Rend. Lincei Mat. Appl. 26 (2015), 151–160
DOI 10.4171/RLM/699

Functional Analysis — *Functions determining locally solid topological Riesz spaces continuously embedded in L^0*, by Paola Cavaliere, Paolo de Lucia and Anna de Simone, communicated on 13 February 2015.

Abstract. — We present a class of non-negative functions, acting on a solid vector subspace X of L^0, enjoying the following property: each member of the class determines on X a locally solid topological Riesz space structure which is continuously embedded into L^0. These functions are neither necessarily monotone, nor subadditive. Special instances are provided by function norms and quasi-norms on X.

Key words: Measurable function, convergence in measure, topological Riesz space, quasi-triangular function, quasi-norm, embedding.

2000 Mathematics Subject Classification: 46E40, 46E30, 28A12, 46A16.

1. Introduction and main results

Given a measure space (S, Σ, μ), we let $L^0(\mu)$ denote the vector space of all (μ-equivalence classes of) extended real-valued functions defined on S, which are Σ-measurable and μ-almost everywhere finite on S. Under the μ-almost everywhere ordering, $L^0(\mu)$ turns out to be a Riesz space, namely, a vector lattice whose standard topology is the topology of convergence in measure on μ-finite sets, hereafter denoted as τ_μ.

This topology plays a central role in many questions in analysis, and, in particular, in the study of function spaces. Its special importance may be said to derive from the fact that Lebesgue spaces are actually continuously embedded into $L^0(\mu)$. In formulas,

$$(L^p(\mu), \tau_{\|\cdot\|_p}) \hookrightarrow (L^0(\mu), \tau_\mu) \tag{1.1}$$

for every $p \in [1, \infty]$. Here, $\tau_{\|\cdot\|_p}$ and $\hookrightarrow$ stand for the norm topology of the Lebesgue space $L^p(\mu)$, and for a continuous embedding, respectively.

The Lebesgue spaces $L^p(\mu)$, for $p \in [1, \infty]$, are in fact prototypal examples of locally solid (see Section 2) topological Riesz spaces of measurable functions, as well as of rearrangement invariant Banach function spaces. The norm $\|\cdot\|_p$ is indeed a lattice norm, i.e. it is monotone on the positive cone of $L^p(\mu)$, and only depends on the measure of level sets of functions. Furthermore, in the case when the measure space (S, Σ, μ) is totally finite, all rearrangement invariant function spaces are continuously embedded into $L^1(\mu)$ (see e.g. [4] for relevant definitions and examples, and the monograph [3] for a comprehensive treatment of the topic). Thus, thanks to (1.1) with $p = 1$, all rearrangement invariant function spaces built upon a totally finite measure space are continuously embedded into $(L^0(\mu), \tau_\mu)$.

In recent years, various contributions have appeared, where generalizations, along different directions, of rearrangement invariant function spaces have been exploited (see e.g. [1, 14, 12, 7]). The generalizations that we have in mind roughly amount to equipping some solid vector subspace X of $L^0(\mu)$ with some 'relaxed' function norm on X, such as, for instance, a quasi-norm. Such function spaces lack a Banach structure, and their topological structure heavily depends on the assumptions on the involved relaxed norm. It is noteworthy that membership of a function to such spaces does not still assure its local integrability. Consequently, the above mentioned embedding into $(L^0(\mu), \tau_\mu)$ of these generalized function spaces is not guaranteed.

The above considerations naturally raise the issue of finding a broad class $\mathscr{T}$ of nonnegative functions, acting on a solid vector subspace $X = X(\mu)$ of $L^0(\mu)$, obeying the following requirements:

i) membership of a function Φ to the class $\mathscr{T}$ depends only on the behavior of the function in question with respect to the lattice operations of X;
ii) each function $\Phi \in \mathscr{T}$ generates on X a locally solid topology, denoted as τ_Φ;
iii) embedding (1.1) continues to hold even if $(L^p(\mu), \tau_{\|\cdot\|_p})$ is replaced by (X, τ_Φ), provided $\Phi \in \mathscr{T}$;
iv) lattice norms on X belong to the class $\mathscr{T}$, and the locally solid topology generated on X by each of them as elements of $\mathscr{T}$ agrees with the norm topology.

