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Functional Analysis — *Moser-Trudinger inequality in grand Lebesgue space*, by Robert Černý, communicated on 9 January 2015.

Abstract. — Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We study sharp constants for the Moser-Trudinger inequality in the Sobolev-type space $W_0 L^{n)}(\Omega)$, where $L^{n)}(\Omega)$ is so called grand L^n space introduced in [9]. In particular, we obtain our results with respect to two quantities introduced in [8].

Key words: Grand Lebesgue space, Sobolev spaces, embedding theorems, sharp constants, Moser-Trudinger inequality.

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1. Introduction

Throughout this note, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded set, ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, $\mathscr{L}_n$ is the n-dimensional Lebesgue measure and $|\Omega|$ stands for $\mathscr{L}_n(\Omega)$. We use the standard notation $n' = \frac{n}{n-1}$.

If $W_0^{1,p}(\Omega)$ denotes the usual completion of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, then it is well known that

$$\begin{aligned} &W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega) \quad \text{if } 1 \leq p < n, \\ &W_0^{1,p}(\Omega) \subset L^\infty(\Omega) \quad \text{if } n < p. \end{aligned}$$

In the borderline case $p = n$ we have

$$\begin{aligned} &W_0^{1,n}(\Omega) \subset L^q(\Omega) \quad \text{for every } q \in [1, \infty), \\ \text{however,} \quad &W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega). \end{aligned}$$

This case was studied more precisely by Trudinger [14] who showed that for every $K \geq 0$ and every $u \in W_0^{1,n}(\Omega)$ we have

$$\int_\Omega \exp((K|u(x)|)^{n'})\, dx < \infty.$$

Moser [12] proved the famous inequality

$$\sup_{\|\nabla u\|_{L^n(\Omega)} \leq 1} \int_\Omega \exp((K|u(x)|)^{n'})\, dx \begin{cases} \leq C(n, K, |\Omega|) & \text{when } K \leq n\omega_n^{\frac{1}{n}} \\ = \infty & \text{when } K > n\omega_n^{\frac{1}{n}}. \end{cases} \tag{1.1}$$

In the last two decades, the Moser-Trudinger inequality became a crucial tool when proving the existence and the regularity of nontrivial weak solutions to elliptic partial differential equations with critical growth (see for example the pioneering works [2] and [3] by Adimurthi). Further applications required several versions and generalizations of the Moser inequality such as a version for unbounded domains (see [1]), a version without boundary conditions (see [6]), the Concentration-Compactness Alternative (see [11] and [5]) and others.

The aim of this note is to obtain an inequality of the same type as (1.1) for the functions having their gradient in the grand Lebesgue space $L^{n)}(\Omega)$. These spaces were introduced in [9] and the condition $|\nabla f| \in L^{n)}(\Omega)$, where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a homeomorphism, is a borderline condition for a pathological behavior of the Jacobian and the failure of the Luzin N condition (see [9, 10, 4]).

In [9], the space $L^{n)}(\Omega)$ was introduced and it was equipped with the norm

$$\|f\|_{L^{n)}(\Omega)} = \sup_{\varepsilon \in (0, n-1)} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f|^{n-\varepsilon} \right)^{\frac{1}{n-\varepsilon}}. \tag{1.2}$$

However, it is a bit uncomfortable to work with this norm (see the last section for more information), therefore we derive our results for two equivalent quantities obtained in [8]. The first one is a quasi-norm

$$\|f\| = \sup_{t \in (0, |\Omega|)} \left(\log^{-1}\left(\frac{e|\Omega|}{t}\right) \int_t^{|\Omega|} (f^*(s))^n \, ds \right)^{\frac{1}{n}} \tag{1.3}$$

and the second one is a norm

$$\|f\| = \sup_{t \in (0, |\Omega|)} \left(\log^{-1}\left(\frac{e|\Omega|}{t}\right) \int_t^{|\Omega|} (f^{**}(s))^n \, ds \right)^{\frac{1}{n}}. \tag{1.4}$$

Let us recall that passing to an equivalent norm (or quasi-norm) in Moser-type inequalities influences the size of the borderline exponent. Our new results are the following.

