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Mathematical Physics — *Some remarks on a linearized Schrödinger equation*, by G. Busoni and L. Prati, communicated on 13 February 2015.

Abstract. — In the present paper we propose four systems of linear Partial Differential Equations that can be deduced from the nonlinear Schrödinger equation for the propagation of light in optical fibers in the frame of the recently-proposed Combined Regular-Logarithmic Perturbation method. The unknown function in the Schrödinger equation is the optical field envelope; it is a complex-valued function. Following the Combined Regular-Logarithmic Perturbation method, proposed by Secondini, Forestieri and Menyuk, we look for complex solutions of the Schrödinger equation in the form of a perturbed continuous wave that relates three unknown real-valued functions. Since the Schrödinger equation is complex, we split it into two real equations, both in the three real unknowns. We linearize these two equations and add a third linear equation that relates the same three unknown quantities. We propose four different choices for the third equation, therefore we obtain four different real systems of linear Partial Differential Equations and we analyze the corresponding systems of Ordinary Differential Equations for the Fourier transforms of the unknowns. One of the four systems we obtain is equivalent to that studied by the quoted authors. We add to it other three choices that could be useful to model different situations. Again, we consider the real part of the Ordinary Differential Equations and we present solutions in recursive form. We also suggest solutions for the complex-valued Fourier transforms by using Bessel functions.

Key words: Linearized Schrödinger equation, PDEs, ODEs, optical fiber, perturbation method.

Mathematics Subject Classification: 35Q55, 35Q94, 35C10, 34A25, 34A30.

1. Introduction

Light propagates in optical fibers according to the nonlinear Schrödinger equation (NLSE) and its variants, [4, 1]. Moving from one variant of the NLSE, in this paper we are going to examine several systems of Partial Differential Equations (PDEs) that can be deduced from it. In the model considered in this paper, the following aspects are taken into account: chromatic dispersion, loss and self-phase modulation. We assume that the time-frame is moving with the signal group velocity. Effects of polarization are neglected. With the above assumptions, the NLSE for a single-mode fiber can be written as follows, [3, 5]:

$$\frac{\partial v}{\partial z} = j\frac{\beta_2}{2}\frac{\partial^2 v}{\partial t^2} - j\gamma|v|^2 v - \frac{\alpha}{2}v \tag{1.1}$$

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where the complex-valued function $v(z,t)$ is the optical field envelope, β_2 is the chromatic dispersion coefficient ($\beta_2 > 0$ for defocusing fibers, $\beta_2 < 0$ for focusing ones), $\gamma > 0$ is the Kerr nonlinear coefficient, $\alpha > 0$ is the power attenuation constant, z represents the coordinate along the fiber axis, t is time and j stands for the unit imaginary number.

REMARK 1 (Dimensional Analysis). The physical dimension of the optical field envelope is $[v] = [u] = M^{1/2}LT^{-3/2}$. The chromatic dispersion coefficient β_2 has real values and its dimension is $[\beta_2] = L^{-1}T^2$; it is $\beta_2 > 0$ for *defocusing* fibers or $\beta_2 < 0$ for *focusing* fibers; the Kerr nonlinear coefficient $\gamma > 0$ has real values and dimension $[\gamma] = M^{-1}L^{-3}T^3$; the power attenuation constant $\alpha > 0$ has dimension $[\alpha] = L^{-1}$. We are going to assume the three coefficients as constants (see also the following Remark 3). The power $P_0 > 0$ has dimension $[P_0] = ML^2T^{-3}$; the real-valued functions a and b represent a-dimensional quantities: $[a] = [b] = M^0L^0T^0$. Also the phase θ is a real-valued function which represents an a-dimensional quantity: $[\theta] = M^0L^0T^0$.

By putting:

$$v(z,t) := \exp\left(\frac{-\alpha z}{2}\right)u(z,t) \tag{1.2}$$

Equation (1.1) becomes:

$$\frac{\partial u}{\partial z} = j\frac{\beta_2}{2}\frac{\partial^2 u}{\partial t^2} - j\gamma\exp(-\alpha z)|u|^2u. \tag{1.3}$$

We are interested in finding solutions $u(z,t)$ belonging to the Hilbert space $L^2(\mathbb{R})$ with respect to t.

Solutions to Equation (1.3) are known in some special cases. For $\gamma = 0$ it is

$$u(z,t) = \int_{\mathbb{R}} \frac{\exp\left[-\frac{(t-t')^2}{2j\beta_2(z-z_0)}\right]}{\sqrt{2\pi j\beta_2(z-z_0)}} u(z=z_0,t')\,dt'$$

where z_0 is the initial coordinate of the fiber. Another well-known solution to Equation (1.3) is for $\beta_2 = 0$ and it is:

$$u(z,t) = u(z=z_0,t)\exp\left[j\frac{\gamma|u(z=z_0,t)|^2}{\alpha}(\exp(-\alpha z) - \exp(-\alpha z_0))\right].$$

Also in a special case, for soliton solutions with $\alpha = 0$, exact analytical solutions are known, see [8]. In general, when $\alpha \neq 0$ numerical methods or analytical approximations are necessary, see [5, 6] and the bibliography therein for an overview.

We are mainly interested in the results regarding the Combined Regular-Logarithmic Perturbation method proposed in [6]. This method is suitable for modeling the presence of nonlinear and dispersive effects because the contribution of the quadratic perturbation terms is preserved even after the linearization. The signal is considered like a continuous wave (CW) and the noise is treated like a perturbation. The authors look for solutions of Equation (1.3) in the form:

$$u(z,t) = \sqrt{P_0}[1 + a(z,t) + jb(z,t)]\exp[-j\theta(z,t)] \tag{1.4}$$

where $P_0 > 0$ is the power of the noise-free solution and $a(z,t)$, $b(z,t)$ and $\theta(z,t)$ are real perturbation functions; these latter are a-dimensional quantities. By putting Equation (1.4) in Equation (1.3) and naming the complex-valued additive perturbation function $c(z,t) := a(z,t) + jb(z,t)$, the authors obtain the following complex differential equation to be solved:

$$\begin{aligned} \frac{\partial c}{\partial z} - j(1+c)\frac{\partial \theta}{\partial z} = j\frac{\beta_2}{2}\left[\frac{\partial^2 c}{\partial t^2} - j(1+c)\frac{\partial^2 \theta}{\partial t^2} - 2j\frac{\partial c}{\partial t}\frac{\partial \theta}{\partial t} - (1+c)\left(\frac{\partial \theta}{\partial t}\right)^2\right] \\ - j\gamma P_0 \exp(-\alpha z)|1+c|^2(1+c), \end{aligned} \tag{1.5}$$

where the dependence of both functions c and θ on z and t has been omitted.