Let us remark that condition i) is justified by the highly non-trivial fact that the topology of convergence in measure on $L^0(\mu)$ can be described just in terms of the Riesz space structure, without any reference to either the underlying measure algebra or to integration (see e.g. [11, Chapt. 36, Sect. 367T]). Accordingly, here our investigation of such an issue calls into play the notion of disjointness for functions in any Riesz subspace $X \subseteq L^0(\mu)$, that the lattice structure of X actually allows.

To be more specific, two functions $f, g \in X$ are called *disjoint* whenever $|f| \wedge |g| = 0$. Here, we adopt the standard notation $f \vee g = \sup\{f, g\}$ and $f \wedge g = \inf\{f, g\}$ for $f, g \in X \subseteq L^0(\mu)$. We shall write $f \perp g$ to denote that f and g are disjoint.

A function $\Phi : X \to [0, \infty[$, with $\Phi(0) = 0$, is then called *quasi-triangular* whenever for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$\begin{aligned} &\Phi(f \vee g) < \epsilon \quad \text{if } \Phi(f), \Phi(g) \in [0, \delta_\epsilon[; \\ &\Phi(g) < \epsilon \quad \text{if } \Phi(f), \Phi(f \vee g) \in [0, \delta_\epsilon[\end{aligned} \tag{1.2}$$

for all $f, g \in X$, with $f \perp g$.

Each quasi-triangular function on X clearly satisfies i). Moreover, lattice norms on X turn out to be quasi-triangular functions. This is also the case for quasi-monotone and quasi-subadditive nonnegative maps on X, namely, those functions $\Psi : X \to [0, \infty[$, with $\Psi(0) = 0$, for which some constants $C_1, C_2 \geq 1$ exist such that

$$\frac{1}{C_1} \max\{\Psi(f), \Psi(g)\} \leq \Psi(f \vee g) \leq C_2 \max\{\Psi(f), \Psi(g)\} \tag{1.3}$$

for every $f, g \in X$, with $f \perp g$. All quasi-norms on X satisfy (1.3). Typical examples are provided by $X = L^p(\mu)$, for $p \in]0, 1[$, and $\Psi(f) = \|f\|_p = \left(\int |f|^p\right)^{1/p}$ for $f \in X$. Further instances can be obtained from functions of the form $A \circ \Psi$, where $\Psi : X \to [0, +\infty[$, $\Psi(0) = 0$, fulfils (1.3) with $C_1 = C_2 = 1$, and $A : [0, +\infty[\to [0, +\infty[$ vanishes at 0 and is either convex (namely, a Young function) and satisfies the Δ_2-condition, or quasi-concave. Recall that A satisfies the Δ_2-condition if a positive constant c exists such that $A(2t) \leq cA(t)$ for $t \geq 0$, whereas A is quasi-concave if $A(0) = 0$, $A(t) > 0$ and increasing for $t > 0$, and $\frac{A(t)}{t}$ is decreasing for $t > 0$.

We refer to [12] for more details on customary, and less standard, examples of quasi-monotone and quasi-subadditive functions.

Our first result shows, in particular, that quasi-triangular functions fulfil requirement ii) as well. In what follow, the notion of Fréchet-Nikodým topology on a solid vector space X will play a role. Recall that the symmetric difference of two function $f, g \in L^0(\mu)$ is defined as $f \triangle g = f \vee g - f \wedge g = |f - g|$, and the Fréchet-Nikodým topology turns $(X, \triangle)$ into a topological abelian group with a base of neighborhoods of 0 consisting of solid subsets of X. For further information and references, we refer to [9, 8, 15].

THEOREM 1.1. *Let X be a solid vector subspace of $L^0(\mu)$. Every quasi-triangular function $\Phi : X \to [0, \infty[$, with $\Phi(0) = 0$, induces on X a locally solid topology τ_Φ whose neighborhood base at each $f \in X$ is the family $\mathscr{B}[f] = \{B(f, r) : r > 0\}$, where*

$$B(f, r) = \{g \in X : \Phi(|u|) < r \text{ for } u \in L^0(\mu) \text{ s.t. } |u| \leq ||f| - |g||\}. \tag{1.4}$$

Moreover, τ_Φ is a Fréchet-Nikodým topology on X.

