \Lambda_p = \Lambda_\infty. \tag{6}$$

Our result proves that on the class of convex sets the first nontrivial Neumann p-Laplacian eigenvalues converge to the first nontrivial Neumann ∞-Laplacian eigenvalue, namely $\Lambda = \Lambda_\infty$ is in fact the first nontrivial eigenvalue in (1).

Therefore we can point out some consequences.

COROLLARY 2. *For convex Ω the first positive Neumann eigenvalue $\Lambda_\infty(\Omega)$ is never larger than the first Dirichlet eigenvalue $\lambda_\infty(\Omega)$. Moreover $\lambda_\infty(\Omega) = \Lambda_\infty(\Omega)$ if and only if Ω is a ball.*

The inequality $\Lambda_2(\Omega) < \lambda_2(\Omega)$ follows from a combination of the Szegö-Weinberger and the Faber-Krahn inequalities, see e.g. the books by Bandle or Kesavan [3, 14]. The strict inequality $\Lambda_p(\Omega) < \lambda_p(\Omega)$ for general p and any convex Ω has been recently proved in [2].

COROLLARY 3. *For convex Ω any Neumann eigenfunction associated with $\Lambda_\infty(\Omega)$ cannot have a closed nodal domain inside Ω.*

Since a Neumann eigenfunction u for the ∞-Laplacian is in general just continuous, a closed nodal line inside Ω means that there exists an open subset $\Omega' \subset \Omega$ such that $u > 0$ in Ω' (or < 0 in Ω') and $u = 0$ on $\partial\Omega'$. Assuming that such a nodal line exists, we can use standard arguments. We observe that u is also a Dirichlet eigenfunction on Ω' with same eigenvalue. We get $\frac{2}{\mathrm{diam}(\Omega)} = \Lambda_\infty(\Omega) = \lambda_\infty(\Omega') = \frac{1}{R(\Omega')} \geq \frac{2}{\mathrm{diam}(\Omega)}$ and notice that the last inequality is strict for all sets other than balls. This proves the Corollary.

Next we recall that the Payne-Weinberger inequality states that on any convex subset $\Omega \subset \mathbb{R}^n$ the first nontrivial Neumann eigenvalue for the Laplacian is bounded from below by the quantity $\frac{\pi^2}{\mathrm{diam}(\Omega)^2}$. Recently such an estimate has been generalized to the first nontrivial Neumann p-Laplacian eigenvalues in [7, 8, 18] to get

$$\Lambda_p \geq (p-1)^{1/p}\left(\frac{2\pi}{p\,\mathrm{diam}(\Omega)\sin\frac{\pi}{p}}\right). \tag{7}$$

As $p \to \infty$ the right hand side in this Payne-Weinberger inequality (7) converges

$$\lim_{p\to\infty}(p-1)^{1/p}\Big(\frac{2\pi}{p\,\mathrm{diam}(\Omega)\sin\frac{\pi}{p}}\Big) = \frac{2}{\mathrm{diam}(\Omega)},$$

and in view of (6) we may therefore conclude that

Corollary 4. *The Payne-Weinberger inequality* (7) *for the first Neumann eigenvalue of the p-Laplacian becomes an identity for* $p = \infty$.

As a byproduct of our proofs we obtain also the following result, which is related to the hot-spot conjecture. The hot spot conjecture, see [4], says that a first nontrivial Neumann eigenfunction for the linear Laplace operator on a convex domain Ω should attain its maximum or minimum on the boundary $\partial\Omega$ and the proof of Lemma 1 will show that u_∞ has this property as well. But there may be more than one eigenfunction associated to Λ_∞.

Corollary 5. *If Ω is convex and smooth, then any first nontrivial Neumann eigenfunction, i.e. any viscosity solution to* (1) *for $\Lambda = \Lambda_\infty$ attains both its maximum and minimum only on the boundary $\partial\Omega$. Moreover the extrema of u are located at points that have maximal distance in $\overline{\Omega}$.*

The proof of our main result, Theorem 1, will be a combination of Theorem 2 in Section 2 on the limiting problem as $p \to \infty$ and Proposition 1 in Section 3. Corollary 5 will be derived at the very end of this paper.