Theorem 1.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that the space $L^{n)}(\Omega)$ is equipped with the quasi-norm* (1.3). *Let us set*

$$K_1 = n\omega_n^{\frac{1}{n}}.$$

Then

$$\sup_{u \in C_0^\infty(\Omega), \|\nabla u\| \leq 1} \int_\Omega \exp(K|u(x)|) \, dx \begin{cases} \leq C(n, |\Omega|, K) & \text{for } K < K_1 \\ = \infty & \text{for } K \geq K_1. \end{cases}$$

Theorem 1.2. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that the space $L^{n)}(\Omega)$ is equipped with the norm* (1.4). *Let us set*

$$K_2 = \frac{n}{n-1} K_1 = \frac{n^2}{n-1} \omega_n^{\frac{1}{n}}.$$

Then

$$\sup_{u \in C_0^\infty(\Omega),\, \|\nabla u\| \le 1} \int_\Omega \exp(K|u(x)|)\, dx \begin{cases} \le C(n, |\Omega|, K) & \text{for } K < K_2 \\ = \infty & \text{for } K \ge K_2. \end{cases}$$

The paper is organized as follows. After Preliminaries we prove Theorem 1.1. Theorem 1.2 is proved in the fourth section. The last section is devoted to some comments concerning the author's unsuccessful attempt to obtain sharp constants corresponding to the Moser-type inequality with respect to the norm (1.2).

2. Preliminaries

Notation. The n-dimensional Lebesgue measure is denoted by $\mathscr{L}_n$ and $|\Omega|$ stands for $\mathscr{L}_n(\Omega)$. By $B(x, R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with the radius $R > 0$. If $x = 0$, we simply write $B(R)$.

By C we denote a generic positive constant which may depend on n, $|\Omega|$ and K. This constant may vary from expression to expression as usual.

Non-increasing rearrangement. The non-increasing rearrangement f^* of a measurable function f on Ω is

$$f^*(t) = \sup\{s \ge 0 : \big|\{x \in \Omega : |f(x)| > s\}\big| > t\} \quad \text{for } t \in (0, \infty).$$

Further, we define the maximal function of f^* by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds \quad \text{for } t \in (0, \infty).$$

Next, we recall an inequality obtained in [13]. If Ω is open and $u \in W_0^{1,1}(\Omega)$, then

$$u^*(t) \le \frac{1}{n\omega_n^{\frac{1}{n}}}\left(t^{-\frac{1}{n'}}\int_0^t |\nabla u|^*(s)\, ds + \int_t^{|\Omega|} |\nabla u|^*(s) s^{-\frac{1}{n'}}\, ds\right) \quad \text{for every } t \in (0, |\Omega|). \tag{2.1}$$

Let us also derive a version of (2.1) for the quantity $|\nabla u|^{**}$. By the Fubini theorem we have

$$\begin{aligned}
&\int_t^\infty s^{-\frac{n-1}{n}} |\nabla u|^{**}(s)\, ds \\
&\quad = \int_t^\infty s^{-2+\frac{1}{n}}\left(\int_0^s |\nabla u|^*(r)\, dr\right) ds \\
&\quad = \int_t^\infty \int_0^s s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, dr\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_t^\infty s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, ds\, dr + \int_t^\infty \int_r^\infty s^{-2+\frac{1}{n}} |\nabla u|^*(r)\, ds\, dr \\
&= -\frac{n}{n-1}\Big([s^{-\frac{n-1}{n}}]_t^\infty \int_0^t |\nabla u|^*(r)\, dr + \int_t^\infty [s^{-\frac{n-1}{n}}]_r^\infty |\nabla u|^*(r)\, dr \Big) \\
&= \frac{n}{n-1}\Big(t^{-\frac{n-1}{n}} \int_0^t |\nabla u|^*(r)\, dr + \int_t^\infty r^{-\frac{n-1}{n}} |\nabla u|^*(r)\, dr \Big).
\end{aligned}
$$

Thus, (2.1) reads

$$
u^*(t) \le \frac{n-1}{n^2 \omega_n^{\frac{1}{n}}} \int_t^{|\Omega|} |\nabla u|^{**}(s) s^{-\frac{1}{n'}}\, ds \quad \text{for every } t \in (0, |\Omega|). \tag{2.2}
$$