By splitting Equation (1.5) into two real differential equations, the authors obtain a system of two equations in three unknowns: a, b and θ. Being such a system under-determined, they arbitrarily add an additional equation which relates the three variables. In order to obtain a linearized model leading to a useful solution of Equation (1.5), they add the following equation:

$$\frac{\partial \theta}{\partial z} = \gamma P_0 \exp(-\alpha z)|1+c|^2 + \frac{\beta_2}{2}\left(\frac{\partial \theta}{\partial t}\right)^2. \tag{1.6}$$

Indeed in [6] it is explained that as in principle all of the choices are equivalent, Equation (1.6) leads to a useful solution because it minimizes the impact of the terms neglected in the linearization process. The system they aim to solve, simplified thanks to Equation (1.6), is therefore the following:

$$\frac{\partial a}{\partial z} = \frac{\beta_2}{2}\left(-\frac{\partial^2 b}{\partial t^2} + \frac{\partial^2 \theta}{\partial t^2} + a\frac{\partial^2 \theta}{\partial t^2} + 2\frac{\partial a}{\partial t}\frac{\partial \theta}{\partial t}\right) \tag{1.7a}$$

$$\frac{\partial b}{\partial z} = \frac{\beta_2}{2}\left(\frac{\partial^2 a}{\partial t^2} + b\frac{\partial^2 \theta}{\partial t^2} + 2\frac{\partial b}{\partial t}\frac{\partial \theta}{\partial t}\right) \tag{1.7b}$$

$$\frac{\partial \theta}{\partial z} = \gamma P_0 \exp(-\alpha z)(1 + 2a + a^2 + b^2) + \frac{\beta_2}{2}\left(\frac{\partial \theta}{\partial t}\right)^2. \tag{1.7c}$$

They define

$$\theta(z,t) = \phi(z,t) + \int_0^z \gamma P_0 \exp(-\alpha s)\, ds, \tag{1.8}$$

where $\int_0^z \gamma P_0 \exp(-\alpha s)\, ds := \phi_{NL}(z)$ is the deterministic time-independent nonlinear phase rotation of the noise-free solution. By neglecting the quadratic terms, the authors obtain a linearized system for the Fourier transforms $A(z,\omega)$, $B(z,\omega)$ and $\Phi(z,\omega)$ of $a(z,t)$, $b(z,t)$ and $\phi(z,t)$ respectively:

$$\frac{\partial A}{\partial z} = \frac{\beta_2}{2}\omega^2(B - \Phi) \tag{1.9a}$$

$$\frac{\partial B}{\partial z} = -\frac{\beta_2}{2}\omega^2 A \tag{1.9b}$$

$$\frac{\partial \Phi}{\partial z} = 2\gamma P_0 \exp(-\alpha z) A. \tag{1.9c}$$

In order to solve system (1.9), the authors suggest the use of a 3×3 transfer matrix. When $\alpha = 0$ and γ and β_2 are constant, the transfer matrix assumes a simple closed-form expression.

In this paper, we go back to the complex differential equation in (1.5), we split it into the real and imaginary parts, and we add a third equation which relates the three unknowns in order to obtain a linearized system. We show four different possibilities for choosing the third equation, see Section 2. One of these leads exactly to the same situation modeled in [6]. Indeed, we obtain four systems of Ordinary Differential Equations (ODEs) for the Fourier transforms A, B and Φ of the real-valued functions a, b and ϕ and one of these systems leads to system (1.9). By handling the real part (or equivalently the imaginary part) of the Fourier transforms, we are able to present solutions in recursive forms even for α not null, see Section 3. In the same section we also suggest solutions for the complex-valued Fourier transforms by using the Bessel functions of the first kind for the case $\beta_2 > 0$ or the modified Bessel functions of the first kind for the case $\beta_2 < 0$. In Section 4 we briefly discuss some conclusive remarks; in the Appendix we compute the solution of a fourth-order equation (see Equation (2.6) in Section 2) for the real-valued function a both in the homogeneous and non-homogeneous cases.

2. The linearized systems

By considering Equation (1.5) and substituting $c(z,t) = a(z,t) + jb(z,t)$, one obtains the two following real differential equations in $a(z,t)$, $b(z,t)$ and $\phi(z,t)$:

$$\frac{\partial a}{\partial z} = \frac{\beta_2}{2}\left[-\frac{\partial^2 b}{\partial t^2} + (1+a)\frac{\partial^2 \phi}{\partial t^2} + 2\frac{\partial a}{\partial t}\frac{\partial \phi}{\partial t} + b\left(\frac{\partial \phi}{\partial t}\right)^2\right] + \gamma P_0 b(2a + a^2 + b^2)\exp(-\alpha z) - b\frac{\partial \phi}{\partial z} \tag{2.1a}$$

(2.1b)
$$\frac{\partial b}{\partial z} = \frac{\beta_2}{2}\left[\frac{\partial^2 a}{\partial t^2} + b\frac{\partial^2 \phi}{\partial t^2} + 2\frac{\partial b}{\partial t}\frac{\partial \phi}{\partial t} - (1+a)\left(\frac{\partial \phi}{\partial t}\right)^2\right] - \gamma P_0(1+a)(2a+a^2+b^2)\exp(-\alpha z) + (1+a)\frac{\partial \phi}{\partial z}.$$

Assume that $z > 0$ is real and $t \in \mathbb{R}$. In the following, a linearized problem associated to system (2.1) is studied. The linear terms in Equations (2.1a) and (2.1b) are respectively:

(2.2a)
$$\frac{\partial a}{\partial z} = \frac{\beta_2}{2}\left[-\frac{\partial^2 b}{\partial t^2} + \frac{\partial^2 \phi}{\partial t^2}\right]$$

(2.2b)
$$\frac{\partial b}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 a}{\partial t^2} - 2\gamma P_0 a\exp(-\alpha z) + \frac{\partial \phi}{\partial z}.$$

Equations (2.2a) and (2.2b) constitute a system of two equations in three unknowns: a, b and ϕ. Being such a system under-determined, one looks for an additional linear equation which relates the three variables.

The second-order terms for Equations (2.1a) and (2.1b) are respectively:

(2.3a)
$$\frac{\beta_2}{2}\left[a\frac{\partial^2 \phi}{\partial t^2} + 2\frac{\partial a}{\partial t}\frac{\partial \phi}{\partial t}\right] + 2\gamma P_0 ab\exp(-\alpha z) - b\frac{\partial \phi}{\partial z}$$

(2.3b)
$$\frac{\beta_2}{2}\left[b\frac{\partial^2 \phi}{\partial t^2} + 2\frac{\partial b}{\partial t}\frac{\partial \phi}{\partial t} - \left(\frac{\partial \phi}{\partial t}\right)^2\right] - \gamma P_0(3a^2+b^2)\exp(-\alpha z) + a\frac{\partial \phi}{\partial z};$$

similarly the third-order terms for the two equations in system (2.1) are respectively:

(2.4a)
$$\frac{\beta_2}{2}b\left(\frac{\partial \phi}{\partial t}\right)^2 + \gamma P_0 b(a^2+b^2)\exp(-\alpha z)$$

(2.4b)
$$-\frac{\beta_2}{2}a\left(\frac{\partial \phi}{\partial t}\right)^2 - \gamma P_0 a(a^2+b^2)\exp(-\alpha z).$$

In order to obtain a third linear equation, to be considered together with equations in system (2.2), one can equal to zero some addends in (2.3a) or in (2.3b). Here it is not evaluated the efficiency of one choice with respect to the others by an engineering point of view.

By starting from the second-order term in (2.3a) one obtains the linear relationships in Equations (2.5a) and (2.5b); by starting from the second-order term in (2.3b) one obtains the linear relationships in Equations (2.5c), (2.5d) and (2.5e). The five alternative equations are therefore the following:

(2.5a) $$\frac{\beta_2}{2}\frac{\partial^2\phi}{\partial t^2} + 2\gamma P_0 b \exp(-\alpha z) = 0 \quad \text{if } a \not\equiv 0$$

(2.5b) $$\frac{\partial\phi}{\partial z} - 2\gamma P_0 a \exp(-\alpha z) = 0 \quad \text{if } b \not\equiv 0$$

(2.5c) $$\frac{\beta_2}{2}\frac{\partial^2\phi}{\partial t^2} - \gamma P_0 b \exp(-\alpha z) = 0 \quad \text{if } b \not\equiv 0$$

(2.5d) $$\frac{\partial\phi}{\partial z} - 3\gamma P_0 a \exp(-\alpha z) = 0 \quad \text{if } a \not\equiv 0$$

(2.5e) $$2\frac{\partial b}{\partial t} - \frac{\partial\phi}{\partial t} = 0 \quad \text{if } \frac{\partial\phi}{\partial t} \not\equiv 0$$

Condition (2.5c) is not admissible. Indeed, if $\beta_2 \to 0$ and $\gamma \neq 0$, condition (2.5c) leads to a contradiction: $b \not\equiv 0$ and $b \equiv 0$ at the same time.