2. The limiting problem as $p \to \infty$

Definition 1. Let Ω be a bounded open connected domain in $\mathbb{R}^n$. The intrinsic diameter of Ω, denoted by $\mathrm{diam}(\Omega)$, is defined as

$$\mathrm{diam}(\Omega) := \sup_{x,y\in\Omega} d_\Omega(x,y) \tag{8}$$

whith d_Ω denoting geodetic distance in Ω.

Consider the eigenvalue problem

$$\Lambda_p^p = \min\left\{\frac{\int_\Omega |\nabla v|^p\,dx}{\int_\Omega |v|^p\,dx} : v \in W^{1,p}(\Omega), \int_\Omega |v|^{p-2}v\,dx = 0\right\}. \tag{9}$$

Let u_p be a minimizer of (9) such that $\|u_p\|_p = 1$, where $\|f\|_p^p = \frac{1}{|\Omega|}\int_\Omega |f|^p\,dx$.

For every $p > 1$ u_p satisfies the Euler equation

$$\begin{cases} -\mathrm{div}(|\nabla u_p|^{p-2}\nabla u_p) = \Lambda_p^p|u_p|^{p-2}u_p & \text{in } \Omega \\ |\nabla u_p|^{p-2}\frac{\partial u_p}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

and

LEMMA 1. *Let Ω be a connected bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, then*

$$\lim_{p\to+\infty} \Lambda_p = \Lambda_\infty := \frac{2}{\operatorname{diam}(\Omega)}. \tag{11}$$

Here $\operatorname{diam}(\Omega)$ *denotes the intrinsic diameter as defined in* (8).

PROOF. **Step 1** $\limsup_{p\to\infty} \Lambda_p \le \dfrac{2}{\operatorname{diam}(\Omega)}$.

We start proving that $\Lambda_\infty \le 2/\operatorname{diam}(\Omega)$. Let $x_0 \in \Omega$. We choose $c_p \in \mathbb{R}$ such that $w(x) = d_\Omega(x, x_0) - c_p$ is a good test function in (9), that is

$$\int_\Omega |w|^{p-2} w\, dx = 0.$$

Using this test function in (9) we get (recalling that $|\nabla d_\Omega(x, x_0)| \le 1$ a.e. in Ω)

$$\Lambda_p \le \frac{1}{\left(\frac{1}{|\Omega|}\int_\Omega |d_\Omega(x, x_0) - c_p|^p\right)^{1/p}}. \tag{12}$$

Now we observe that $0 \le c_p \le \operatorname{diam}(\Omega)$ and thus up to a subsequence $c_p \to c$, with $0 \le c \le \operatorname{diam}(\Omega)$, then we obtain

$$\liminf_{p\to\infty} \left(\frac{1}{|\Omega|}\int_\Omega |d(x, x_0) - c_p|^p\right)^{1/p} = \sup_{x\in\Omega} |d_\Omega(x, x_0) - c| \ge \operatorname{diam}(\Omega)/2$$

and then from (12) the Step 1 is proved.

Step 2 $\liminf_{p\to\infty} \Lambda_p \ge \dfrac{2}{\operatorname{diam}(\Omega)}$.

By definition we get

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^p\, dx\right)^{1/p} = \Lambda_p.$$

Let us fix $m > n$. For $p > m$ by Hölder inequality we have

$$\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^m dx\right)^{1/m} \le \Lambda_p.$$

We can deduce that $\{u_p\}_{p\ge m}$ is uniformly bounded in $W^{1,m}(\Omega)$ and then assume that, up to a subsequence, u_p converges weakly in $W^{1,m}(\Omega)$ and in $C^0(\Omega)$ to a function $u_\infty \in W^{1,m}(\Omega)$. For $q > m$, by semicontinuity and Hölder inequality, we get

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \le \liminf_{p\to\infty} \frac{\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^q\, dx\right)^{1/q}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_p(x)|^q\, dx\right)^{1/q}} \le \liminf_{p\to\infty} \frac{\left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p(x)|^p\, dx\right)^{1/p}}{\left(\frac{1}{|\Omega|}\int_\Omega |u_p(x)|^q\, dx\right)^{1/q}}.$$