When Φ, in addition, vanishes only at functions which equal 0 a.e., then the topology τ_Φ is Hausdorff, and the positive cone $X_+ = \{f \in X : f \geq 0\}$ of X is closed in (X, τ_Φ).

We observe that, being X a solid vector subspace of $L^0(\mu)$, the set $B(f, r)$ described in (1.4) is well-defined, since functions $u \in L^0(\mu)$ actually belong to X. In the special case when Φ is a lattice norm on X in Theorem 1.1, one has that $B(f, r) = \{g \in X : \Phi(f - g) < r\}$, i.e. $B(f, r)$ is the standard open ball centered at f, with radius r, of normed Riesz spaces. This follows from the solidity of X and from the monotonicity of a lattice norm on X_+. The first part of condition iv) is thus satisfied by quasi-triangular functions.

It is however clear that condition iii)—and, consequently, the second part of iv)—entails additional assumptions on the behavior of quasi-triangular functions with respect to multiplication by scalars, as well as to the μ-almost everywhere ordering. Next result provides them in the context of σ-finite measure spaces. Letting χ_A denote the characteristic function of any $A \in \Sigma$, we state the following

THEOREM 1.2. *Let (S, Σ, μ) be a σ-finite measure space, and let X be a solid Riesz subspace of $L^0(\mu)$ satisfying the following property*

(N0) $\chi_{K_j} \in X$ *for all* $j \in \mathbb{N}$, *and for some* $(K_j)_{j \in \mathbb{N}} \subset \Sigma$ *such that* $\mu(K_j) < \infty$ *and* $K_j \nearrow S$.

Then

$$(X, \tau_\Phi) \hookrightarrow (L^0(\mu), \tau_\mu)$$

for every quasi-triangular function $\Phi : X \to [0, \infty[$ *such that*

(N1) *there exist a function* $\omega : [0, \infty[\to [0, \infty[$, *with* $\omega(0) = 0$, *such that*

$$\Phi(\alpha f) \le \omega(\alpha)\Phi(f) \quad \textit{for all } \alpha \in [0, \infty[,\ f \in X,$$

(N2) $\Phi(f) > 0$ *if* f *does not vanish* μ-*a.e.;*
(N3) *for any* $\lambda > 0$ *there exists some* $r_\lambda > 0$ *such that* $\Phi(g) < \lambda$ *whenever* $f, g \in X_+$, $g \le f$ *and* $\Phi(f) < r_\lambda$.

To summarize, an answer to the issue stated above can be formulated in the setting of σ-finite measure spaces (S, Σ, μ) as follows. For any solid vector subspace $X = X(\mu)$ of $L^0(\mu)$ obeying condition (N0), the class $\mathscr{T}$ consists of all those non-negative functions defined on X which are quasi-triangular and fulfil assumptions (N1)–(N3).

Since σ-finiteness of a measure space (S, Σ, μ) is equivalent to metrizability of the topology of convergence in measure on $L^0(\mu)$, τ_μ can be effectively described in terms of convergent sequences. Our last result thus relates τ_Φ-sequential convergence to sequential convergence in measure in X (and thus with μ-a.e. sequential pointwise convergence in S).

COROLLARY 1.3. *Let (S, Σ, μ), X and Φ be as in Theorem 1.2. If $(f_k)_{k \in \mathbb{N}} \subset X$ converges to some $f \in X$ with respect to τ_Φ, then every subsequence of $(f_k)_{k \in \mathbb{N}}$ has a further subsequence which converges to f μ-a.e. in S.*

Let us mention that Theorems 1.1–1.2 improve [12, Theorem 1.6], where the case of quasi-additive and quasi-monotone functions fulfilling conditions (N1)–(N2) is taken into account. In particular, [12, Definition 1.2] introduces a definition of topology induced by a nonnegative function vanishing at the zero of the abelian group $(X, +)$. A specialization of our Theorem 1.1 to the framework of [12] shows that there is no need for such an assumption. Indeed, any quasi-triangular function does generate a Fréchet-Nikodým topology on a solid vector subspace of $L^0(\mu)$.