Finally we recall the Hölder-type inequality for Grand Lebesgue spaces obtained in [7]. It reads

$$
\frac{1}{|\Omega|} \int_\Omega |fg|\, dx \le \|f\|_{L^{n)}(\Omega)} \|g\|_{L^{(n'}(\Omega)}, \tag{2.3}
$$

where the first norm is (1.2) and the second one is

$$
\|g\|_{L^{(n'}(\Omega)} = \inf_{g=\sum g_k} \Big\{ \sum_{k=1}^\infty \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}} \Big(\frac{1}{|\Omega|} \int_\Omega |g_k|^{(n-\varepsilon)'} \Big)^{\frac{1}{(n-\varepsilon)'}} \Big\}.
$$

3. Proof of theorem 1.1

Proof of Theorem 1.1: Case $K < K_1$. By (2.1) we have for every $t \in (0, |\Omega|)$

$$
u^*(t) \le \frac{1}{n \omega_n^{\frac{1}{n}}} \Big(t^{-\frac{1}{n'}} \int_0^t |\nabla u|^*(s)\, ds + \int_t^{|\Omega|} |\nabla u|^*(s) s^{-\frac{1}{n'}}\, ds \Big) =: \frac{1}{n \omega_n^{\frac{1}{n}}} (I_1 + I_2).
$$

Let us estimate the integrals. When estimating I_2 we apply Hölder's inequality and the assumption $\|\nabla u\| \le 1$ with the quasi-norm (1.3) to obtain for every $t \in (0, |\Omega|)$

$$
\begin{aligned}
I_2 &= \int_t^{|\Omega|} |\nabla u|^*(s) s^{-\frac{1}{n'}}\, ds \le \Big(\int_t^{|\Omega|} (|\nabla u|^*(s))^n\, ds \Big)^{\frac{1}{n}} \Big(\int_t^{|\Omega|} \frac{1}{s}\, ds \Big)^{\frac{n-1}{n}} \\
&\le \log^{\frac{1}{n}}\Big(\frac{e|\Omega|}{t} \Big) \log^{\frac{n-1}{n}}\Big(\frac{|\Omega|}{t} \Big) \le \log\Big(\frac{e|\Omega|}{t} \Big).
\end{aligned} \tag{3.1}
$$

Next, let us estimate I_1. We use inequality (2.3), the assumption $\|\nabla u\| \le 1$ and the fact that the quasi-norm (1.3) and the norm (1.2) are equivalent (it was proved in [8], we should also note that the constants concerning this equivalence depend on $|\Omega|$)

$$
\begin{aligned}
I_1 &= t^{-\frac{n-1}{n}} \int_0^t |\nabla u|^*(s)\,ds = t^{-\frac{n-1}{n}} \int_0^{|\Omega|} |\nabla u|^*(s)\chi_{(0,t)}(s)\,ds \\
&\le t^{-\frac{n-1}{n}} \|\,|\nabla u|^*\|_{L^{n)}((0,|\Omega|))} \|\chi_{(0,t)}\|_{L^{(n'}((0,|\Omega|))} \le Ct^{-\frac{n-1}{n}} \|\chi_{(0,t)}\|_{L^{(n'}((0,|\Omega|))}.
\end{aligned}
$$

Next, we have for $t > 0$ small enough

$$
\begin{aligned}
\|\chi_{(0,t)}\|_{L^{(n'}((0,|\Omega|))} &= \inf_{g=\sum g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}} \left(\frac{1}{|\Omega|} \int_0^{|\Omega|} |g_k|^{(n-\varepsilon)'} \right)^{\frac{1}{(n-\varepsilon)'}} \right\} \\
&\le \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}} \left(\frac{1}{|\Omega|} \int_0^{|\Omega|} \chi_{(0,t)}^{(n-\varepsilon)'} \right)^{\frac{1}{(n-\varepsilon)'}} \\
&\le C \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}} t^{\frac{1}{(n-\varepsilon)'}} \le Ct^{\frac{1}{n'}} \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}} \\
&\le Ct^{\frac{1}{n'}} \left(\log^{-1}\left(\frac{1}{t} \right) \right)^{-\frac{1}{n-\log^{-1}\left(\frac{1}{t}\right)}} \le Ct^{\frac{1}{n'}} \log^{\frac{1}{n-\frac{1}{2}}}\left(\frac{1}{t} \right).
\end{aligned}
$$