Thus

$$\frac{\|\nabla u_\infty\|_q}{\|u_\infty\|_q} \leq \frac{\|u_\infty\|_\infty}{\|u_\infty\|_q} \liminf_{p\to\infty} \Lambda_p, \tag{13}$$

and letting $q \to \infty$ we get

$$\frac{\|\nabla u_\infty\|_\infty}{\|u_\infty\|_\infty} \leq \liminf_{p\to\infty} \Lambda_p. \tag{14}$$

Now we observe that condition $\int_\Omega |u_p|^{p-2} u_p = 0$ leads to

$$\sup u_\infty = -\inf u_\infty, \tag{15}$$

infact we have

$$\begin{aligned} 0 &\leq | \|(u_\infty)^+\|_{p-1} - \|(u_\infty)^-\|_{p-1}| \\ &= | \|(u_\infty)^+\|_{p-1} - \|(u_p)^+\|_{p-1} + \|(u_p)^-\|_{p-1} - \|(u_\infty)^-\|_{p-1}| \\ &\leq | \|(u_\infty)^+\|_{p-1} - \|(u_p)^+\|_{p-1}| + | \|(u_\infty)^-\|_{p-1} - \|(u_p)^-\|_{p-1}| \\ &\leq \|(u_\infty)^+ - (u_p)^+\|_{p-1} + \|(u_\infty)^- - (u_p)^-\|_{p-1}. \end{aligned} \tag{16}$$

Letting $p \to \infty$ we obtain (15). Using the following inequality (see for instance [5], p. 269)

$$|u_\infty(x) - u_\infty(y)| \leq d_\Omega(x,y)\|\nabla u_\infty\|_\infty \leq \mathrm{diam}(\Omega)\|\nabla u_\infty\|_\infty,$$

we can conclude the proof by (14) observing that

$$2\|u\|_\infty = \sup u_\infty - \inf u_\infty \leq \mathrm{diam}(\Omega)\|\nabla u_\infty\|_\infty. \qquad \square$$

REMARK 1. Our proof shows that u_∞ increases with constant slope $\Lambda_\infty\|u_\infty\|_\infty$ along the geodesic between two points spanning $\mathrm{diam}(\Omega)$. In a rectangle this would be a diagonal.

Before proving Theorem 2 we recall the definition of viscosity super (sub) solution to

$$\begin{cases} F(u,\nabla u,\nabla^2 u) = \min\{|\nabla u| - \Lambda|u|, -\Delta_\infty u\} = 0 & \text{in } \{u>0\}\cap\Omega \\ G(u,\nabla u,\nabla^2 u) = \max\{\Lambda|u| - |\nabla u|, -\Delta_\infty u\} = 0 & \text{in } \{u<0\}\cap\Omega \\ H(\nabla^2 u) = -\Delta_\infty u = 0, & \text{in } \{u=0\}\cap\Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

DEFINITION 2. An upper semicontinuous function u is a viscosity subsolution to (17) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$u(x_0) = \phi(x_0), \quad \text{and} \quad u(x) < \phi(x) \quad \text{if } x \neq x_0, \quad \text{then}$$

$$F(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0 \quad \text{if } u(x_0) > 0 \tag{18}$$

$$G(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0 \quad \text{if } u(x_0) < 0 \tag{19}$$

$$H(\nabla^2\phi(x_0)) \leq 0 \quad \text{if } u(x_0) = 0, \tag{20}$$

while if $x_0 \in \partial\Omega$ and $\phi \in C^2(\overline{\Omega})$ are such that

$$u(x_0) = \phi(x_0), \quad \text{and} \quad u(x) < \phi(x) \quad \text{if } x \neq x_0, \quad \text{then}$$

$$\min\left\{ F(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \leq 0 \quad \text{if } u(x_0) > 0 \tag{21}$$

$$\min\left\{ G(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \leq 0 \quad \text{if } u(x_0) < 0 \tag{22}$$

$$\min\left\{ H(\nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \leq 0 \quad \text{if } u(x_0) = 0. \tag{23}$$