We finally point out that the proof of Theorem 1.2 depends in a key way on the equivalence on sets of finite measure between the notion of 0-continuity and that of (ϵ, δ)-continuity of the measure μ—of the underlying measure space (S, Σ, μ)—with respect to the non-additive measure ϕ determined on Σ by the quasi-triangular function Φ on X which is taken into account (see the end of Sect. 2 for definitions). Such an equivalence is established in a more general setting by [5, Theorem 1.2], whose proof makes use of Fréchet-Nikodým topologies and of standard methods of measure theory. Our approach is therefore completely different from that of [12], where the theory of capacities, and specifically certain capacitary estimates on semigruppoids are instead exploited (see [12, Sect. 2] and [13]).

Proofs of our results will be given in Sect. 3. The next section contains definitions, notation and preliminary results needed in our discussion.

2. Background

We collect here some definitions and properties from the theory of Riesz spaces. We refer to the monographs [2], [10, Chap. 24] and [11, Chap. 35–36] for more details and proofs.

Let (S, Σ, μ) be a measure space, i.e. Σ is σ-algebra of subsets of a set S, and $\mu : \Sigma \to [0, +\infty]$ is an extended non-negative measure. Hereafter, χ_A stands for the characteristic function of $A \in \Sigma$.

We let $L^0(\mu) = L^0(S, \Sigma, \mu)$ denote the vector space of all (μ-equivalence classes of) extended real-valued functions defined on S, which are Σ-measurable and μ-almost everywhere finite on S. The space $L^0(\mu)$ is a Riesz space (or vector lattice) under the μ-almost everywhere ordering, defined by saying that $f \leq g$ whenener $f(x) \leq g(x)$ for all $x \in S \backslash N$, with $N \in \Sigma$ and $\mu(N) = 0$.

Following the classic lattice notation, we let $f \vee g = \sup\{f, g\}$, $f \wedge g = \inf\{f, g\}$ and $f \triangle g = f \vee g - f \wedge g = |f - g|$ for every $f, g \in L^0(\mu)$. Then two functions $f, g \in L^0(\mu)$ are called *disjoint*, in symbols $f \perp g$, if $|f| \wedge |g| = 0$. Note that for $f \perp g$, then $f \triangle g = f \vee g = f + g$. Moreover, a subset Y of $L^0(\mu)$ is said to be *solid* if, for every $g \in Y$,

$$\{f \in L^0(\mu) : |f| \leq |g|\} \subseteq Y.$$

Solid sets of $L^0(\mu)$ contain absolute values of their elements. Thus, solid vector subspaces (usually called '*ideals*') of $L^0(\mu)$ are Riesz subspaces of $L^0(\mu)$, i.e. vector subspaces of $L^0(\mu)$ closed under lattice operations. The converse, however, fails. A straightforward example is given by the Riesz subspace $C^0[0, 1]$ of all continuous real functions in $L^0[0, 1]$, which is clearly not an ideal. Here, $L^0[0, 1]$ simply denotes $L^0(\mu)$ in the case when (S, Σ, μ) is the Lebesgue measure space on $[0, 1]$.

Throughout, we assume $L^0(\mu)$ equipped with the topology of *convergence in measure*, that we shall denote as τ_μ. This topology is induced by the semimetrics

$$\mathsf{p}_K : (f,g) \in L^0(\mu) \times L^0(\mu) \mapsto \int |f-g| \wedge \chi_K \in [0,\infty[\tag{2.1}$$

for $K \in \Sigma$, with $\mu(K) < \infty$.

The metrizability of τ_μ depends on the underlying measure space. Indeed, τ_μ is metrizable if, and only if, (S,Σ,μ) is σ-finite. When (S,Σ,μ) is σ-finite, if $(K_j)_{j\in\mathbb{N}} \subset \Sigma$ is some increasing sequence such that $S = \bigcup_{j\in\mathbb{N}} K_j$, with $\mu(K_j) < \infty$ for all j, the function

$$\rho(f,g) = \sum_{j\in\mathbb{N}} \frac{\mathsf{p}_{K_j}(f,g)}{1+2^j\mu(K_j)} \quad \text{for } (f,g) \in L^0(\mu) \times L^0(\mu)$$

is a metric on $L^0(\mu)$ inducing τ_μ, and the sets

$$U_{j,\epsilon} = \{f \in L^0(\mu) : \mu(\{x \in K_j; |f(x)| \geq \epsilon\}) < \epsilon\}, \tag{2.2}$$

with $j \in \mathbb{N}$ and $\epsilon > 0$, form a base of solid neighborhoods at zero. So a sequence $(f_k)_{k\in\mathbb{N}}$ in $L^0(\mu)$ converges in measure to $f \in L^0(\mu)$ ($f_k \xrightarrow{\mu} f$, for short) if, and only if,

$$\lim_k \mu(\{x \in S : |f_k(x) - f(x)| \geq \epsilon\}) = 0 \tag{2.3}$$

for every $\epsilon > 0$.

