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Partial Differential Equations — *A failing in the Calderon-Zygmund theory of Dirichlet problems for linear equations with discontinuous coefficients*, by LUCIO BOCCARDO, communicated on 13 February 2015.

Dedicated to Haim Brezis[1]*, a beloved mentor and friend*[2]

ABSTRACT. — This note is devoted to the Calderon-Zygmund theory for linear differential operators with discontinuous coefficients. It is known that the theory holds if the datum $f(x)$, in (1.1), belongs to the Lebesgue space $L^m(\Omega)$, with $1 < m < \frac{2N}{N+2}$ (see [6]). In this paper we prove that the theory fails if $m > \frac{N}{2}$.

KEY WORDS: Failing in the Calderon-Zygmund theory, Dirichlet problems, linear equations with discontinuous coefficients.

MATHEMATICS SUBJECT CLASSIFICATION: 35B30, 35J25.

1. INTRODUCTION

In this note, we study the lack summability of the gradient of the solution of the linear boundary value problem, with discontinuous coefficients,

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, $f(x)$ is a function belonging to some Lebesgue space, and M is a bounded elliptic matrix; i.e., there exist $0 < \alpha \le \beta$ such that

$$\alpha|\xi|^2 \le M(x)\xi\xi, \quad |M(x)| \le \beta, \tag{1.2}$$

for every ξ in $\mathbb{R}^N$, for almost every x in Ω.

Since we assume that Ω is bounded, note that the Lebesgue spaces are ordered: that is $L^p(\Omega) \subset L^q(\Omega)$, if $p > q$.

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[1] for his 70th-birthday.
[2] see [1].

Under the above assumptions, this paper is concerned with the regularity theorem

$$(1.3)\qquad f \in L^m(\Omega), \quad 1 < m < N, \quad \text{implies } \nabla u \in (L^{m^*}(\Omega))^N,$$

where $m^* = \frac{mN}{N-m}$.

In particular, we recall classical results and we prove a new theorem about the statement (1.3). If the right hand side belongs to the Marcinkiewicz space the following similar result is proved in [4]

$$(1.4)\qquad f \in M^m(\Omega), \quad 1 < m < \frac{2N}{N+2}, \quad \text{implies } \nabla u \in (M^{m^*}(\Omega))^N.$$

2. Weak solutions

Proposition 2.1. *The following results about the summability of the solutions of Dirichlet problems for equations with discontinuous coefficients are nowadays classical, since the paper* [18].

(1) If $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, thanks to Lax-Milgram Theorem and Sobolev embedding, there exist a weak solutions $u \in W_0^{1,2}(\Omega)$ of (1.1); that is

$$(2.1)\qquad u \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\nabla u \nabla v = \int_\Omega f(x)v(x), \quad \forall v \in W_0^{1,2}(\Omega).$$

(2) If $f \in L^m(\Omega)$, $\frac{2N}{N+2} < m \leq \frac{N}{2}$, the summability of u (which belongs to $L^{m^{**}}(\Omega)$, $m^{**} = \frac{mN}{N-2m}$, if $\frac{2N}{N+2} < m < \frac{N}{2}$ and it has exponential summability if $m = \frac{N}{2}$), was proved by Guido Stampacchia ([17], [18]).

(3) If $f \in L^m(\Omega)$, $m > \frac{N}{2}$, the boundedness of u, was proved by Guido Stampacchia ([18]).

(4) About the gradients, Norman Meyers proved in [13] (see also [11]) that there exists $\varepsilon > 0$, only depending on $M(x)$, such that, for all $m \in \left(\frac{2N}{N+2}, \frac{2N}{N+2} + \varepsilon\right)$, then $u \in W_0^{1,m^*}(\Omega)$ with

$$\|u\|_{W_0^{1,m^*}(\Omega)} \leq C_m \|f\|_{L^m(\Omega)},$$

where $C_m > 0$ only depends on $M(x)$ and m.

On the other hand, the situation is quite different if the coefficients of $M(x)$ are smooth enough.

Here we repeat the statement of Lemma 1 of the paper [1], by Haim Brezis, about the standard L^p-regularity theory for elliptic equations in divergence form (see also [14], or [10]).