REMARK 2. By deriving the terms in Equation (2.2a) with respect to z and changing order of derivatives (assuming it is allowed) and referring to Equation (2.2b), one obtains the following fourth-order equation:

$$\frac{\partial^2 a}{\partial z^2} = -\left(\frac{\beta_2}{2}\right)^2\frac{\partial^4 a}{\partial t^4} + \beta_2\gamma P_0 \exp(-\alpha z)\frac{\partial^2 a}{\partial t^2}$$

or

(2.6) $$\frac{\partial^2 a}{\partial z^2} = \frac{\partial^2}{\partial t^2}\left\{-\left(\frac{\beta_2}{2}\right)^2\frac{\partial^2 a}{\partial t^2} + \beta_2\gamma P_0 \exp(-\alpha z) a\right\}.$$

An explicit solution can be given when $\gamma = 0$. The reader can find the steps in the Appendix. The NLSE has a known solution for $\gamma = 0$, but solving Equation (2.6) for $\gamma = 0$ let one determine the unknown real-valued function $a(z,t)$ in this special case. Equations in (2.2) and one of (2.5a), (2.5b), (2.5d) or (2.5e) let us determine also the real-valued functions $b(z,t)$ and $\phi(z,t)$ when $\gamma = 0$, therefore we can obtain the three perturbation components for the solution in Equation (1.4). But Equation (2.6) when $\gamma = 0$ can be seen like the homogeneous equation associated to the non-homogeneous one:

$$\frac{\partial^2 a}{\partial z^2} + \left(\frac{\beta_2}{2}\right)^2\frac{\partial^4 a}{\partial t^4} = q(z,t)$$

where $q(z,t)$ is a real-valued source term. The steps for finding $a(z,t)$ in this case are given in the Appendix. The Appendix concludes with an implicit form of Equation (2.6), relating $a(z,t)$ and its second-order partial derivative $\frac{\partial^2 a}{\partial t^2}(z,t)$, that can be transformed into a Volterra integral equation in the unknown $a(z,t)$.

By considering Equations (2.2a), (2.2b) and one of (2.5a), (2.5b), (2.5d) or (2.5e), one obtains four different systems, each constituted by three PDEs. Such systems are difficult to handle; therefore it may be interesting to consider $A(z,\omega)$, $B(z,\omega)$ and $\Phi(z,\omega)$, the Fourier transforms, with respect to $t \in \mathbb{R}$, of $a(z,t)$, $b(z,t)$ and $\phi(z,t)$ respectively. The new equations contain the real variable $z > 0$ and $\omega \in \mathbb{R}$ (dimension T^{-1}) as a parameter. Differentiation is with respect to z. Functions $A(z,\omega)$, $B(z,\omega)$ and $\Phi(z,\omega)$ have complex values but the new equations can be seen like equations either in $(\mathrm{Re}(A), \mathrm{Re}(B), \mathrm{Re}(\Phi))$ or in $(\mathrm{Im}(A), \mathrm{Im}(B), \mathrm{Im}(\Phi))$, according to the standard notation $F(z,\omega) = \mathrm{Re}(F(z,\omega)) + j\,\mathrm{Im}(F(z,\omega))$ for any complex-valued function $F(z,\omega)$. For the sake of simplicity, the real functions $\mathrm{Re}(A)$, $\mathrm{Re}(B)$, $\mathrm{Re}(\Phi)$ (or equivalently $\mathrm{Im}(A)$, $\mathrm{Im}(B)$, $\mathrm{Im}(\Phi)$) are denoted by means of letters A, B and Φ; they are a-dimensional quantities, like a, b and ϕ.

The real coefficients β_2, γ, α, P_0 are constant. The derivative with respect to z of a function $f(z,\omega)$ is represented as $f'(z,\omega)$; z and ω are omitted in the following four real systems:

$$(2.7)\qquad \begin{cases} A' = \frac{\beta_2}{2}\omega^2 B - \frac{\beta_2}{2}\omega^2 \Phi \\ B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A + \Phi' \\ -\frac{\beta_2}{2}\omega^2 \Phi + 2\gamma P_0 \exp(-\alpha z) B = 0 & \text{with } A \not\equiv 0 \end{cases}$$

$$(2.8)\qquad \begin{cases} A' = \frac{\beta_2}{2}\omega^2 B - \frac{\beta_2}{2}\omega^2 \Phi \\ B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A + \Phi' \\ \Phi' = 2\gamma P_0 \exp(-\alpha z) A & \text{with } B \not\equiv 0 \end{cases}$$

$$(2.9)\qquad \begin{cases} A' = \frac{\beta_2}{2}\omega^2 B - \frac{\beta_2}{2}\omega^2 \Phi \\ B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A + \Phi' \\ \Phi' = 3\gamma P_0 \exp(-\alpha z) A & \text{with } A \not\equiv 0 \end{cases}$$

$$(2.10)\qquad \begin{cases} A' = \frac{\beta_2}{2}\omega^2 B - \frac{\beta_2}{2}\omega^2 \Phi \\ B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A + \Phi' \\ \omega(2B - \Phi) = 0 & \text{with } \Phi \not\equiv 0 \end{cases}$$

Note that system (2.8) is equal to system (1.9), obtained in [6], with the request that $B \not\equiv 0$. System (2.10) becomes easily:

$$(2.11)\qquad \begin{cases} A' = -\frac{\beta_2}{2}\omega^2 B \\ B' = \left[\frac{\beta_2}{2}\omega^2 + 2\gamma P_0 \exp(-\alpha z)\right] A \\ \Phi = 2B & \text{with } \Phi \not\equiv 0. \end{cases}$$

The four systems are equipped with the initial conditions at $z = 0$:

$$(2.12a)\qquad A(z=0,\omega) = A_0(\omega)$$

$$(2.12b)\qquad B(z=0,\omega) = B_0(\omega)$$

$$(2.12c)\qquad \Phi(z=0,\omega) = \Phi_0(\omega)$$

or, by omitting the dependence on ω:

$$A(z=0) = A_0 \tag{2.13a}$$

$$B(z=0) = B_0 \tag{2.13b}$$

$$\Phi(z=0) = \Phi_0. \tag{2.13c}$$

REMARK 3. The reader can note that the coefficients β_2 and γ could be dependent on z (but not on t) without changing the expressions for systems (2.7)–(2.10).

From the first and the second equations of all the four systems (2.7)–(2.10):

$$A' = \frac{\beta_2}{2}\omega^2 B - \frac{\beta_2}{2}\omega^2 \Phi, \tag{2.14}$$

$$B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A + \Phi' \tag{2.15}$$

one obtains the following equation (see Equation (7) in [2]):

$$A'' = -\frac{\beta_2}{2}\omega^2 \left(\frac{\beta_2}{2}\omega^2 + 2\gamma P_0 \exp(-\alpha z)\right) A \tag{2.16}$$

where f'' represents the second-order derivative of any function $f(z, \omega)$ with respect to z. Indeed, Equation (2.15) can be written as:

$$B' - \Phi' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A, \tag{2.17}$$

and Equation (2.14) can be re-arranged as:

$$A' = \frac{\beta_2}{2}\omega^2 (B - \Phi). \tag{2.18}$$

The previous equation can be derived with respect to z:

$$A'' = \frac{\beta_2}{2}\omega^2 (B' - \Phi') \tag{2.19}$$

which becomes, by referring to Equation (2.17),

$$A'' = \frac{\beta_2}{2}\omega^2 \left(-\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z) A\right) \tag{2.20}$$

and Equation (2.16) follows directly. Once ODE (2.16) is solved, A can be put in the four systems, obtaining easily both B and Φ, also assuming: $\omega \neq 0$ for systems (2.7) and (2.10); $-\frac{\beta_2}{2}\omega^2 + 2\gamma P_0 \exp(-\alpha z) \neq 0$ for system (2.7).