Hence, focusing on σ-finite measure spaces, the topology τ_μ of convergence in measure can be effectively described in terms of convergent sequences, and the following sharp characterization of sequential convergence in measure becomes relevant (see e.g. [10, Chapt. 24, Sect. 245K]).

Proposition 2.1. *Let (S,Σ,μ) be a σ-finite measure space. A sequence $(f_k)_{k\in\mathbb{N}} \subset L^0(\mu)$ converges in measure to $f \in L^0(\mu)$ if, and only if, every subsequence of $(f_k)_{k\in\mathbb{N}}$ has a sub-subsequence converging to f μ-a.e. on S.*

In Sect. 1 we have introduced the definition of a quasi-triangular function acting on Riesz subspace X of $L^0(\mu) = L^0(S,\Sigma,\mu)$, emphasizing—throught requirement i)—that such notion just depends on the behavior of the function in question with respect to the lattice operations of X. It is therefore obvious that the notion of quasi-triangular function defined on any Boolean ring (and thus on the σ-algebra Σ of the measure space taken into account) can be analogously formulated. Precisely, a function $\phi : \Sigma \to [0,\infty[$, with $\phi(\emptyset) = 0$, is said to be *quasi-triangular* (on Σ) whenever for any $\epsilon > 0$ there exists some $\delta_\epsilon > 0$ such that

$$\begin{aligned} &\phi(E \cup F) < \epsilon \quad \text{if } \phi(E), \phi(F) \in [0,\delta_\epsilon[; \\ &\phi(F) < \epsilon \qquad\quad \text{if } \phi(E), \phi(E \cup F) \in [0,\delta_\epsilon[\end{aligned} \tag{2.4}$$

for all $E, F \in \Sigma$, with $E \cap F = \emptyset$. For several concrete examples of quasi-triangular functions acting on Boolean rings, we refer the reader to [6]. Here we just mention that non-negative classical measures, finitely additive measures, k-triangular functions and quasi-submeasures (i.e. quasi-subadditive and quasi-monotone in a sense similar to (1.3) above) acting on Σ are all relevant instances of quasi-triangular functions on Σ.

Quasi-triangular functions may fail to be both monotone and subadditive. Then the *kernel* of a quasi-triangular $\phi : \Sigma \to [0, \infty[$ is defined as

$$\mathcal{N}(\phi) = \{A \in \Sigma : \phi(E) = 0 \text{ for all } E \in \Sigma_A\}, \tag{2.5}$$

where $\Sigma_A = \{E \cap A : E \in \Sigma\}$.

Given a measure space (S, Σ, μ), and two quasi-triangular functions ϕ, ν on Σ, we recall that ν is said to be 0-*continuous with respect to* ϕ, in symbols $\nu \ll \phi$, if $\mathcal{N}(\phi) \subseteq \mathcal{N}(\nu)$; whereas ν is called (ε, δ)-*continuous with respect to* ϕ, in symbols $\nu[AC]\phi$, whenever for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\phi(\Sigma_A) \subseteq [0, \delta[$ for some $A \in \Sigma$, then $\nu(A) < \varepsilon$.

Of course, $\nu[AC]\phi$ implies $\nu \ll \phi$. On specializing [5, Theorem 1.2], the non-trivial reverse implication also holds for ν being a classical measure on Σ. This equivalence will play a key role in the proof of Theorem 1.2.

3. Proofs

Proof of Theorem 1.1. We first show that, taken any $f \in X$, the sets

$$B(f, r) = \{g \in X : \Phi(|u|) < r \text{ for } u \in L^0(\mu) \text{ such that } |u| \le ||f| - |g||\},$$

for $r \in]0, \infty[$, form a neighbourhood base at f. Notice that, being X a solid vector subspace of $L^0(\mu)$, each function $u \in L^0(\mu)$ appearing in the above description of $B(f, r)$ does belong to X as well its absolute value.