DEFINITION 3. A lower semicontinuous function u is a viscosity supersolution to (17) if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that

$$u(x_0) = \phi(x_0), \quad \text{and} \quad u(x) > \phi(x) \quad \text{if } x \neq x_0, \quad \text{then}$$

$$F(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0 \quad \text{if } u(x_0) > 0 \tag{24}$$

$$G(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0 \quad \text{if } u(x_0) < 0 \tag{25}$$

$$H(\nabla^2\phi(x_0)) \geq 0 \quad \text{if } u(x_0) = 0, \tag{26}$$

while if $x_0 \in \partial\Omega$ and $\phi \in C^2(\overline{\Omega})$ are such that

$$u(x_0) = \phi(x_0), \quad \text{and} \quad u(x) > \phi(x) \quad \text{if } x \neq x_0,$$

then

$$\max\left\{ F(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \geq 0 \quad \text{if } u(x_0) > 0 \tag{27}$$

$$\max\left\{ G(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \geq 0 \quad \text{if } u(x_0) < 0 \tag{28}$$

$$\max\left\{ H(\nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0) \right\} \geq 0 \quad \text{if } u(x_0) = 0. \tag{29}$$

DEFINITION 4. A continuous function u is a solution to (17) iff it is both a supersolution and a subsolution to (17).

Remark 2. It is instructive to use the definition for checking that the one-dimensional function $u(x) = x_1$ on the square $\Omega = (-1,1) \times (-1,1)$ is a viscosity solution of (17). In fact, $u \in C^2(\Omega)$, and $-\Delta_\infty u = 0$ in Ω.

So the first PDE in (17) is satisfied if also $1 = |\nabla u| \geq \Lambda u$ on $\{u > 0\}$, and that implies $\Lambda \leq 1$.

The Neumann boundary condition is satisfied in classical sense on horizontal parts of $\partial\Omega$. However, for Neumann condition to hold in the viscosity sense on the right part, we must verify

$$\min\{\min\{|\nabla\phi| - \Lambda\phi, -\Delta_\infty\phi\}, \partial\phi/\partial v\}(x_0) \leq 0$$

for any C^2 test function ϕ touching u in $x_0 \in \partial\Omega$ from above, and

$$\max\{\min\{|\nabla\psi| - \Lambda\psi, -\Delta_\infty\psi\}, \partial\psi/\partial v\}(x_0) \geq 0$$

for any smooth test function ψ touching u from below.

Recall $|\nabla u| = \partial u/\partial v = 1$ everywhere. Therefore only the very first constraint is active on the boundary and implies

$$\Lambda \geq 1.$$

This shows that $u(x) = x_1$ is a viscosity solution to (17) with eigenvalue $\Lambda = 1$, but

$$\Lambda = 1 > \frac{1}{\sqrt{2}} = \frac{2}{\operatorname{diam}(\Omega)} = \Lambda_\infty.$$

In what follows we will use the notation

$$F_p(u, \nabla u, \nabla^2 u) = -(p-2)|\nabla u|^{p-4}\Delta_\infty u - |\nabla u|^{p-2}\Delta u - \Lambda_p^p |u|^{p-2}u$$

with

$$\Delta_\infty u = \sum_{i,j=1}^n u_{x_i} u_{x_i x_j} u_{x_j}.$$

Lemma 2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Lambda_p^p |u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{30}$$

then u is a viscosity solution to

$$\begin{cases} F_p(u, \nabla u, \nabla^2 u) = 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{31}$$

PROOF. That u is a viscosity solution to the differential equation $F_p = 0$ in Ω was shown in [13], Lemma 1.8. It remains to show that the Neumann boundary condition is satisfied in the viscosity sense as defined for instance in [10]. Let $x_0 \in \partial\Omega$, $\phi \in C^2(\overline{\Omega})$ such that $u(x_0) = \phi(x_0)$ and $\phi(x) < u(x)$ when $x \neq x_0$. Assume by contradiction that

$$\max\left\{|\nabla\phi(x_0)|^{p-2}\frac{\partial\phi}{\partial\nu}(x_0), F_p(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0))\right\} < 0. \tag{32}$$