This yields for every $t \in (0, |\Omega|)$ (it is easy to see that $I_1 \le C$ whenever t is bounded away from 0)

$$
I_1 \le C + C \log^{\frac{1}{n-\frac{1}{2}}}\left(\frac{1}{t} \right)
$$

and thus we obtain from (3.1)

$$
u^*(t) \le \frac{1}{n\omega_n^{\frac{1}{n}}} (I_1 + I_2) \le \frac{1}{n\omega_n^{\frac{1}{n}}} \left(C + C \log^{\frac{1}{n-\frac{1}{2}}}\left(\frac{1}{t} \right) + \log\left(\frac{e|\Omega|}{t} \right) \right).
$$

Hence, if $K = (1 - 2\varepsilon)K_1$ for some $\varepsilon > 0$, we have for some $t_0 > 0$ small enough

$$
\begin{aligned}
\int_\Omega \exp(K|u(x)|)\,dx &= \int_0^{|\Omega|} \exp(K|u|^*(t))\,dt \\
&\le \int_0^{t_0} \exp\left((1-\varepsilon)K_1 n^{-1} \omega_n^{-\frac{1}{n}} \log\left(\frac{e|\Omega|}{t} \right) \right) dt + \int_{t_0}^{|\Omega|} C\,dt \\
&= \int_0^{t_0} \exp\left((1-\varepsilon) \log\left(\frac{e|\Omega|}{t} \right) \right) dt + C \\
&= \int_0^{t_0} \frac{C}{t^{1-\varepsilon}}\,dt + C = C.
\end{aligned}
$$

Thus, we are done. $\square$

Proof of Theorem 1.1: Case $K \geq K_1$. We can suppose that $0 \in \Omega$. Let $R > 0$ be so small that $B(R) \subset \Omega$. Let us set

$$u(x) = \begin{cases} \omega_n^{-\frac{1}{n}} \log\left(\frac{R}{|x|}\right) & \text{for } x \in B(R)\setminus\{0\} \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B(R). \end{cases}$$

Then we have

$$|\nabla u|(x) = \omega_n^{-\frac{1}{n}} \frac{1}{|x|} \quad \text{for } x \in B(R)\setminus\{0\}$$

and, as $|B(|x|)| = \omega_n |x|^n$,

$$|\nabla u|^*(t) = \omega_n^{-\frac{1}{n}} \frac{1}{\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}} = \frac{1}{t^{\frac{1}{n}}} \quad \text{for } t \in (0, |B(R)|).$$

Now, for every $t \in (0, |B(R)|)$ we obtain

$$\begin{aligned} \log^{-1}\left(\frac{e|B(R)|}{t}\right) \int_t^{|B(R)|} (|\nabla u|^*(s))^n \, ds &= \log^{-1}\left(\frac{e\omega_n R^n}{t}\right) \int_t^{|B(R)|} \frac{1}{s} ds \\ &= \log^{-1}\left(\frac{e\omega_n R^n}{t}\right) \log\left(\frac{\omega_n R^n}{t}\right) \leq 1. \end{aligned}$$

Thus, $\|\nabla u\| \leq 1$. On the other hand

$$\begin{aligned} \int_{B(R)} \exp(K_1|u(x)|)\, dx &= n\omega_n \int_0^R y^{n-1} \exp\left(K_1 \omega_n^{-\frac{1}{n}} \log\left(\frac{R}{y}\right)\right) dy \\ &= n\omega_n \int_0^R y^{n-1} \exp\left(n \log\left(\frac{R}{y}\right)\right) dy = C \int_0^R \frac{1}{y} dy = \infty. \end{aligned}$$

Now it is easy to see that for each $\delta \in \left(0, \frac{1}{4}\right)$ there is $\tilde{\delta} \in (0, \delta)$ and a radially symmetric function $u_\delta \in C_0^\infty(B(R))$ such that

$$u_\delta(x) \begin{cases} = u_\delta|_{\partial B(\delta)} & \text{for } x \in B(\delta) \\ \in (u|_{\partial B(2\delta)}, u|_{\partial B(\delta)}) & \text{for } x \in B(2\delta)\setminus B(\delta) \\ = u(x) & \text{for } x \in B(R-\tilde{\delta})\setminus B(2\delta) \\ \in (0, u(x)) & \text{for } x \in B(R)\setminus B(R-\tilde{\delta}), \end{cases}$$

$$|\nabla u_\delta(x)| \begin{cases} = 0 & \text{for } x \in B(\delta) \\ \in (0, |\nabla u(x)|) & \text{for } x \in B(2\delta)\setminus B(\delta) \\ = |\nabla u(x)| & \text{for } x \in B(R-\tilde{\delta})\setminus B(2\delta) \\ \in (0, 2|\nabla u(x)|) & \text{for } x \in B(R)\setminus B(R-\tilde{\delta}), \end{cases}$$