3. Solution of the ordinary differential equation

Consider the ODE (2.16) equipped with two initial conditions, see Equation (2.13a):

$$A''(z) = -\frac{\beta_2}{2}\omega^2\left(\frac{\beta_2}{2}\omega^2 + 2\gamma P_0 \exp(-\alpha z)\right)A(z) \tag{3.1a}$$

$$A(z=0) = A_0 \tag{3.1b}$$

$$A'(z=0) = A_0' \tag{3.1c}$$

where it has been omitted the dependence of A, A_0 and A_0' on ω.

The ODE can be solved by a development in power series for z with coefficients depending on ω and on the constant real parameters β_2, γ and α. For the sake of simplicity we put:

$$\delta := \frac{\beta_2}{2}\omega^2 \tag{3.2}$$

and

$$k := 2\gamma P_0. \tag{3.3}$$

Note that δ is positive if $\beta_2 > 0$ (defocusing fiber) and it is negative if $\beta_2 < 0$ (focusing fiber); k is positive. Equation (3.1a) becomes:

$$A''(z) = -\delta^2 A(z) - \delta k \exp(-\alpha z)A(z). \tag{3.4}$$

Recall that it is:

$$\exp(-\alpha z) = \sum_{i=0}^{+\infty} \frac{(-\alpha z)^i}{i!}. \tag{3.5}$$

Look for solutions A having the following form:

$$A(z) = \sum_{m=0}^{+\infty} a_m z^m; \tag{3.6}$$

note that from problem (3.1) it follows:

$$A(z=0) = a_0 = A_0(\omega) \tag{3.7}$$

and

$$A'(z=0) = a_1 = A_0'(\omega). \tag{3.8}$$

From Equations (3.5) and (3.6) one obtains:

$$\exp(-\alpha z)A(z) = \sum_{n=0}^{+\infty}\Big(\sum_{i=0}^{n}\frac{(-\alpha)^i}{i!}a_{n-i}\Big)z^n \tag{3.9}$$

and

$$A''(z) = \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}z^n. \tag{3.10}$$

Therefore Equation (3.4) becomes

$$\sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}z^n = -\delta^2\sum_{n=0}^{+\infty}a_n z^n - \delta k\sum_{n=0}^{+\infty}\Big(\sum_{i=0}^{n}\frac{(-\alpha)^i}{i!}a_{n-i}\Big)z^n \tag{3.11}$$

and this gives the following recursive relationship for $n \geq 0$:

$$(n+2)(n+1)a_{n+2} = -\delta^2 a_n - \delta k\sum_{i=0}^{n}\frac{(-\alpha)^i}{i!}a_{n-i}, \tag{3.12}$$

where

$$a_0 = A_0(\omega)$$

and

$$a_1 = A_0'(\omega).$$

Here some values for a_n when $n \geq 0$ are explicitly written. For $n = 0$, it is

$$a_2 = -\frac{1}{2!}\delta(\delta+k)a_0. \tag{3.13}$$

For $n = 1$ it results

$$a_3 = -\frac{1}{3!}\delta(\delta+k)a_1 + \frac{1}{3!}\alpha\delta k a_0. \tag{3.14}$$

For $n = 2$ it is

$$a_4 = \frac{2\delta k\alpha}{4!}a_1 + \frac{[\delta(\delta+k)]^2 - \delta k\alpha^2}{4!}a_0. \tag{3.15}$$

Similarly, for $n = 3$ it is:

$$a_5 = \frac{[\delta(\delta+k)]^2 - 3\delta k\alpha^2}{5!}a_1 - \frac{4\delta^2 k(\delta+k)\alpha - \delta k\alpha^3}{5!}a_0. \tag{3.16}$$

Therefore, according to Equation (3.6), one obtains:

$$\begin{aligned} A(z) &= \sum_{n=0}^{5} a_n z^n + R_5(z) \\ &= a_0\left\{1 - \frac{\delta(\delta+k)}{2!}z^2 + \frac{\delta k\alpha}{3!}z^3 + \frac{[\delta(\delta+k)]^2 - \delta k\alpha^2}{4!}z^4 \right. \\ &\qquad \left. - \frac{4\delta^2 k(\delta+k)\alpha - \delta k\alpha^3}{5!}z^5\right\} \\ &\quad + a_1\left\{z - \frac{\delta(\delta+k)}{3!}z^3 + \frac{2\delta k\alpha}{4!}z^4 + \frac{[\delta(\delta+k)]^2 - 3\delta k\alpha^2}{5!}z^5\right\} + R_5(z) \end{aligned} \tag{3.17}$$

where $R_5(z)$ is the rest of the series.

Remark 4. Note that $\alpha > 0$ has its effects on the terms of order three or more.

Recall that $\beta_2 > 0$ or $\beta_2 < 0$. It is useful to treat the two cases separately. First let us consider the case when $\alpha = 0$ and $\beta_2 > 0$ (therefore $\delta > 0$). The second order Equation (3.4) becomes, for $\alpha = 0$:

$$A''(z) = -\delta^2 A(z) - \delta k A(z) \tag{3.18}$$

or therefore, with $\beta_2 > 0$:

$$A''(z) = -h^2 A(z) \tag{3.19}$$

by putting

$$h := \sqrt{\delta(\delta+k)}. \tag{3.20}$$

The reader can easily recognize the equation of harmonic motion where $h > 0$ is the pulsation. By taking into account the initial conditions in problem (3.1), the bounded solutions are:

$$\begin{aligned} A(z,\omega) &= A_0(\omega)\cos\left(\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}z\right) \\ &\quad + \frac{A_0'(\omega)}{\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}}\sin\left(\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}z\right), \end{aligned} \tag{3.21}$$

where it is reintroduced the dependence for A on both z and ω. Formally, the previous solution is true for every $\omega \in \mathbb{R}$ and for every $z \in \mathbb{R}$ (even if the fiber is defined for $z > 0$).

If α is null, but the fiber is focusing, that is $\beta_2 < 0$ (and therefore $\delta < 0$), one still obtains solutions like those in Equation (3.21) provided that $\delta + k < 0$, that is $\omega^2 > 4\gamma P_0|\beta_2|^{-1}$. For the values of ω such that $\omega^2 < 4\gamma P_0|\beta_2|^{-1}$, solutions $A(z,\omega)$ are given by combinations of real exponential functions which diverge for $z \to \infty$. Indeed, put

$$r := \sqrt{-\delta(\delta + k)}, \tag{3.22}$$

where $-\delta(\delta + k) > 0$ because of the above assumptions; Equation (3.18) can be written as follows:

$$A''(z) = r^2 A(z); \tag{3.23}$$

according to the initial conditions in problem (3.1) and re-introducing the dependence for A on both z and ω, the solutions are:

$$\begin{aligned} A(z,\omega) = {} & \frac{1}{2}\left\{A_0(\omega) + \frac{A_0'(\omega)}{\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}}\right\} \\ & \times \exp\left(\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}\,z\right) \\ & + \frac{1}{2}\left\{A_0(\omega) - \frac{A_0'(\omega)}{\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}}\right\} \\ & \times \exp\left(-\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}\,z\right) \end{aligned} \tag{3.24}$$