Proposition 2.2. *Assume that the coefficients of $M(x)$ are continuous functions on $\overline{\Omega}$ and $1 < m < N$. Then $u \in W_{loc}^{1,m^*}(\Omega)$ and, for $\omega \subset\subset \Omega$,*

$$\|u\|_{W^{1,m^*}(\omega)} \leq C_0(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^m(\Omega)}),$$

where C_0 depends on α, β, m, ω, Ω and the modulus of continuity of the coefficients of $M(x)$.

REMARK 2.3. First of all, we note that the above result does not only concern the case $m \geq \frac{2N}{N+2}$.

REMARK 2.4. Remark that the interval $\frac{2N}{N+2} < m < \frac{2N}{N+2} + \varepsilon$ is present in Proposition 2.1 (4) and in Proposition 2.2.

3. INFINITE ENERGY SOLUTIONS

About the existence of solutions, the framework is completely different if

$$f \in L^m(\Omega), \quad 1 < m < \frac{2N}{N+2}. \tag{3.1}$$

First of all, the above assumption on the summability of $f(x)$ does not allow the use of Lax-Milgram Theorem, in order to prove the existence of weak solutions. Then in [6] is proved the following existence result, concerning infinite energy solutions.

PROPOSITION 3.1. *Assume* (1.2), (3.1). *Then there exists a distributional solution $u \in W_0^{1,m^*}(\Omega)$ of* (1.1)*; that is*

$$u \in W_0^{1,m^*}(\Omega) : \int_\Omega M(x)\nabla u \nabla \varphi = \int_\Omega f(x)\varphi(x), \quad \forall \varphi \in W_0^{1,(m^*)'}(\Omega). \tag{3.2}$$

REMARK 3.2. If $m = 1$, the previous statement is not true; in this case, it is proved in [5] and [3] that the above Dirichlet problems has a distributional solution u which belongs to the Marcinkiewicz space $M^{\frac{N}{N-2}}(\Omega)$ and $\nabla u \in (M^{\frac{N}{N-2}}(\Omega))^N$.

REMARK 3.3. Note that $1 < m < \frac{2N}{N+2}$ implies $\frac{N}{N-1} < m^* < 2$ and that it is not possible to take $\varphi = u$ in the previous definition of distributional solution.

REMARK 3.4. We recall the duality method of Guido Stampacchia.

REMARK 3.5. The existence result stated in Proposition 3.1 is proved in [6] for nonlinear differential operators; here we use it in the easier linear case.

Further developments for the existence theory of infinite energy solutions in Dirichlet problems can be found in [5], [6], [7], [8].

Now we recall a non-uniqueness result (fundamental in our discussion) by James Serrin ([16], see also [15]).

PROPOSITION 3.6. *Let Ω be the unit ball of $\mathbb{R}^N$. There exist*
- *a discontinuous matrix $M_S(x)$, which satisfies* (1.2),
- *a function $u_S \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$,*

$$u_S \text{ not identically zero}, \tag{3.3}$$

such that u_S *is distributional solution of the boundary value problem*

$$(3.4)\qquad \begin{cases} -\mathrm{div}(M_S(x)\nabla u_S) = 0, & \text{in } \Omega; \\ u_S = 0, & \text{on } \partial\Omega; \end{cases}$$

in the sense of (3.2)*; that is*

$$(3.5)\qquad u_S \in W_0^{1,q}(\Omega) : \int_\Omega M_S(x)\nabla u_S \nabla\varphi = 0, \quad \forall\varphi \in W_0^{1,q'}(\Omega),\ q' > N.$$

4. A failing in the Calderon-Zygmund theory of Dirichlet problems for equations with discontinuous coefficients

In this section, we prove that the statement (1.3), can be false for $m > \frac{N}{2}$. The proof uses a duality method.

Theorem 4.1. *There exist a matrix* $M(x)$*, which satisfies* (1.2)*,* $f \in L^m(\Omega)$*, with* $m > \frac{N}{2}$ *and* Ω*, such that the unique weak solution in* $W_0^{1,2}(\Omega)$ *of the boundary value problem* (1.1) *does not belong to* $W_0^{1,m^*}(\Omega)$*.*

Proof. First of all, note that if $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then the unique weak solution exists (by Lax-Milgram Theorem); moreover it is bounded (by the Stampacchia's boundedness Theorem). Thus our result concerns the gradient of the solution: ∇u does not belong to $(L^{m^*}(\Omega))^N$.