Then there exists a ball $B_r(x_0)$, centered at x_0 with radius $r > 0$, such that (32) holds true $\forall x \in \overline{\Omega} \cap B(x_0, r)$. Denote by $0 < m = \inf_{\overline{\Omega}\cap B_r(x_0)}(u(x) - \phi(x))$ and by $\psi(x) = \phi(x) + \frac{m}{2}$. Using $(\psi - u)^+$ as test function in the weak formulation we have both

$$\int_{\psi>u} |\nabla\psi|^{p-2}\nabla\psi\nabla(\psi - u)\,dx < \Lambda_p^p \int_{\psi>u} |\phi|^{p-2}\phi(\psi - u)\,dx$$

and

$$\int_{\psi>u} |\nabla u|^{p-2}\nabla u\nabla(\psi - u)\,dx = \Lambda_p^p \int_{\psi>u} |u|^{p-2}u(\psi - u)\,dx.$$

Subtraction yields the contradiction

$$\begin{aligned} C\int_{\psi>u} |\nabla(\psi - u)|^p\,dx &\le \int_{\psi>u} (|\nabla\psi|^{p-2}\nabla\psi - |\nabla u|^{p-2}\nabla u, \nabla(\psi - u))\,dx \\ &< \Lambda_p^p \int_{\psi>u} (|\phi|^{p-2}\phi - |u|^{p-2}u)(\psi - u)\,dx < 0. \end{aligned} \tag{33}$$ □

THEOREM 2. *Let Ω be a bounded open connected set of $\mathbb{R}^n$. If u_∞ and Λ_∞ are defined as above then u_∞ satisfies* (17) *in the viscosity sense with $\Lambda = \Lambda_\infty$.*

PROOF. First we observe that in fact there exists a subsequence u_{p_i} uniformly converging to u_∞ in Ω. Now let us prove that u_∞ is a viscosity supersolution to (17) in Ω. Let $x_0 \in \Omega$ and let $\phi \in C^2(\Omega)$ be such that $\phi(x_0) = u_\infty(x_0)$ and $\phi(x) < u_\infty(x)$ for $x \in \Omega\backslash\{x_0\}$. Since $u_{p_i} \to u_\infty$ uniformly in $B_r(x_0)$ one can prove that $u_{p_i} - \phi$ has a local minimum in x_i, with $\lim_i x_i = x_0$. Recalling that u_{p_i} is a viscosity solution to (31), choosing $\psi(x) = \phi(x) - \phi(x_i) + u_{p_i}(x_i)$ as test function we obtain

$$\begin{aligned} &-[(p_i - 2)|\nabla\phi(x_i)|^{p_i-4}\Delta_\infty\phi(x_i) + |\nabla\phi(x_i)|^{p_i-2}\Delta\phi(x_i)] \\ &\qquad \ge \Lambda_{p_i}^{p_i}|u_{p_i}(x_i)|^{p_i-2}u_{p_i}(x_i). \end{aligned} \tag{34}$$

Three cases can occur.

- $u_\infty(x_0) > 0$. In this case (34) implies that $|\nabla\phi(x_i)| > 0$, hence dividing (34) by $|\nabla\phi(x_i)|^{p_i-4}(p_i-2)$ we have

$$-\frac{|\nabla\phi(x_i)|^2\Delta\phi(x_i)}{p_i-2} - \Delta_\infty\phi(x_i) \geq \left(\frac{\Lambda_{p_i}u_{p_i}(x_i)}{|\nabla\phi(x_i)|}\right)^{p_i-4}\frac{\Lambda_{p_i}^4 u_{p_i}^3(x_i)}{p_i-2}. \tag{35}$$

Letting p_i go to $+\infty$ we have $\dfrac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} \leq 1$ and $-\Delta_\infty\phi(x_0) \geq 0$ hence

$$\min\{|\nabla\phi(x_0)| - \Lambda_\infty|\phi(x_0)|, -\Delta_\infty\phi(x_0)\} \geq 0.$$

- $u_\infty(x_0) < 0$. Also in this case (34) implies that $|\nabla\phi(x_i)| > 0$, and dividing by $|\nabla\phi(x_i)|^{p_i-4}(p_i-2)$ we have again (35). If $\dfrac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} < 1$, letting p_i go to ∞, we have $-\Delta_\infty\phi(x_0) \geq 0$, otherwise $\dfrac{\Lambda_\infty\phi(x_0)}{|\nabla\phi(x_0)|} \geq 1$. In both cases we have

$$\max\{\Lambda_\infty|\phi(x_0)| - |\nabla\phi(x_0)|, -\Delta_\infty\phi(x_0)\} \geq 0.$$

- $u_\infty(x_0) = 0$. If $|\nabla\phi(x_0)| = 0$ then, by definition, we have $-\Delta_\infty\phi(x_0) = 0$. If $|\nabla\phi(x_0)| > 0$ then $\lim_i \dfrac{\Lambda_{p_i}|u_{p_i}(x_i)|}{|\nabla\phi(x_i)|} = 0$ hence (35) implies

$$-\Delta_\infty\phi(x_0) \geq 0.$$

It remains to prove that u_∞ satisfies the boundary conditions in the viscosity sense.