One obtains the following limits, when $r = r(\omega) = \sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)} \to 0^+$:

$$\lim_{\omega\to 0} A(z,\omega) = A_0(\omega = 0) + A_0'(\omega = 0)z, \tag{3.25}$$

$$\lim_{\omega\to\left(\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)^-} A(z,\omega) = A_0\left(\omega = \sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) + A_0'\left(\omega = \sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)z \tag{3.26}$$

and

$$\lim_{\omega \to \left(-\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)^+} A(z,\omega) = A_0\left(\omega = -\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) + A_0'\left(\omega = -\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)z. \tag{3.27}$$

Note that for $z \to +\infty$, it is $A(z,\omega) \to +\infty$, maybe except for $\omega = 0$ and $\omega = \pm\sqrt{\frac{4\gamma P_0}{|\beta_2|}}$. For such values of ω, Equation (3.18) becomes:

$$A''(z) = 0 \tag{3.28}$$

whose solutions are, by taking into account the initial conditions in problem (3.1):

$$A(z,\omega = 0) = A_0(\omega = 0) + A_0'(\omega = 0)z, \tag{3.29}$$

$$A\left(z,\omega = \sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) = A_0\left(\omega = \sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) + A_0'\left(\omega = \sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)z \tag{3.30}$$

and

$$A\left(z,\omega = -\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) = A_0\left(\omega = -\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right) + A_0'\left(\omega = -\sqrt{\frac{4\gamma P_0}{|\beta_2|}}\right)z. \tag{3.31}$$

We are now going to consider the case when $\alpha > 0$ and $\beta_2 > 0$ (and therefore $\delta > 0$). By referring to Equations (3.6), (3.17) and (3.21), the solution can be written in the following form:

$$\begin{aligned} A(z,\omega) &= A_0(\omega)\cos\left(\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}z\right) \\ &\quad + \frac{A_0'(\omega)}{\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}}\sin\left(\sqrt{\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)}z\right) \\ &\quad + A_0(\omega)\left\{\frac{\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha}{3!}z^3 - \frac{\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^2}{4!}z^4\right. \\ &\quad \left. - \frac{4\left(\frac{\beta_2\omega^2}{2}\right)^2 2\gamma P_0\left(\frac{\beta_2\omega^2}{2} + 2\gamma P_0\right)\alpha - \frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^3}{5!}z^5 + \cdots\right\} \\ &\quad + A_0'(\omega)\left\{\frac{2\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha}{4!}z^4 - \frac{3\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^2}{5!}z^5 + \cdots\right\} \end{aligned} \tag{3.32}$$

Similarly, when $\alpha > 0$ and $\beta_2 < 0$ (and therefore $\delta < 0$), by referring to Equations (3.6), (3.17) and (3.24), one obtains:

$$
\begin{aligned}
(3.33)\quad A(z,\omega) = {} & \frac{1}{2}\left\{ A_0(\omega) + \frac{A_0'(\omega)}{\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2}+2\gamma P_0\right)}} \right\} \\
& \times \exp\Big(\sqrt{-\frac{\beta_2\omega^2}{2}\Big(\frac{\beta_2\omega^2}{2}+2\gamma P_0\Big)}\,z\Big) \\
& + \frac{1}{2}\left\{ A_0(\omega) - \frac{A_0'(\omega)}{\sqrt{-\frac{\beta_2\omega^2}{2}\left(\frac{\beta_2\omega^2}{2}+2\gamma P_0\right)}} \right\} \\
& \times \exp\Big(-\sqrt{-\frac{\beta_2\omega^2}{2}\Big(\frac{\beta_2\omega^2}{2}+2\gamma P_0\Big)}\,z\Big) \\
& + A_0(\omega)\left\{ \frac{\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha}{3!}z^3 - \frac{\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^2}{4!}z^4 \right. \\
& \left. - \frac{4\left(\frac{\beta_2\omega^2}{2}\right)^2 2\gamma P_0\left(\frac{\beta_2\omega^2}{2}+2\gamma P_0\right)\alpha - \frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^3}{5!}z^5 + \cdots \right\} \\
& + A_0'(\omega)\left\{ \frac{2\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha}{4!}z^4 - \frac{3\frac{\beta_2\omega^2}{2}2\gamma P_0\alpha^2}{5!}z^5 + \cdots \right\}.
\end{aligned}
$$

Once it has been solved the ODE in (3.1), systems (2.7)–(2.10) can be solved, too.

3.1. Solution of the real systems

In the previous Section the ODE in problem (3.1) has been solved. Now that $A(z,\omega)$ has been determined, one can solve the real systems (2.7)–(2.10). The focus is put on system (2.8) because it is the same as in [6]. Being $A(z)$ known, one has to solve:

$$
(3.34)\quad \begin{cases} B' = -\frac{\beta_2}{2}\omega^2 A - 2\gamma P_0 \exp(-\alpha z)A + \Phi' \\ \Phi' = 2\gamma P_0 \exp(-\alpha z)A \qquad\qquad \text{with } B \not\equiv 0 \end{cases}
$$

that is, by substituting the second equation in the first one:

$$
(3.35)\quad \begin{cases} B' = -\frac{\beta_2}{2}\omega^2 A \\ \Phi' = 2\gamma P_0 \exp(-\alpha z)A \quad \text{with } B \not\equiv 0 \end{cases}
$$

and therefore it results, by re-introducing the dependence of A, B and Φ on z and ω:

$$B(z,\omega) = B_0(\omega) - \int_0^z \frac{\beta_2}{2}\omega^2 A(z',\omega)\,dz', \tag{3.36}$$

$$\Phi(z,\omega) = \Phi_0(\omega) + 2\gamma P_0 \int_0^z \exp(-\alpha z')A(z',\omega)\,dz'. \tag{3.37}$$

Systems (2.7) and (2.9)–(2.10) can be treated similarly.

For system (2.7) one obtains if $\beta_2 < 0$ (focusing case):

$$B(z,\omega) = \frac{A'(z)}{\frac{\beta_2}{2}\omega^2 - 2\gamma P_0 \exp(-\alpha z)}, \tag{3.38}$$

$$\Phi(z,\omega) = \frac{2\gamma P_0 \exp(-\alpha z)}{\frac{\beta_2}{2}\omega^2} \frac{A'(z)}{\frac{\beta_2}{2}\omega^2 - 2\gamma P_0 \exp(-\alpha z)}. \tag{3.39}$$

For system (2.9) it is:

$$B(z,\omega) = B_0(\omega) - \int_0^z \frac{\beta_2}{2}\omega^2 A(z',\omega)\,dz' + \gamma P_0 \int_0^z \exp(-\alpha z')A(z',\omega)\,dz', \tag{3.40}$$

$$\Phi(z,\omega) = \Phi_0(\omega) + 3\gamma P_0 \int_0^z \exp(-\alpha z')A(z',\omega)\,dz'. \tag{3.41}$$

For system (2.10) it results:

$$\begin{aligned} B(z,\omega) = {} & B_0(\omega) + \int_0^z \frac{\beta_2}{2}\omega^2 A(z',\omega)\,dz' \\ & + 2\gamma P_0 \int_0^z \exp(-\alpha z')A(z',\omega)\,dz', \end{aligned} \tag{3.42}$$

$$\begin{aligned} \Phi(z,\omega) = {} & 2B_0(\omega) + \int_0^z \beta_2\omega^2 A(z',\omega)\,dz' \\ & + 4\gamma P_0 \int_0^z \exp(-\alpha z')A(z',\omega)\,dz'. \end{aligned} \tag{3.43}$$

3.2. Solution of the Ordinary Differential Equation for complex-valued functions

So far it was considered the Cauchy problem for A in (3.1) with the assumption that the unknowns were the real part (or on the imaginary part) of A, because handling real-valued functions is often useful. Moreover, by means of this approach it results that the power attenuation constant α does not affect the terms of order less than 3 in the power series solution (3.6).

