Calculus of Variations — *New approximations of the total variation and filters in Imaging*, by HAIM BREZIS, communicated on 9 January 2015.

ABSTRACT. — In this paper I present several results concerning the approximation of the BV-norm by non-local functionals. Some of these functionals are convex, others are non-convex. The mode of convergence introduces mysterious novelties and numerous problems remain open. The original motivation comes from Image Processing.

1. INTRODUCTION

Throughout this paper I assume that Ω is either a bounded, smooth open subset of $\mathbb{R}^N$, or that $\Omega = \mathbb{R}^N$. The case $N = 1$ is already of great interest, see e.g. Section 3 and [BN1], since many difficulties (and open problems!) occur even when $N = 1$. My goal is to present several techniques for approximating the total variation, i.e., the BV-norm of a function u, $\int_\Omega |\nabla u|$, by non-local functionals.

I will first discuss, in Section 2, a very simple and general formula originally discovered by J. Bourgain, P. Mironescu and myself (see [BBM1], [Br1]). The functionals approximating the total variation are convex and of the form:

$$(1.1) \qquad \Phi_n(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_n(|x - y|) \, dx \, dy,$$

where $u \in L^1(\Omega)$, and ρ_n is a sequence of radial mollifiers converging to the Dirac δ at 0 (see Section 2.1 for more details). I will discuss pointwise convergence, i.e., convergence of $\Phi_n(u)$ for *fixed u*, and also Γ-convergence (in the sense of De Giorgi) of the functionals Φ_n to the total variation. The results are somewhat natural because $|\nabla u|$ is approximated by "finite differences". Perhaps the main surprise is that such an approximation had never been noticed earlier!

Then, I will discuss, in Section 3, a rather unusual approximation of the total variation by non-local *non-convex* functionals. The idea grew out of a formal

This paper is related to a talk given by the author at the Accademia dei Lincei on November 26, 2013.

computation that I made around 2002, inspired from related computations in [BBM1]. Here the functionals are of the form

$$(1.2) \qquad \Lambda(u) = \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{N+1}} \, dx \, dy \quad \text{and} \quad \Lambda_\delta(u) = \delta\Lambda(u/\delta),$$

where $u \in L^1(\Omega)$, δ is a small parameter and φ is a specified function. The example I had originally in mind for φ was the Heaviside function

$$(1.3) \qquad \varphi(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ 1 & \text{if } t > 1. \end{cases}$$

I proved that if $u \in C_c^\infty(\mathbb{R}^N)$, then $\Lambda_\delta(u)$ converges, as $\delta \to 0$, to $K_N \int_\Omega |\nabla u|$, where K_N is an explicit constant depending only on N; see Theorem 3.1.

My computation, reproduced below, was never published. Instead I asked some students to convert this formal computation into a rigorous theorem for more general functions u. The task turned out to be *much more complicated* than I had expected. The first shock came when A. Ponce exhibited a function $u \in W^{1,1}(\Omega)$ such that $\Lambda_\delta(u)$ tends to infinity (see Remark 3.1 below). At this stage it occured to me that pointwise convergence was doomed to be flawed and I asked H.-M. Nguyen to study the asymptotics of Λ_δ as $\delta \to 0$ in the spirit of Γ-convergence. My hope was that the "natural" limit would be restored, i.e., that Λ_δ would Γ-converge to the functional $K_N \int_\Omega |\nabla \cdot|$. In a remarkable piece of work H.-M. Nguyen ([Ng2], [Ng3]) established that I was partly wrong and partly right: Λ_δ does Γ-converge—however the Γ-limit is $k \int_\Omega |\nabla \cdot|$ for some mysterious constant $k < K_N$!

I learned in 2011 from my (part-time) colleague R. Kimmel at the Technion that the total variation and some non-local functionals were used as filters in Image Processing (see Sections 2.6 and 3.4 below). In particular, the Yaroslavky filter (and some of its descendants) renewed my interest in the study of $\Lambda_\delta(u)$ for a general function φ, not just the Heaviside function (1.3). The outcome is the joint paper [BN1] whose results are summarized in Section 3.

To conclude, let me mention that a different type of approximation of the BV-norm of a function u, especially suited when u is the characteristic function of a set A—so that its BV-norm is the perimeter of A—has been recently developed in [ABBF1] and [ABBF2] (with roots in [BBM2]).