It is immediate that $f \in B(f, r)$ for every $r > 0$ and that $B(f, r_1) \cap B(f, r_2) \supseteq B(f, r)$ for every $r_1, r_2 \in]0, \infty[$ and $r \in]0, \min\{r_1, r_2\}]$.

We now claim that for any $r > 0$ there exist some numbers $\rho, \rho' > 0$ such that $B(g, \rho') \subseteq B(f, r)$ when $g \in B(f, \rho)$.

Indeed, given $r > 0$, the quasi-triangularity of Φ provides the existence of some $\delta \le r$ for which (2.4) holds with ϵ replaced by r. Let $g \in B(f, \delta)$, and take $h \in B(g, \delta)$. Note that $||h| - |f|| = |h| \triangle |f|$, and $|f| \triangle |h| \in X$. Moreover, the absolute value of any $u \in L^0(\mu)$ fulfilling the estimate $|u| \le ||f| - |g||$ can be rewritten as $|u| = v_1 \vee v_2$, where

$$v_1 = |u| \wedge \{|h| \triangle |g| - [(|h| \triangle |g|) \wedge (|g| \triangle |f|)]\},$$
$$v_2 = |u| \wedge \{|g| \triangle |f| - [(|h| \triangle |g|) \wedge (|g| \triangle |f|)]\}.$$

Observe that $v_1, v_2 \in X_+$, $v_1 \perp v_2$. Moreover, $v_1 \in B(g, \delta)$ and $v_2 \in B(f, \delta)$. So the quasi-triangularity of Φ guarantees that $\Phi(|u|) < r$. Thus, $h \in B(f, r)$. This proves the claim with $\rho = \rho' = \delta$.

Second, each $B(f,r)$, $r > 0$, is a solid set in X. This follows at once observing that $|h| \triangle |f| \le |f| \triangle |g|$ for every $g \in B(f,r)$ and $h \in X$ with $|h| \le |g|$.

Next, we show that $(X, \triangle, \tau_\Phi)$ is actually a topological group, where τ_Φ stands for the topology induced on X by the sets $B(f,r)$, with $f \in X$ and $r > 0$. For this, since $f \triangle f = 0$ for every $f \in X$, it is enough to prove the continuity on X of the symmetric difference operation $\triangle$. Take $f, g \in X$, and consider $B(f \triangle g, r)$ for some $r > 0$. Again, the quasi-triangularity of Φ yields the existence of some $\delta \le r$ such that (2.4) holds with ϵ replaced by r. Easy computations provide that $f_1 \triangle g_1 \in B(f \triangle g, r)$ whenever $f_1 \in B(f,\delta)$ and $g_1 \in B(g,\delta)$. That is, the desired continuity of $\triangle$ on X.

Finally, to see that τ_Φ is a Fréchet-Nikodým topology, it suffices to show that functions $\pi_g : f \in X \mapsto f \wedge g \in X$ are τ_Φ-continuous, uniformly with respect to $g \in X$. But this follows by observing that, for any $f_1 \in X$, then $(f \wedge g) \triangle (f_1 \wedge g) = (f \triangle f_1) \wedge g$ for all $f, g \in X$, and $B(f_1 \wedge g, r) \subseteq B(f_1, r)$ for every $r > 0$ and $g \in X$.

When Φ, in addition, vanishes only at functions which equal 0 a.e., it is easy to check that τ_Φ is Hausdorff. Hence, in particular, $\{0\}$ is closed in (X, τ_Φ). Being τ_Φ a Fréchet-Nikodým topology, the map $f \mapsto f \wedge 0$ is continuous in (X, τ_Φ). Then X_+ is closed in (X, τ_Φ), since $X_+ = \{f \in X : f \wedge 0 = 0\}$. □

Proof of Theorem 1.2. Let $(K_j)_{j\in\mathbb{N}} \subset \Sigma$ be an increasing sequence of sets of finite measure such that $S = \bigcup_j K_j$ and $\chi_{K_j} \in X$ for all j, accordingly to assumption (N0).