Anyhow, one can also solve the Cauchy problem for the complex-valued function A by using the Bessel functions in the non trivial case when $\omega \neq 0$ (and therefore $\delta \neq 0$). Bessel Functions are often used in problems related to fiber optics, see for instance [2] where the authors report a solution to a similar equation in terms of Hankel functions (or Bessel functions of third kind). Anyhow, we distinguish between two different cases: $\beta_2 > 0$ and $\beta_2 < 0$. This distinction leads us to the Bessel equation and to the modified Bessel equation respectively. Indeed, by putting

$$A(z) := y(\xi(z)) \tag{3.44}$$

where

$$\xi(z) := \frac{2}{\alpha}\sqrt{|\delta|k}\exp\left(-\frac{\alpha}{2}z\right) \tag{3.45}$$

with δ and k as defined in (3.2) and (3.3) and α positive, Equation (3.4) becomes:

$$\xi^2 y''(\xi) + \xi y'(\xi) + \left[\left(\frac{2|\delta|}{\alpha}\right)^2 + \operatorname{sgn}(\delta)\xi^2\right] y(\xi) = 0. \tag{3.46}$$

The previous Equation can be rewritten as follows:

$$\xi^2 y''(\xi) + \xi y'(\xi) + \left[\operatorname{sgn}(\delta)\xi^2 - \left(\frac{2|\delta|}{\alpha}j\right)^2\right] y(\xi) = 0. \tag{3.47}$$

Put

$$v := \frac{2|\delta|}{\alpha}j, \tag{3.48}$$

therefore Equation (3.47) becomes:

$$\xi^2 y''(\xi) + \xi y'(\xi) + [\operatorname{sgn}(\delta)\xi^2 - v^2] y(\xi) = 0. \tag{3.49}$$

Two cases are possible: either $\operatorname{sgn}(\delta) = \operatorname{sgn}(\beta_2) = 1$ or $\operatorname{sgn}(\delta) = \operatorname{sgn}(\beta_2) = -1$. Consider the case $\beta_2 > 0$. Therefore Equation (3.49) becomes

$$\xi^2 y''(\xi) + \xi y'(\xi) + [\xi^2 - v^2] y(\xi) = 0 \tag{3.50}$$

that is the Bessel equation of complex order v (here v is imaginary). It is well known that solutions of this equation are given by:

$$y(\xi) = c_1 J_v(\xi) + c_2 J_{-v}(\xi) \tag{3.51}$$

where c_1 and c_2 are constant with respect to ξ (and therefore also with respect to z) and J_μ is the Bessel function of the first kind of complex (non integral) order μ.

By taking into account Equations (3.2), (3.3), (3.44) and (3.48), solutions of Equation (2.16) are:

$$
\begin{aligned}
(3.52) \qquad A(z) &= c_1 J_{\frac{|\beta_2|\omega^2}{\alpha} j}\Big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big)\Big) \\
&\quad + c_2 J_{-\frac{|\beta_2|\omega^2}{\alpha} j}\Big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big)\Big)
\end{aligned}
$$

In order to determine c_1 and c_2, compute the derivative of $A(z)$ with respect to z:

$$
\begin{aligned}
(3.53) \quad A'(z) &= -c_1\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big) J'_{\frac{|\beta_2|\omega^2}{\alpha} j}\Big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big)\Big) \\
&\quad - c_2\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big) J'_{-\frac{|\beta_2|\omega^2}{\alpha} j}\Big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\Big(-\frac{\alpha}{2}z\Big)\Big)
\end{aligned}
$$

where $J'_\mu(\cdot)$ represents the derivative of $J_\mu(\cdot)$ with respect to its argument.

By substituting Equations (3.52) and (3.53) in the initial conditions in problem (3.1), one obtains:

$$
\begin{cases}
c_1 J_{\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) + c_2 J_{-\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) = A_0(\omega) \\
-c_1\sqrt{|\beta_2|\omega^2\gamma P_0}\, J'_{\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) \\
\qquad -c_2\sqrt{|\beta_2|\omega^2\gamma P_0}\, J'_{-\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) = A'_0(\omega).
\end{cases}
$$

Say K the following non null quantity:

$$
(3.54) \quad K := -\sqrt{|\beta_2|\omega^2\gamma P_0}\begin{vmatrix} J_{\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) & J_{-\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) \\ J'_{\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) & J'_{-\frac{|\beta_2|\omega^2}{\alpha} j}\big(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\big) \end{vmatrix} \neq 0.
$$

It is $K \neq 0$ because it is assumed $\omega \neq 0$ and because of the property of the Bessel functions of the first kind when the order is not integral. One is therefore able to compute the unique c_1 and c_2 in order to obtain the solution A for problem (3.1). Denote $\mathscr{W}(\xi)$ the Wronskian of $J_\nu(\xi)$ and $J_{-\nu}(\xi)$; it holds, see [7]:

$$
(3.55) \qquad \mathscr{W}(\xi) := \begin{vmatrix} J_\nu(\xi) & J_{-\nu}(\xi) \\ J'_\nu(\xi) & J'_{-\nu}(\xi) \end{vmatrix} = -\frac{2\sin(\nu\pi)}{\pi\xi}
$$

and therefore Equation (3.54) becomes:

$$
(3.56) \qquad K = \frac{\alpha}{\pi}\sin\Big(\frac{|\beta_2|\omega^2}{\alpha}\pi j\Big).
$$

Finally

(3.57)
$$c_1(\omega) = -\frac{A_0(\omega)\sqrt{|\beta_2|\omega^2\gamma P_0}J'_{-\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)+A'_0(\omega)J_{-\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)}{\frac{\alpha}{\pi}\sin\left(\frac{|\beta_2|\omega^2}{\alpha}\pi j\right)},$$

(3.58)
$$c_2(\omega) = \frac{A'_0(\omega)J_{\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)+A_0(\omega)\sqrt{|\beta_2|\omega^2\gamma P_0}J'_{\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)}{\frac{\alpha}{\pi}\sin\left(\frac{|\beta_2|\omega^2}{\alpha}\pi j\right)}.$$

By putting these last two expressions for c_1 and c_2 in Equation (3.52), one obtains the solution for problem (3.1).