Assume that $x_0 \in \partial\Omega$ and let $\phi \in C^2(\overline{\Omega})$ be such that $\phi(x_0) = u_\infty(x_0)$ and $\phi(x) < u_\infty(x)$ for $x \in \overline{\Omega}\backslash\{x_0\}$. Using again the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \phi$ has a minimum point $x_i \in \overline{\Omega}$, with $\lim_i x_i = x_0$.

If $x_i \in \Omega$ for infinitely many i arguing as before we get

$$\begin{gathered}
\min\{|\nabla\phi(x_0)| - \Lambda_\infty|\phi(x_0)|, -\Delta_\infty\phi(x_0)\} \geq 0 \quad \text{if } u(x_0) > 0\\
\max\{\Lambda_\infty|\phi(x_0)| - |\nabla\phi(x_0)|, -\Delta_\infty\phi(x_0)\} \geq 0 \quad \text{if } u(x_0) < 0\\
-\Delta_\infty\phi(x_0) \geq 0, \quad \text{if } u(x_0) = 0.
\end{gathered}$$

If $x_i \in \partial\Omega$, since u_{p_i} is viscosity solution to (31), for infinitely many i we have

$$|\nabla\phi(x_i)|^{p_i-2}\frac{\partial\phi}{\partial\nu}(x_i) \geq 0$$

which concludes the proof.

Arguing in the same way we can prove that u_∞ is a viscosity subsolution to (17) in Ω. □

3. Λ_∞ IS THE FIRST NON TRIVIAL EIGENVALUE

PROPOSITION 1. *Let Ω be a smooth bounded open convex set in $\mathbb{R}^n$. If for some $\Lambda > 0$ problem* (17) *admits a nontrivial eigenfunction u, then $\Lambda \geq \Lambda_\infty$.*

The main idea is to use a test function involving the distance from a suitable point $x_0 \in \Omega$. This function is smooth everywhere except x_0. For the nonconvex case one may want to use intrinsic distance instead, which however is not of class C^2, as pointed out in [1].

LEMMA 3. *Let Ω, Λ and u be as in the statement of Proposition* 1. *Let Ω_1 be an open connected subset of Ω such that $u \geq m$ in $\overline{\Omega}_1$ for some positive constant m. Then $u > m$ in Ω_1.*

PROOF. Let x_0 be any point in Ω_1. Our aim is to show that $u(x_0) > m$. Obviously, for any given $R > 0$ such that $B_R(x_0) \subset \Omega_1$ we have $u \not\equiv m$ in $B_R(x_0)$ otherwise we have in $B_R(x_0)$ that $|\nabla u| - \Lambda|u| < 0$ (in the viscosity sense) which violates the first equation in (17). This means that for any $R > 0$ such that $B_R(x_0) \subset \Omega_1$ it is possible to find $x_1 \in B_{R/4}(x_0)$ such that $u(x_1) > m$. The continuity of u implies that for some $\varepsilon > 0$ small enough, there exists $r \leq \operatorname{dist}(x_0, x_1)$ such that $u > m + \varepsilon$ on $\partial B_r(x_1)$. Therefore the function

$$v(x) = m + \frac{\varepsilon}{\frac{R}{2} - r}\left(\frac{R}{2} - |x - x_1|\right) \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1)$$

is such that

$$-\Delta_\infty v = 0 \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1).$$

Since

$$-\Delta_\infty u \geq 0 \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1)$$

in the viscosity sense, and

$$u \geq v \quad \text{on } \partial B_{R/2}(x_1) \cup \partial B_r(x_1)$$

the comparison principle, see Theorem 2.1 in [11], implies that $u \geq v > m$ in $B_{R/2}(x_1)\backslash B_r(x_1)$ and therefore $u(x_0) > m$. □