2. THE BBM FORMULA; SOME VARIANTS AND APPLICATIONS

2.1. The BBM formula

Let $(\rho_\varepsilon(r))_{\varepsilon > 0}$ be sequence of *radial mollifiers*, more precisely

$$(2.1) \qquad \rho_\varepsilon(r) : [0, +\infty) \to [0, +\infty) \text{ is measurable,}$$

$$(2.2) \qquad \int_0^{+\infty} \rho_\varepsilon(r) r^{N-1} \, dr = 1 \quad \forall \varepsilon > 0,$$

$$(2.3) \qquad \int_\delta^{+\infty} \rho_\varepsilon(r) r^{N-1} \, dr \to 0, \quad \text{as } \varepsilon \to 0, \forall \delta > 0.$$

The standard example of such mollifiers is

$$(2.4) \qquad \rho_\varepsilon(r) = \frac{1}{\varepsilon^N} \rho\left(\frac{r}{\varepsilon}\right)$$

with $\rho \geq 0$, ρ smooth, $\rho(r) = 0$ for $r \geq 1$, and

$$(2.5) \qquad \int_0^\infty \rho(r) r^{N-1} \, dr = 1.$$

Other examples, especially with non-smooth mollifiers are of interest (see Section 2.4, Example 1).

Given $u \in L^1(\Omega)$ and $\varepsilon > 0$ set

$$(2.6) \qquad \Phi_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|) \, dx \, dy,$$

$$(2.7) \qquad \Phi_0(u) = \begin{cases} K_N \int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

where

$$(2.8) \qquad K_N = \int_{S^{N-1}} |\sigma \cdot e| \, d\sigma \quad (\text{any } e \in S^{N-1}),$$

and $K_N = 2$ when $N = 1$.

The main result is the following

THEOREM 2.1 ([BBM1], [Da]). *Under the above assumptions we have*

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u) = \Phi_0(u) \quad \forall u \in L^1(\Omega).$$

Theorem 2.1 contains *two* assertions:

(a) If $u \in BV(\Omega)$, then

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u) = K_N \int_\Omega |\nabla u|$$

(b) If $u \in L^1(\Omega)$ satisfies

$$\Phi_{\varepsilon_n}(u) \leq C \quad \text{for some sequence } \varepsilon_n \to 0,$$

then $u \in BV(\Omega)$ and thus (a) applies.

REMARK 2.1. To be precise, assertion (a) was established in [BBM1] only for $u \in W^{1,1}(\Omega)$; for a general $u \in BV(\Omega)$ it was only proved that $\Phi_\varepsilon(u) \simeq \int |\nabla u|$ as $\varepsilon \to 0$. The full assertion (a) was raised as an open problem, which was eventually settled by J. Davila [Da]; a simpler proof may be found in [vSW].

　　The sketch of the proof of Theorem 2.1 will be presented in Section 2.3. We start with

2.2. A suggestive computation

For simplicity we assume here that $\Omega = \mathbb{R}^N$ and $u \in C_c^\infty(\mathbb{R}^N)$. Write

$$(2.9) \qquad \Phi_\varepsilon(u) = \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|} \rho_\varepsilon(|h|)\, dh.$$

Taylor's expansion as $|h| \to 0$ is "natural" because ρ_ε "lives" near 0 (consider e.g. the special form (2.4)). We have

$$u(x+h) - u(x) = h \cdot \nabla u(x) + O(|h|^2),$$

so that

$$\frac{|u(x+h) - u(x)|}{|h|} = \frac{|h \cdot \nabla u(x)|}{|h|} + O(|h|)$$

and therefore, as $\varepsilon \to 0$,

$$(2.10) \quad \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|} \rho_\varepsilon(|h|)\, dh = \int_{\mathbb{R}^N} \frac{|h \cdot \nabla u(x)|}{|h|} \rho_\varepsilon(|h|)\, dh + o(1).$$

Next we compute the integral in the RHS of (2.10) using polar coordinates: $r = |h|$ and $\sigma = \frac{h}{|h|} \in S^{N-1}$. This yields

$$(2.11) \quad \int_{\mathbb{R}^N} \frac{|h \cdot \nabla u(x)|}{|h|} \rho_\varepsilon(|h|)\, dh = \int_0^\infty dr \int_{S^{N-1}} |\sigma \cdot \nabla u(x)| \rho_\varepsilon(r) r^{N-1}\, d\sigma.$$

　　Observe that for any $V \in \mathbb{R}^N$,

$$(2.12) \qquad \int_{S^{N-1}} |\sigma \cdot V|\, d\sigma = |V| \int_{S^{N-1}} |\sigma \cdot e|\, d\sigma \quad \forall e \in S^{N-1}.$$

Combining (2.9), (2.10), (2.11) and (2.12) we find

$$(2.13) \qquad \Phi_\varepsilon(u) = K_N \int_{\mathbb{R}^N} |\nabla u(x)|\, dx + o(1).$$

　　We now turn to

2.3. Sketch of the proof of Theorem 2.1

For simplicity, assume again that $\Omega = \mathbb{R}^N$.