Turn now to the second case, when $\beta_2 < 0$: Equation (3.49) becomes

$$\xi^2 y''(\xi) + \xi y'(\xi) - [\xi^2 + \nu^2]y(\xi) = 0 \tag{3.59}$$

that is the modified Bessel equation of complex (non integral) order ν. The solutions of this equation are

$$y(\xi) = k_1 I_\nu(\xi) + k_2 I_{-\nu}(\xi) \tag{3.60}$$

where k_1 and k_2 are constant with respect to ξ (and therefore with respect to z) and I_μ is the modified Bessel function of the first kind of complex (non integral) order μ. Thanks to Equations (3.2), (3.3), (3.44), (3.45) and (3.48), one obtains:

$$\begin{aligned}A(z) = {} & k_1 I_{\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\left(-\frac{\alpha}{2}z\right)\right)\\ & + k_2 I_{-\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\exp\left(-\frac{\alpha}{2}z\right)\right).\end{aligned} \tag{3.61}$$

The computation of k_1 and k_2 in order to obtain the solution of problem (3.1) is similar to the computation of c_1 and c_2 in the previous case: say $\tilde{\mathscr{W}}(\xi)$ the Wronskian of $I_\nu(\xi)$ and $I_{-\nu}(\xi)$; it holds, see [7]:

$$\tilde{\mathscr{W}}(\xi) := \begin{vmatrix} I_\nu(\xi) & I_{-\nu}(\xi) \\ I'_\nu(\xi) & I'_{-\nu}(\xi) \end{vmatrix} = -\frac{2\sin(\nu\pi)}{\pi\xi} \tag{3.62}$$

and therefore

(3.63)
$$k_1(\omega) = -\frac{A_0(\omega)\sqrt{|\beta_2|\omega^2\gamma P_0}I'_{-\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)+A'_0(\omega)I_{-\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)}{\frac{\alpha}{\pi}\sin\left(\frac{|\beta_2|\omega^2}{\alpha}\pi j\right)},$$

(3.64)
$$k_2(\omega) = \frac{A'_0(\omega)I_{\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)+A_0(\omega)\sqrt{|\beta_2|\omega^2\gamma P_0}I'_{\frac{|\beta_2|\omega^2}{\alpha}j}\left(\frac{2}{\alpha}\sqrt{|\beta_2|\omega^2\gamma P_0}\right)}{\frac{\alpha}{\pi}\sin\left(\frac{|\beta_2|\omega^2}{\alpha}\pi j\right)}.$$

4. Conclusions

In the framework of the Combined Regular-Logarithmic Perturbation method proposed in [6] (see Section 1 for a brief summary), it was looked for solutions of the nonlinear Schrödinger equation (1.3) for the propagation of light in optical fibers. In the Combined Regular-Logarithmic Perturbation method, solutions have the form of a perturbed continuous wave where perturbation is due to three real functions to be determined. In Section 2 they were proposed four systems of Partial Differential Equations that can be deduced from the Schrödinger equation after a linearization process that includes the addition of an equation arbitrarily chosen. By taking the Fourier transform of the real perturbation functions involved, four systems of Ordinary Differential Equations have been obtained; their analytical solutions can be given by recurrence. One of the four systems obtained is exactly the same approached in [6] by the means of a transfer matrix. The other three systems proposed can be considered as useful alternatives to model different situations and mutual relationships that relates the perturbation functions in play in the model. In Section 3 the proposed systems have been solved: analytical solutions have been given by means of recurrence formulae. By using this technique it has been proven that the power attenuation $\alpha > 0$ does not affect the terms of order less than three. Moreover, solutions which involve a combination of Bessel functions have been proposed: Bessel functions of the first kind when the fiber is defocusing, and modified Bessel functions of the first kind for focusing fibers.

A future research will be aimed at comparing the proposed systems in order to determine, from a practical point of view, which one is a better approximation for light propagation in optical fibers. Another crucial aspect is the solution of the problem for multi-span optical fibers instead of single-span ones: the presence of amplifiers along the fiber constitutes an independent source of complex additive white Gaussian noise to be accounted for.

A. Appendix

Consider Equation (2.6). The homogeneous PDE for $\gamma = 0$ becomes, for $z > 0$, $t \in \mathbb{R}$:

$$\frac{\partial^2 a}{\partial z^2}(z,t) + \left(\frac{\beta_2}{2}\right)^2 \frac{\partial^4 a}{\partial t^4}(z,t) = 0. \tag{A.1}$$

Consider the following initial conditions for $a(z,t)$:

$$a(z=0,t) = \varphi(t), \tag{A.2}$$

$$\frac{\partial a}{\partial z}(z=0,t) = \psi(t). \tag{A.3}$$

Equation (A.1) can be rewritten as follows by using an operator notation:

$$\left[\frac{\partial^2}{\partial z^2}+\left(\frac{\beta_2}{2}\right)^2\frac{\partial^4}{\partial t^4}\right]a(z,t)=0; \tag{A.4}$$

the previous Equation can be splitted as follows:

$$\left[\frac{\partial}{\partial z}-j\frac{|\beta_2|}{2}\frac{\partial^2}{\partial t^2}\right]\left[\frac{\partial}{\partial z}+j\frac{|\beta_2|}{2}\frac{\partial^2}{\partial t^2}\right]a(z,t)=0. \tag{A.5}$$

Say L the operator that acts on functions $f(z,t)$ as:

$$Lf(z,t):=j\frac{|\beta_2|}{2}\frac{\partial^2}{\partial t^2}f(z,t). \tag{A.6}$$

Therefore Equation (A.5) becomes

$$\left[\frac{\partial}{\partial z}-L\right]\left[\frac{\partial}{\partial z}+L\right]a(z,t)=0. \tag{A.7}$$

Say $a_1(z,t)$ and $a_2(z,t)$ respectively the solutions of the following two equations:

$$\left[\frac{\partial}{\partial z}-L\right]a_1(z,t)=0, \tag{A.8}$$

$$\left[\frac{\partial}{\partial z}+L\right]a_2(z,t)=0. \tag{A.9}$$

The solutions are:

$$\begin{aligned} a_1(z,t)&=\frac{1}{\sqrt{2\pi j|\beta_2|z}}\int_{\mathbb{R}}\exp\left[j\frac{(t-t')^2}{2|\beta_2|z}\right]a_1(z=0,t')\,dt' \\ &:=\mathrm{e}^{Lz}a_1(z=0,\cdot)(t) \end{aligned} \tag{A.10}$$

$$\begin{aligned} a_2(z,t)&=\frac{1}{\sqrt{-2\pi j|\beta_2|z}}\int_{\mathbb{R}}\exp\left[-j\frac{(t-t')^2}{2|\beta_2|z}\right]a_2(z=0,t')\,dt' \\ &:=\mathrm{e}^{-Lz}a_2(z=0,\cdot)(t) \end{aligned} \tag{A.11}$$

by using the semigroup notation.

The solution $a(z,t)$ of Equation (A.1) can be written as

$$a(z,t)=a_1(z,t)+a_2(z,t) \tag{A.12}$$

and conditions in Equations (A.2) and (A.3) let one determine functions $a_1(z=0,t)$ and $a_2(z=0,t)$ that appear in Equations (A.10) and (A.11) and therefore in (A.12): indeed, by substituting (A.12) in Equations (A.2) and (A.3), one obtains the following system:

$$\text{(A.13)} \qquad \begin{cases} a_1(z=0,t)+a_2(z=0,t)=\varphi(t) \\ La_1(z=0,t)-La_2(z=0,t)=\psi(t) \end{cases}$$

or rather, by applying operator L^{-1} to the second equation:

$$\text{(A.14)} \qquad \begin{cases} a_1(z=0,t)+a_2(z=0,t)=\varphi(t) \\ a_1(z=0,t)-a_2(z=0,t)=(L^{-1}\psi)(t). \end{cases}$$

By addiction and subtraction, one obtains $a_1(z=0,t)$ and $a_2(z=0,t)$:

$$\text{(A.15)} \qquad \begin{cases} a_1(z=0,t)=\frac{1}{2}\varphi(t)+\frac{1}{2}(L^{-1}\psi)(t) \\ a_2(z=0,t)=\frac{1}{2}\varphi(t)-\frac{1}{2}(L^{-1}\psi)(t). \end{cases}$$