LEMMA 4. *Let Ω, Λ and u be as in the statement of Proposition* 1. *Then u certainly changes sign.*

PROOF. Since u is a nontrivial solution to (17), we can always assume, possibly changing the sign of the eigenfunction u, that it is positive somewhere. We shall prove that the minimum of u in $\overline{\Omega}$ is negative. We argue by contradiction and we assume that the minimum m is nonnegative. In view of Lemma 3 a positive minimum can not be attained in Ω. On the other hand zero as well can not be attained as minimum in Ω. If so, since $u \not\equiv 0$, there would exist a point $x_0 \in \Omega$ and a ball $B_R(x_0) \subset \Omega$ such that $u(x_0) = 0$ and $\max_{B_{R/4}(x_0)} u > 0$. Let $x_1 \in B_{R/4}(x_0)$ be such that $u(x_1) > 0$. The continuity of u implies that there exists $r \leq \operatorname{dist}(x_0, x_1)$ such that $u > u(x_1)/2$ on $\partial B_r(x_1)$. Therefore the function

$$v(x) = \frac{u(x_1)}{R - 2r}\Big(\frac{R}{2} - |x - x_1|\Big) \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1)$$

is such that

$$-\Delta_\infty v = 0 \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1).$$

Since

$$-\Delta_\infty u \geq 0 \quad \text{in } B_{R/2}(x_1)\backslash B_r(x_1)$$

in the viscosity sense, and

$$u \geq v \quad \text{on } \partial B_{R/2}(x_1) \cup \partial B_r(x_1)$$

the comparison principle, see Theorem 2.1 in [11], implies that $u \geq v > 0$ in $B_{R/2}(x_1)\backslash B_r(x_1)$ and therefore $u(x_0) > 0$.

Therefore the only possibility is that there exists $x_0 \in \partial\Omega$ nonnegative minimum point of u. We shall prove that $\frac{\partial u}{\partial \nu}(x_0) < 0$ in the viscosity sense in contradiction to (24)–(26). Indeed there certainly exist $\bar{x} \in \Omega$ and $r > 0$ such that the ball $B_r(\bar{x}) \subset \Omega$ is inner tangential to $\partial\Omega$ at x_0 and $\partial B_r(\bar{x}) \cap \partial\Omega = \{x_0\}$. Then the function

$$v(x) = u(\bar{x}) - \Big(\frac{u(\bar{x}) - u(x_0)}{r}\Big)(|x - \bar{x}|) \quad \text{in } B_r(\bar{x})\backslash\{\bar{x}\}$$

satisfies

$$-\Delta_\infty v = 0 \quad \text{in } B_r(\bar{x})\backslash\{\bar{x}\}$$

since

$$-\Delta_\infty u \geq 0 \quad \text{in } B_r(\bar{x})\backslash\{\bar{x}\}$$

in the viscosity sense, and

$$u \geq v \quad \text{on } \partial B_r(\bar{x}) \cup \{\bar{x}\}.$$

Using again the comparison principle, see Theorem 2.1 in [11], we get $u \geq v$ in $\overline{\Omega}$. Therefore the function

$$\phi = u(\bar{x}) - (u(\bar{x}) - u(x_0))\Big(\frac{|x - \bar{x}|}{r}\Big)^{\frac{1}{2}}$$

is such that $\phi \in C^2(\overline{\Omega} - \{\bar{x}\})$,

$$\phi < v \leq u \quad \text{in } B_r(\bar{x}) - \{\bar{x}\},$$
$$\phi(x) < u(x_0) \leq u(x) \quad \text{in } \Omega \backslash B_r(\bar{x}),$$

and

$$u(x_0) = \phi(x_0).$$

However

$$\max\left\{F(\phi(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)), \frac{\partial\phi}{\partial\nu}(x_0)\right\} < 0 \tag{36}$$

contradicts (24)–(26). □

Proof of Proposition 1. Let u be a non trivial eigenfunction of (17) and let us denote by $\Omega_+ = \{x \in \Omega : u(x) > 0\}$ and by $\Omega_- = \{x \in \Omega : u(x) < 0\}$. Lemma 4 ensures that they are both nonempty sets. Let us normalize the eigenfunction u such that

$$\max_{\bar{\Omega}} u = \frac{1}{\Lambda}.$$

Then $\Lambda u \leq 1$ which implies that

$$\min\{|\nabla u| - 1, -\Delta_\infty u\} \leq 0 \quad \text{in } \Omega_+ \tag{37}$$

in the viscosity sense.