ASSERTION (a). By the triangle inequality we have

$$(2.14) \qquad |\Phi_\varepsilon(u) - \Phi_\varepsilon(v)| \le \Phi_\varepsilon(u - v) \quad \forall u, v.$$

On the other hand, it is well-known (see e.g. [Br2], Proposition 9.3) that for every w,

$$\int_{\mathbb{R}^N} |w(x + h) - w(x)|\, dx \le |h| \int_{\mathbb{R}^N} |\nabla w(x)|\, dx,$$

and therefore we obtain the important estimate

$$(2.15) \qquad \Phi_\varepsilon(w) \le \int_{\mathbb{R}^N} |\nabla w(x)|\, dx \int_{\mathbb{R}^N} \rho_\varepsilon(|h|)\, dh = C_N \int_{\mathbb{R}^N} |\nabla w(x)|\, dx.$$

Combining (2.14) and (2.15) we find

$$(2.16) \qquad |\Phi_\varepsilon(u) - \Phi_\varepsilon(v)| \le C_N \int_{\mathbb{R}^N} |\nabla(u - v)|.$$

Using (2.13), (2.16) and a standard density argument we conclude that

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u) = K_N \int_{\mathbb{R}^N} |\nabla u(x)|\, dx \quad \forall u \in W^{1,1}(\Omega).$$

Since smooth functions are *not* dense in BV the proof of assertion (a) for BV functions is more delicate; see [Da] and [vSW].

ASSERTION (b). We follow a suggestion of E. Stein who simplified our original proof. Let $u \in L^1(\mathbb{R}^N)$ be such that

$$(2.17) \qquad \Phi_{\varepsilon_n}(u) \le C \quad \text{for some sequence } \varepsilon_n \to 0.$$

Take any sequence of *smooth* mollifiers (μ_δ). Since the functional Φ_ε is *convex* we have

$$(2.18) \qquad \Phi_\varepsilon(\mu_\delta * u) \le \Phi_\varepsilon(u).$$

Next we *fix $\delta > 0$*. By assertion (a) we know that

$$(2.19) \qquad \lim_{\varepsilon \to 0} \Phi_\varepsilon(\mu_\delta * u) = K_N \int_{\mathbb{R}^N} |\nabla(\mu_\delta * u)|.$$

Applying (2.19) and assumption (2.17) we find that

$$(2.20) \qquad \int_{\mathbb{R}^N} |\nabla(\mu_\delta * u)| \le C, \quad \text{where } C \text{ is independent of } \delta.$$

Finally we pass to the limit in (2.20) as $\delta \to 0$ and conclude that $u \in BV$.

2.4. Two examples

We present here two simple choices of mollifiers ρ_ε, each one having its own interest.

EXAMPLE 1. Choose

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon/r^{N-\varepsilon} & \text{if } 0 < r < 1 \\ 0 & \text{if } r \ge 1. \end{cases}$$

We deduce from Theorem 2.1 that

$$(2.21) \qquad \lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{N+1-\varepsilon}} \, dx \, dy = K_N \int_\Omega |\nabla u|$$

Indeed Theorem 2.1 gives

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_{\substack{\Omega \\ |x-y|<1}} \frac{|u(x) - u(y)|}{|x - y|^{N+1-\varepsilon}} \, dx \, dy = K_N \int_\Omega |\nabla u|.$$

But

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_{\substack{\Omega \\ |x-y|\ge 1}} \frac{|u(x) - u(y)|}{|x - y|^{N+1-\varepsilon}} \, dx \, dy = 0,$$

since

$$\int_\Omega \int_{\substack{\Omega \\ |x-y|\ge 1}} \frac{|u(x) - u(y)|}{|x - y|^{N+1-\varepsilon}} \, dx \, dy \le 2 \int_\Omega |u(x)| \, dx \int_{\substack{\Omega \\ |y-x|\ge 1}} \frac{dy}{|y - x|^{N+1-\varepsilon}}$$

and

$$\int_{\substack{\mathbb{R}^N \\ |y-x|\ge 1}} \frac{dy}{|y - x|^{N+1-\varepsilon}} \le C \quad \text{independent of } \varepsilon, \text{ as } \varepsilon \to 0.$$