$\|\nabla u_\delta\| \leq 1$ and

$$\int_{B(R)} \exp(K_1|u_\delta(x)|)\,dx \geq \int_{B(R)\setminus B(2\delta)} \exp(K_1|u(x)|)\,dx - C \xrightarrow{\delta\to 0_+} \infty. \qquad \square$$

4. Proof of Theorem 1.2

Proof of Theorem 1.2: Case $K < K_2$. By (2.2) we have for every $t \in (0, |\Omega|)$

$$u^*(t) \leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \int_t^{|\Omega|} |\nabla u|^{**}(s) s^{-\frac{1}{n'}}\,ds.$$

Next we apply Hölder's inequality and the assumption $\|\nabla u\| \leq 1$ with the norm (1.4) to obtain for every $t \in (0, |\Omega|)$

$$\begin{aligned} u^*(t) &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \Big(\int_t^{|\Omega|} (|\nabla u|^{**}(s))^n\,ds \Big)^{\frac{1}{n}} \Big(\int_t^{|\Omega|} \frac{1}{s}\,ds \Big)^{\frac{n-1}{n}} \\ &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \log^{\frac{1}{n}}\Big(\frac{e|\Omega|}{t}\Big) \log^{\frac{n-1}{n}}\Big(\frac{|\Omega|}{t}\Big) \\ &\leq \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \log\Big(\frac{e|\Omega|}{t}\Big). \end{aligned}$$

Hence, if $K = (1-\varepsilon)K_2$ for some $\varepsilon > 0$, we have

$$\begin{aligned} \int_\Omega \exp(K|u|)\,dx &= \int_0^{|\Omega|} \exp(K|u|^*(t))\,dt \\ &\leq \int_0^{|\Omega|} \exp\Big((1-\varepsilon)K_2 \frac{n-1}{n^2\omega_n^{\frac{1}{n}}} \log\Big(\frac{e|\Omega|}{t}\Big)\Big)\,dt \\ &= \int_0^{|\Omega|} \exp\Big((1-\varepsilon)\log\Big(\frac{e|\Omega|}{t}\Big)\Big)\,dt \\ &= \int_0^{|\Omega|} \frac{C}{t^{1-\varepsilon}}\,dt = C \end{aligned}$$

and we are done. $\square$

Proof of Theorem 1.2: Case $K \geq K_2$. We can suppose that $0 \in \Omega$. Let $R > 0$ be so small that $B(R) \subset \Omega$. Let us set

$$u(x) = \begin{cases} \frac{n-1}{n}\omega_n^{-\frac{1}{n}} \log\big(\frac{R}{|x|}\big) & \text{for } x \in B(R)\setminus\{0\} \\ 0 & \text{for } x \in \mathbb{R}^n\setminus B(R). \end{cases}$$

Then we have

$$|\nabla u|(x) = \frac{n-1}{n}\omega_n^{-\frac{1}{n}}\frac{1}{|x|} \quad \text{for } x \in B(R)\setminus\{0\},$$

$$|\nabla u|^*(t) = \frac{n-1}{n}\omega_n^{-\frac{1}{n}}\frac{1}{\left(\frac{t}{\omega_n}\right)^{\frac{1}{n}}} = \frac{n-1}{n}\frac{1}{t^{\frac{1}{n}}} \quad \text{for } t \in (0, |B(R)|)$$

and

$$|\nabla u|^{**}(t) = \frac{n-1}{n}\frac{1}{t}\int_0^t \frac{1}{s^{\frac{1}{n}}}\,ds = \frac{n-1}{n}\frac{1}{t}\left[\frac{n}{n-1}s^{\frac{n-1}{n}}\right]_0^t = \frac{1}{t^{\frac{1}{n}}} \quad \text{for } t \in (0, |B(R)|).$$