Operator L^{-1} acts on functions $\psi(t)$ such that $\psi(t)=j\frac{|\beta_2|}{2}\frac{d^2}{dt^2}f(t)$ where $f(t)$, $\frac{d}{dt}f(t)$ and $\frac{d^2}{dt^2}f(t)$ have limits for $t\to\pm\infty$ that guarantee the existence of the finite norm in $L^2(\mathbb{R})$. One can therefore conclude that $f(t)=L^{-1}\psi(t)$ solving the above equation is as follows

$$\text{(A.16)} \qquad f(t)=\int_{\mathbb{R}}\frac{|t-t'|}{2}\frac{2\psi(t')}{j|\beta_2|}dt'$$

and system (A.15) becomes:

$$\text{(A.17)} \qquad \begin{cases} a_1(z=0,t)=\frac{1}{2}\varphi(t)+\frac{1}{2}\int_{\mathbb{R}}|t-t'|\frac{\psi(t')}{j|\beta_2|}dt' \\ a_2(z=0,t)=\frac{1}{2}\varphi(t)-\frac{1}{2}\int_{\mathbb{R}}|t-t'|\frac{\psi(t')}{j|\beta_2|}dt'. \end{cases}$$

By putting Equations in (A.17) in the solution $a(z,t)$ of Equation (A.1) given by Equation (A.12) and reordering, one obtains:

$$\text{(A.18)} \qquad a(z,t)=\int_{\mathbb{R}}\frac{1}{2}\left\{\frac{\exp\left[j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{2\pi j|\beta_2|z}}+\frac{\exp\left[-j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{-2\pi j|\beta_2|z}}\right\}\varphi(t')\,dt'$$
$$+\int_{\mathbb{R}}\frac{1}{2}\left\{\frac{\exp\left[j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{2\pi j|\beta_2|z}}-\frac{\exp\left[-j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{-2\pi j|\beta_2|z}}\right\}\int_{\mathbb{R}}|t'-t''|\frac{\psi(t'')}{j|\beta_2|}dt''\,dt'$$

or

$$a(z,t) = \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] + \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \varphi(t')\, dt'$$
$$+ \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} j \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \int_{\mathbb{R}} |t'-t''| \frac{\psi(t'')}{j|\beta_2|}\, dt''\, dt'$$

that is, by eliding j in the second addend:

$$a(z,t) = \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] + \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \varphi(t')\, dt'$$
$$+ \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\}$$
$$\times \int_{\mathbb{R}} |t'-t''| \frac{\psi(t'')}{|\beta_2|}\, dt''\, dt'. \tag{A.19}$$

Note that $a(z,t)$ is real if $\varphi(t)$ and $\psi(t)$ are real.

The non-homogeneous form of Equation (A.1):

$$\frac{\partial^2 a}{\partial z^2}(z,t) + \left(\frac{\beta_2}{2}\right)^2 \frac{\partial^4 a}{\partial t^4}(z,t) = q(z,t) \tag{A.20}$$

with the same initial conditions (A.2) and (A.3) has the solution:

$$a(z,t) = \int_{\mathbb{R}} \frac{1}{2} \left\{ \frac{\exp\left[j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{2\pi j|\beta_2|z}} + \frac{\exp\left[-j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{-2\pi j|\beta_2|z}} \right\} \varphi(t')\, dt'$$
$$+ \int_{\mathbb{R}} \frac{1}{2} \left\{ \frac{\exp\left[j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{2\pi j|\beta_2|z}} - \frac{\exp\left[-j\frac{(t-t')^2}{2|\beta_2|z}\right]}{\sqrt{-2\pi j|\beta_2|z}} \right\} \int_{\mathbb{R}} |t'-t''| \frac{\psi(t'')}{j|\beta_2|}\, dt''\, dt'$$
$$+ \int_0^z \int_{\mathbb{R}} \frac{1}{2} \left\{ \frac{\exp\left[j\frac{(t-t')^2}{2|\beta_2|(z-s)}\right]}{\sqrt{2\pi j|\beta_2|(z-s)}} - \frac{\exp\left[-j\frac{(t-t')^2}{2|\beta_2|(z-s)}\right]}{\sqrt{-2\pi j|\beta_2|(z-s)}} \right\}$$
$$\times \int_{\mathbb{R}} |t'-t''| \frac{q(s,t'')}{j|\beta_2|}\, dt''\, dt'\, ds \tag{A.21}$$

or rather:

$$
\begin{aligned}
\text{(A.22)} \quad a(z,t) = &\int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] + \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \varphi(t')\, dt' \\
&+ \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \\
&\times \int_{\mathbb{R}} |t'-t''| \frac{\psi(t'')}{|\beta_2|}\, dt''\, dt' + \int_0^z \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|(z-s)}} \\
&\times \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right] \right\} \\
&\times \int_{\mathbb{R}} |t'-t''| \frac{q(s,t'')}{|\beta_2|}\, dt''\, dt'\, ds.
\end{aligned}
$$

Again, $a(z,t)$ is real if φ, ψ and q are real. One can conclude by giving one implicit equation for $a(z,t)$ (see Equation (2.6)):

$$
\begin{aligned}
\text{(A.23)} \quad a(z,t) = &\int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] + \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \varphi(t')\, dt' \\
&+ \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|z}} \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] \right\} \\
&\times \int_{\mathbb{R}} |t'-t''| \frac{\psi(t'')}{|\beta_2|}\, dt''\, dt' + \int_0^z \int_{\mathbb{R}} \frac{1}{2} \frac{1}{\sqrt{\pi|\beta_2|(z-s)}} \\
&\times \left\{ \sin\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right] \right\} \\
&\times \int_{\mathbb{R}} |t'-t''| \frac{\beta_2\gamma P_0 \exp(-\alpha s)}{|\beta_2|} \frac{\partial^2 a(s,t'')}{\partial t''^2}\, dt''\, dt'\, ds.
\end{aligned}
$$

Recall that $a(z,t)$ belongs to the Hilbert space $L^2(\mathbb{R})$ with respect to t. Under the assumptions on $a(z,t)$, it results:

$$
\text{(A.24)} \qquad \int_{\mathbb{R}} |t'-t''| \frac{\partial^2 a(s,t'')}{\partial t''^2}\, dt'' = 2a(s,t')
$$

and one obtains the following equation:

$$
\text{(A.25)} \quad a(z,t) = s(z,t) + \frac{\beta_2\gamma P_0}{|\beta_2|} \int_0^z \int_{\mathbb{R}} k(z-s,t-t') \exp(-\alpha s) 2a(s,t')\, dt'\, ds
$$

where $s(z,t)$ is the following source term:

$$\text{(A.26)}\quad s(z,t) := \int_{\mathbb{R}} \frac{1}{2}\frac{1}{\sqrt{\pi|\beta_2|z}}\left\{\cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right] + \sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right]\right\}\varphi(t')\,dt'$$
$$+ \int_{\mathbb{R}} \frac{1}{2}\frac{1}{\sqrt{\pi|\beta_2|z}}\left\{\sin\left[\frac{(t-t')^2}{2|\beta_2|z}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|z}\right]\right\}$$
$$\times \int_{\mathbb{R}} |t'-t''|\frac{\psi(t'')}{|\beta_2|}\,dt''\,dt'$$

and

$$\text{(A.27)}\quad k(z-s,t-t') := \frac{1}{2}\frac{1}{\sqrt{\pi|\beta_2|(z-s)}}\left\{\sin\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right] - \cos\left[\frac{(t-t')^2}{2|\beta_2|(z-s)}\right]\right\}.$$

By putting

$$\text{(A.28)}\quad (\mathscr{K}a)(z,s,t) := \int_{\mathbb{R}} k(z-s,t-t')2a(s,t')\,dt'$$

Equation (A.25) becomes:

$$\text{(A.29)}\quad a(z,t) = s(z,t) + \frac{\beta_2\gamma P_0}{|\beta_2|}\int_0^z \exp(-\alpha s)(\mathscr{K}a)(z,s,t)\,ds$$

that is, finally, a Volterra integral equation with respect to z, for t fixed, in the unknown $a(z,t)$.

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