In the special case where $u = \mathbf{1}_A$, the characteristic function of a measurable set $A \subset \Omega$, we obtain

$$(2.22) \quad \lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_\Omega \frac{|\mathbf{1}_A(x) - \mathbf{1}_A(y)|}{|x - y|^{N+1-\varepsilon}} \, dx \, dy = K_N \int_\Omega |\nabla \mathbf{1}_A| = K_N \, \mathrm{Per} \, (A; \Omega).$$

In recent years there has been much interest in the convergence of non-local functionals such as (2.22) to the perimeter; see e.g. [CV] and [ADM] (it seems that the authors of [CV] were not aware of the paper [BBM1]).

EXAMPLE 2. Choose

$$\rho_\varepsilon(r) = \begin{cases} \frac{(N+1)r}{\varepsilon^{N+1}} & \text{if } 0 < r < \varepsilon, \\ 0 & \text{if } r > \varepsilon. \end{cases}$$

We deduce from Theorem 2.1 that

$$(2.23) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N+1}} \int_\Omega \int_{\substack{\Omega \\ |x-y|<\varepsilon}} |u(x) - u(y)| \, dx \, dy = \frac{K_N}{(N+1)} \int_\Omega |\nabla u|,$$

and in the special case where $u = \mathbf{1}_A$ we find

$$(2.24) \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{N+1}} \, \mathrm{meas} \, \{x \in A, y \in \Omega \backslash A; |x - y| < \varepsilon\} = \frac{K_N}{2(N+1)} \, \mathrm{Per} \, (A; \Omega).$$

2.5. *Where Γ-convergence enters*

So far we have been concerned with *pointwise* convergence of Φ_ε, i.e., the existence of $\lim_{\varepsilon \to 0} \Phi_\varepsilon(u)$ for *fixed* u. It is natural to study the convergence of Φ_ε in the sense of Γ-*convergence*. Let us recall

DEFINITION. A sequence (F_n) of functionals is said to be Γ-convergent to F_0 in $L^1(\Omega)$ if

$$(\Gamma_1) \qquad \forall u \in L^1(\Omega), \quad \forall u_n \to u \text{ in } L^1(\Omega), \text{ one has } \liminf_{n \to \infty} F_n(u_n) \geq F_0(u)$$

and

$$(\Gamma_2) \qquad \forall u \in L^1(\Omega), \quad \exists \tilde{u}_n \to u \text{ in } L^1(\Omega) \text{ such that } \limsup_{n \to \infty} F_n(\tilde{u}_n) \leq F_0(u).$$

THEOREM 2.2 (A. Ponce [Po]). *Let Φ_ε and Φ_0 be defined by* (2.6) *and* (2.7), *then, as $\varepsilon \to 0$,*

$$\Phi_\varepsilon \to \Phi_0 \text{ in the sense of } \Gamma\text{-convergence.}$$

PROOF. Property (Γ_2) follows from Theorem 2.1 with the choice $\tilde{u}_\varepsilon \equiv u$. We now turn to the proof of (Γ_1) and for simplicity we treat only the case $\Omega = \mathbb{R}^N$. Let (μ_δ) be a sequence of smooth mollifiers. By (2.18) we have

$$(2.25) \qquad \Phi_\varepsilon(\mu_\delta * u_\varepsilon) \le \Phi_\varepsilon(u_\varepsilon),$$

where (u_ε) is a given sequence such that $u_\varepsilon \to u$ in $L^1(\Omega)$. From (2.16) we have

$$(2.26) \qquad |\Phi_\varepsilon(\mu_\delta * u_\varepsilon) - \Phi_\varepsilon(\mu_\delta * u) \le C_N \int |\nabla(\mu_\delta * (u_\varepsilon - u))|$$

$$\le C_N \|\nabla \mu_\delta\|_{L^1} \|u_\varepsilon - u\|_{L^1}.$$

On the other hand if we apply Theorem 2.1 to $\mu_\delta * u$ we find that for *fixed* δ

$$(2.27) \qquad \lim_{\varepsilon \to 0} \Phi_\varepsilon(\mu_\delta * u) = K_N \int |\nabla(\mu_\delta * u)|.$$