For each $j \in \mathbb{N}$, set

$$\mu_j : E \in \Sigma \mapsto \mu(E \cap K_j) \in [0, \infty[, \tag{3.1}$$

$$\phi_j : E \in \Sigma \mapsto \Phi(\chi_{E\cap K_j}) \in [0, \infty[. \tag{3.2}$$

Each μ_j defined in (3.1) is a finite non-negative measure on Σ. Note that functions ϕ_j described in (3.2) are well-defined. In fact, since X is a solid Riesz space containing each χ_{K_j}, functions $\chi_{E\cap K_j}$ belong to X for every $E \in \Sigma$. It is easy to check that all ϕ_j are quasi-triangular functions on Σ, and we leave it to the reader.

We now point out that, for each $j \in \mathbb{N}$, $\mu_j \ll \phi_j$, namely, $\mathcal{N}(\phi_j) \subseteq \mathcal{N}(\mu_j)$.

For this, take any $A \in \mathcal{N}(\phi_j)$. Then $\Phi(\chi_{E\cap K_j}) = 0$ for every $E \in \Sigma_A$. Thus, combining (3.2), (N3) and (3.1) provide that $\mu_j(E) = 0$ for every $E \in \Sigma_A$. This means that A belongs to $\mathcal{N}(\mu_j)$, as desired.

Specializing [5, Theorem 1.2] tells us that, for each $j \in \mathbb{N}$, the 0-continuity property $\mu_j \ll \phi_j$ is actually equivalent to the (ϵ,δ)-continuity property $\mu_j[AC]\phi_j$, namely, corresponding to each $\epsilon > 0$ there exists some $\delta = \delta(\epsilon, j) > 0$ such that, for any $A \in \Sigma$,

$$\Phi(\chi_{E\cap K_j}) < \delta \quad \text{for all } E \in \Sigma_A \quad \Rightarrow \quad \mu_j(A) < \epsilon. \tag{3.3}$$

Now, let $j \in \mathbb{N}$ and $\epsilon > 0$ be arbitrarily given, and consider any $f \in X$. Set

$$M = \{x \in K_j : |f(x)| \ge \epsilon\}.$$

Notice that for each $E \in \Sigma_M$, then $\chi_E \in X$. This follows from assumption (N0) and from the solidity of X. Moreover, $\epsilon\chi_E \leq |f|$ on E.

Assumption (N2) thus implies that

$$\Phi(\chi_E) \leq \omega(\epsilon^{-1})\Phi(\epsilon\chi_E). \tag{3.4}$$

On the other hand, condition (N4) with $\lambda = \delta/\omega(\epsilon^{-1})$ provides us with the existence of some $r = r(\epsilon, j) > 0$ for which

$$\Phi(\epsilon\chi_E) < \delta/\omega(\epsilon^{-1}) \quad \text{whenever } \Phi(|f|) < r. \tag{3.5}$$

Hence, coupling (3.4) and (3.5) entails

$$\Phi(|f|) < r \quad \Rightarrow \quad \Phi(\chi_E) < \delta \quad \text{for every } E \in \Sigma_M.$$

This together with (3.3)—where A has to be replaced by M—assures that

$$\Phi(|f|) < r \quad \Rightarrow \quad \mu_j(M) = \mu(M) = \mu(\{x \in K_j : |f(x)| \geq \epsilon\}) < \epsilon.$$

Therefore, according to (1.4) and (2.2), we can conclude that, for every $j \in \mathbb{N}$ and $\epsilon > 0$, some $r > 0$ exists such that

$$f \in B(0, r) \quad \Rightarrow \quad f \in U_{\epsilon,j}(0),$$

ending the proof. □

Proof of Corollary 1.3. Assume that (S, Σ, μ), X and Φ are as in Theorem 1.2. If $(f_k)_{k \in \mathbb{N}} \subset X$ converges to $f \in X$ with respect to the topology τ_Φ described in Theorem 1.1, then $(f_k)_{k \in \mathbb{N}}$ converges in measure to f, according to Theorem 1.2. Hence Proposition 2.1 concludes. □

Acknowledgment. This research was partly supported by GNAMPA of the Italian INdAM (National Institute of High Mathematics) and the PRIN research project "Metodi logici per il trattamento dell'informazione" (2010) of MIUR (Italian Ministry of University and Research).

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Received 22 October 2014,
and in revised form 16 January 2015.

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