Step 1. In the first part of the proof, we make the more restrictive assumption $\frac{N}{2} < m < N$.

By contradiction, we assume that, if $f \in L^m(\Omega)$, $m > \frac{N}{2}$, the unique weak solution u of the Dirichlet problem (1.1) belongs to $W_0^{1,m^*}(\Omega)$ (note that here m^* is well defined, since $\frac{N}{2} < m < N$) and not only to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Let $\phi \in W_0^{1,(m^*)'}(\Omega)$ and let $\{\phi_n(x)\}$ be a sequence of functions belonging to $W_0^{1,2}(\Omega)$ and converging to ϕ in $W_0^{1,(m^*)'}(\Omega)$; we take $\phi_n(x)$ as test function in (2.1) we to pass to the limit (as $n \to \infty$) and we deduce that

$$(4.1)\qquad \int_\Omega M_S(x)\nabla u \nabla\phi = \int_\Omega f\phi, \quad \forall\phi \in W_0^{1,(m^*)'}(\Omega).$$

Let now $q < \frac{N}{N-1}$ and take

$$f(x) = |u_S|^{q^*-2}u_S,$$

where u_S (with M and Ω) is the solution of (3.4) defined in Proposition 3.6. Note that $u_S \in L^{q^*}(\Omega)$, $q^* < \frac{N}{N-2}$, implies that $|u_S|^{q^*-2}u_S$ belongs to $L^{\frac{q^*}{q^*-1}}(\Omega)$ and $q^* < \frac{N}{N-2}$ implies $\frac{q^*}{q^*-1} > \frac{N}{2}$. Thus the function $f(x) = |u_S|^{q^*-2}u_S$ belongs to $L^\rho(\Omega)$, $\rho > \frac{N}{2}$.

Note that $m > \frac{N}{2}$ implies $m^* > N$. Then, by (4.1), there exists u^* solution of

$$u^* \in W_0^{1,m^*}(\Omega) : -\mathrm{div}(M_S^*(x)\nabla u^*) = |u_S|^{q^*-2} u_S. \tag{4.2}$$

In (4.1) we can take $\phi = u_S$ and we have

$$\int_\Omega M_S^*(x)\nabla u^* \nabla u_S = \int_\Omega |u_S|^{q^*-2} u_S u_S. \tag{4.3}$$

On the other hand, it is possible to use u^* as test function in (3.5), since $u^* \in W_0^{1,m^*}(\Omega)$ and $m^* > N$. Thus we have

$$\int_\Omega M_S(x)\nabla u_S \nabla u^* = 0.$$

This equality and (4.3) give

$$0 = \int_\Omega |u_S|^{q^*-2} u_S u_S = \int_\Omega |u_S|^{q^*},$$

which is in conflict with (3.3).

Step 2. Here we assume $m > \frac{N}{2}$. Let now u^* be the solution of

$$u^* \in W_0^{1,m^*}(\Omega) : -\mathrm{div}(M_S^*(x)\nabla u^*) = \frac{u_S}{|u_S|}. \tag{4.4}$$

Working as in Step 1, we have that

$$0 = \int_\Omega M_S(x)\nabla u_S \nabla u^* = \int_\Omega |u_S|,$$

a contradiction as before. □

THEOREM 4.2. *There exists a matrix $M(x)$, which satisfies (1.2) and $F \in (L^p(\Omega))^N$, $p > N$, such that the unique weak solution w in $W_0^{1,2}(\Omega)$ of the boundary value problem*

$$\begin{cases} -\mathrm{div}(M(x)\nabla w) = -\mathrm{div}(F), & \text{in } \Omega; \\ w = 0, & \text{on } \partial\Omega; \end{cases} \tag{4.5}$$

does not belong to $W_0^{1,p}(\Omega)$.

PROOF. Let Φ be the unique weak solution of the Dirichlet problem $\Phi \in W_0^{1,2}(\Omega) : -\Delta(\Phi) = f$, where f is defined in Theorem 4.1. The Calderon-

Zygmund theory says that $F = \nabla\Phi$ belongs to $(L^p(\Omega))^N$, $p = m^* > N$. But it results that $z = u$, where u is the solution stated in Theorem 4.1 and we proved that u does not belong to $W_0^{1,p}(\Omega)$.

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