For every $x_0 \in \Omega \backslash \Omega_+$ and for every $\epsilon > 0$ and $\gamma > 0$ the function $g_{\epsilon,\gamma}(x) = (1+\epsilon)|x - x_0| - \gamma|x - x_0|^2$ belongs to $C^2(\Omega \backslash B_\rho(x_0))$ for every $\rho > 0$. If γ is small enough compared to ϵ, it verifies

$$\min\{|\nabla g_{\epsilon,\gamma}| - 1, -\Delta_\infty g_{\epsilon,\gamma}\} \geq 0 \quad \text{in } \Omega_+. \tag{38}$$

Therefore (a comparison) Theorem 2.1 in [11] ensures that

$$m = \inf_{x \in \Omega_+} (g_{\epsilon,\gamma}(x) - u(x)) = \inf_{x \in \partial\Omega_+} (g_{\epsilon,\gamma}(x) - u(x)). \tag{39}$$

Now $\partial\Omega_+$ contains certainly points in Ω and possibly on $\partial\Omega$. To rule out that the infimum in the right hand side of (39) is attained on $\partial\Omega$, assume that there exists $\bar{x} \in \partial\Omega \cap \partial\Omega_+$ such that $g_{\epsilon,\gamma}(\bar{x}) - u(\bar{x}) = m$ and choose $g_{\epsilon,\gamma} - m$ as test function in (21). By construction for every $x \in \partial\Omega \cap \partial\Omega_+$ and $\gamma < \frac{\epsilon}{2\,\mathrm{diam}(\Omega)}$ it results that

$$|\nabla g_{\epsilon,\gamma}|(x) = 1 + \epsilon - 2\gamma|x - x_0| > 1,$$
$$\frac{\partial g_{\epsilon,\gamma}}{\partial\nu}(x) = ((1+\epsilon) - 2\gamma|x - x_0|)\left(\frac{x - x_0}{|x - x_0|}, \nu(x)\right) > 0,$$

and

$$-\Delta_\infty g_{\epsilon,\gamma} = 2\gamma|\nabla g_{\epsilon,\gamma}|^2 > 0$$

which give a contradiction to (21). Together with (39) this implies that

$$m = \inf_{x \in \Omega_+} (g_{\epsilon,\gamma}(x) - u(x)) = \inf_{x \in \partial\Omega_+ \cap \Omega} (g_{\epsilon,\gamma}(x) - u(x)) \geq 0.$$

Letting ϵ and γ go to zero we have that

$$|x - x_0| \geq u(x) \quad \forall x \in \{y : u(y) \geq 0\}, \quad \forall x_0 \in \{y : u(y) \leq 0\} \tag{40}$$

hence

$$d^+ = \sup_{x \in \bar{\Omega}_+} \operatorname{dist}(x, \{u = 0\}) \geq \frac{1}{\Lambda}.$$

Arguing in the same way we obtain

$$d^- = \sup_{x \in \bar{\Omega}_-} \operatorname{dist}(x, \{u = 0\}) \geq \frac{1}{\Lambda}$$

hence

$$\operatorname{diam}(\Omega) \geq d^+ + d^- \geq \frac{2}{\Lambda},$$

which concludes the proof of our proposition. □

Corollary 5 follows now easily. Returning to (40) pick $x = \bar{x}$ as the point in which u attains its maximum and correspondingly $x = \underline{x}$ as the point in which u attains its minimum. Then $d(\bar{x}, \Omega_-) \geq \frac{1}{\Lambda}$ and $d(\underline{x}, \Omega_+) \geq \frac{1}{\Lambda}$, so that $\operatorname{diam}(\Omega) \geq |\bar{x} - \underline{x}| \geq \frac{2}{\Lambda}$. Since $\Lambda = \Lambda_\infty$, equality holds and the max and min of u are attained in boundary points which have farthest distance from each other.

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