Now, for every $t \in (0, |B(R)|)$ we obtain

$$\begin{aligned}\log^{-1}\left(\frac{e|\Omega|}{t}\right)\int_t^{|\Omega|}(|\nabla u|^{**}(s))^n\,ds &= \log^{-1}\left(\frac{e|\Omega|}{t}\right)\int_t^{|\Omega|}\frac{1}{s}\,ds \\ &= \log^{-1}\left(\frac{e|\Omega|}{t}\right)\log\left(\frac{|\Omega|}{t}\right) \leq 1.\end{aligned}$$

Thus $\|\nabla u\| \leq 1$. We also have

$$\begin{aligned}\int_{B(R)} \exp(K_2|u(x)|)\,dx &= n\omega_n\int_0^R y^{n-1}\exp\left(K_2\frac{n-1}{n}\omega_n^{-\frac{1}{n}}\log\left(\frac{R}{y}\right)\right)dy \\ &= n\omega_n\int_0^R y^{n-1}\exp\left(n\log\left(\frac{R}{y}\right)\right)dy = C\int_0^R\frac{1}{y}\,dy = \infty.\end{aligned}$$

We conclude the proof using the smoothing procedure shown in the proof of Theorem 1.1. □

5. Open problem

The author was not able to obtain sharp constants concerning the Moser-type inequality with respect to the norm (1.2). In this section we give some comments concerning this open problem.

First, we suggest to replace the norm (1.2) by

$$\|f\| := |\Omega|^{\frac{1}{n}}\sup_{\varepsilon\in(0,n-1)}\left(\frac{\varepsilon}{|\Omega|}\int_\Omega |f|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}}. \tag{5.1}$$

This new norm is just a multiple of the norm (1.2), but the constants in the Moser-type inequality are independent of $|\Omega|$ now.

It is easy to obtain the following partial result.

LEMMA 5.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Suppose that the space $L^{n)}(\Omega)$ is equipped with the norm* (5.1). *Then for every $K \geq n^2 \omega_n^{\frac{1}{n}}$ we have*

$$\sup_{u \in C_0^\infty(\Omega),\, \|\nabla u\| \leq 1} \int_\Omega \exp(K|u(x)|)\, dx = \infty.$$

PROOF. We can suppose that $0 \in \Omega$. Let $R > 0$ be so small that $B(R) \subset \Omega$. Let us set

$$u(x) = \begin{cases} n^{-1} \omega_n^{-\frac{1}{n}} \log\left(\frac{R}{|x|}\right) & \text{for } x \in B(R) \setminus \{0\} \\ 0 & \text{for } x \in \mathbb{R}^n \setminus B(R). \end{cases}$$

We have

$$|\nabla u|(x) = n^{-1} \omega_n^{-\frac{1}{n}} \frac{1}{|x|} \quad \text{for } x \in B(R) \setminus \{0\}.$$

Hence for every $\varepsilon \in (0, n-1]$

$$\begin{aligned} &|B(R)|^{\frac{1}{n}} \left(\frac{\varepsilon}{|B(R)|} \int_{B(R)} |\nabla u|^{n-\varepsilon} \right)^{\frac{1}{n-\varepsilon}} \\ &= (\omega_n R^n)^{\frac{1}{n}} \left(\frac{\varepsilon}{\omega_n R^n} \int_0^R n \omega_n y^{n-1} \left(n^{-1} \omega_n^{-\frac{1}{n}} \frac{1}{y} \right)^{n-\varepsilon} dy \right)^{\frac{1}{n-\varepsilon}} \\ &= R \left(\frac{n^{1-n+\varepsilon} \varepsilon}{R^n} \int_0^R y^{-1+\varepsilon}\, dy \right)^{\frac{1}{n-\varepsilon}} = R (n^{1-n+\varepsilon} R^{\varepsilon - n})^{\frac{1}{n-\varepsilon}} = n^{\frac{1-n+\varepsilon}{n-\varepsilon}} \leq 1 \end{aligned}$$

(it can be easily seen that the worst case is $\varepsilon = n - 1$) and thus $\|\nabla u\| \leq 1$.

Finally

$$\int_\Omega \exp(n^2 \omega_n^{\frac{1}{n}} |u(x)|)\, dx = n \omega_n \int_0^R y^{n-1} \exp\left(n \log\left(\frac{R}{y} \right) \right) dy = C \int_0^R \frac{1}{y}\, dy = \infty.$$

Now, the proof can be easily completed using a suitable smoothing procedure. □

The author was not able to prove the boundedness of the supremum for $K < n^2 \omega_n^{\frac{1}{n}}$ (since the functions with the logarithmic growth played the crucial role in the sharp estimates concerning the norm (1.4) and the quasi-norm (1.3), the author believes that the borderline constant is the number $n^2 \omega_n^{\frac{1}{n}}$ obtained in Remark 5.1). The problem rests upon the fact that the author was not able to modify the application of Hölder's inequality in (3.1) so that the resulting integrals had an appearance compatible with the assumption $\|\nabla u\| \leq 1$, where the norm comes from (5.1).

The first idea was to read $\|\nabla u\| \leq 1$ as

$$|\Omega|^{\frac{1}{n}}\left(\frac{\varepsilon}{|\Omega|}\int_{\Omega}|f|^{n-\varepsilon}\right)^{\frac{1}{n-\varepsilon}} \leq 1 \quad \text{for every } \varepsilon \in (0, n-1)$$

and to apply Hölder's inequality corresponding to each ε in (3.1). That is, for every $t \in (0, |\Omega|)$ we obtain a collection of estimates of $u^*(t)$ indexed by $\varepsilon \in (0, n-1)$ and then we use the most restrictive one. Unfortunately, the resulting estimate of $u^*(t)$ has too large power of $\log(\frac{1}{t})$.

Another option is to use the Hölder-type inequality (2.3). The relevant version with respect to the norm (5.1) reads

$$\int_t^{|\Omega|} |fg|\,dx \leq \|f\|_{L^{n)}((t,|\Omega|))}\|g\|_{L^{(n'}((t,|\Omega|))},$$

where the second norm is

$$\|g\|_{L^{(n'}((t,|\Omega|))} = (|\Omega|-t)^{\frac{n-1}{n}} \inf_{g=\sum g_k}\left\{\sum_{k=1}^{\infty} \inf_{0<\varepsilon<n-1} \varepsilon^{-\frac{1}{n-\varepsilon}}\left(\frac{1}{|\Omega|-t}\int_t^{|\Omega|}|g_k|^{(n-\varepsilon)'}\right)^{\frac{1}{(n-\varepsilon)'}}\right\}.$$

Our problem would be solved if we were able to show that for every η there is $t_0 > 0$ so small that

$$\|s^{-\frac{n-1}{n}}\|_{L^{(n'}((t,|\Omega|))} \leq (1+\eta)\frac{1}{n}\log\left(\frac{1}{t}\right) \quad \text{for every } t \in (0, t_0). \tag{5.2}$$

However, the author was not able to prove this estimate because he did not find the corresponding decomposition of the function $s^{-\frac{n-1}{n}}$. Let us note that numerical approximations show that if $g_1(s) := s^{-\frac{n-1}{n}}$ and other functions from the decomposition are trivial, then we obtain an estimate with too large multiplicative constant

$$\|s^{-\frac{n-1}{n}}\|_{L^{(n'}((t,|\Omega|))} \gtrapprox \log\left(\frac{1}{t}\right).$$

Decomposing $s^{-\frac{n-1}{n}}$ into vertical slices leads to an useless estimate

$$\|s^{-\frac{n-1}{n}}\|_{L^{(n'}((t,|\Omega|))} \leq \infty$$

and horizontal slices (after some computation) give us an estimate with too large power of the logarithm

$$\|s^{-\frac{n-1}{n}}\|_{L^{(n'}((t,|\Omega|))} \leq C\log^{1+\frac{1}{n}}\left(\frac{1}{t}\right).$$

Let us also note that it is not obvious that a decomposition leading to (5.2) actually exists, since we do not know whether inequality (2.3) is sharp. Indeed, in [7, Lemma 2.9] it is shown that for every $f \in L^\infty$ there are $g_k \in L^\infty$ such that

$$\frac{\int |fg_k|}{\|f\|_{L^{n)}}\|g_k\|_{L^{(n'}}} \xrightarrow{k\to\infty} 1$$

(beware of the gap in the proof given in [7], for many functions the worst case is $\sigma(f) = p - 1$). However, we need a statement with fixed function g, that is, for every $g \in L^\infty$, there are $f_k \in L^\infty$ such that

$$\frac{\int |f_k g|}{\|f_k\|_{L^{n)}}\|g\|_{L^{(n'}}} \xrightarrow{k\to\infty} 1.$$

The author did not find such a result in the literature and it also does not look as easy to be proved as the statement concerning a fixed function f.

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