Putting (2.25) and (2.27) together yields

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon) \ge K_N \int |\nabla(\mu_\delta * u)|.$$

Finally we let $\delta \to 0$, use the fact that

$$\lim_{\varepsilon \to 0} \int |\nabla(\mu_\delta * u)| = \int |\nabla u|,$$

and conclude that

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(u_\varepsilon) \ge K_N \int |\nabla u|.$$

Inspired by [GO1] and [GO2], G. Leoni and D. Spector have introduced in [LS1] a variant of Φ_ε: for any given $p \ge 1$, set for $u \in L^1(\Omega)$,

$$\Psi_{\varepsilon,p}(u) = \int_\Omega dx \left[\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) \, dy \right]^{1/p},$$

$$\Psi_{0,p}(u) = \begin{cases} K_{N,p} \int |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

where

$$K_{N,p}^p = \int_{S^{N-1}} |\sigma \cdot e|^p \, d\sigma \quad (\text{any } e \in S^{N-1}).$$

Following the computation in Section 2.2 we see that when $\Omega = \mathbb{R}^N$ and $p \geq 1$,

$$\lim_{\varepsilon \to 0} \Psi_{\varepsilon,p}(u) = K_{N,p} \int_\Omega |\nabla u|, \quad \forall u \in C_c^\infty(\Omega).$$

However the analogue of inequality (2.15) *may fail* when $p > 1$ as noted in [LS2] and [BN2]. In fact, the properties of the functional $\Psi_{\varepsilon,p}$ are *very sensitive* to the choice of ρ_ε. For every $p > 1$, there exist (see [BN2]) *special mollifiers* ρ_ε such that for every w, and every ε,

$$\Psi_{\varepsilon,p}(w) \leq C \int_\Omega |\nabla w(x)| \, dx,$$

for some constant C. In this case we have

$$\lim_{\varepsilon \to 0} \Psi_{\varepsilon,p}(u) = \Psi_{0,p}(u) \quad \forall u \in L^1(\Omega).$$

On the other hand (see [BN2]), for every $p > 1$, it is possible to construct mollifiers ρ_ε and a function $u \in W^{1,1}(\Omega)$ such that

(2.28)
$$\lim_{\varepsilon \to 0} \Psi_{\varepsilon,p}(u) = +\infty.$$

In other words *pointwise convergence may fail* when $p > 1$: it is *not always* true that

$$\lim_{\varepsilon \to 0} \Psi_{\varepsilon,p}(u) = \Psi_{0,p}(u) \quad \forall u \in BV(\Omega).$$

Interestingly, in the framework of Γ-convergence, the "natural" expected conclusion is "restored" for all mollifiers:

THEOREM 2.3 ([LS2]). *For every $p \geq 1$*

$$\Psi_{\varepsilon,p} \to \Psi_{0,p} \quad \text{in the sense of } \Gamma\text{-convergence in } L^1 \text{ as } \varepsilon \to 0.$$

We refer the reader to the proofs in [LS2] and [BN2]. Note that Theorem 2.3 implies in particular, in view of (Γ_1), that

(2.29)
$$\liminf_{\varepsilon \to 0} \Psi_{\varepsilon,p}(u) \geq \Psi_{0,p}(u) \quad \forall u \in BV$$

and we know from (2.28) that this inequality can be strict for some functions $u \in BV$.

2.6. Connections to filters in Image Processing

A fundamental challenge in Image Processing is to improve images of poor quality. Denoising is an immense subject, see, e.g., the excellent survey by A. Buades,

B. Coll and J. M. Morel [BCM]. A popular strategy is to introduce a functional F, called a filter, and use a variational formulation

$$(2.30) \qquad \inf_u \left\{ \lambda \int_\Omega |u - f|^2 + F(u) \right\},$$

or, alternatively, the associated Euler equation

$$(2.31) \qquad 2\lambda(u - f) + F'(u) = 0.$$

Here f represents the given image of poor quality, $\lambda > 0$ is the fidelity parameter (fixed by experts) which governs how much the filtering is desirable. Minimizers of (2.30) or solutions to (2.31) are the denoised images.

Many types of filters are used in Image Processing. Here we present three filters and another one will be described in Section 3. The first one is the celebrated ROF filter due L. Rudin, S. Osher and E. Fatemi [ROF]:

$$F(u) = \int_\Omega |\nabla u|$$

and the corresponding minimization problem is

$$(ROF) \qquad \min_u \left\{ \lambda \int_\Omega |u - f|^2 + \int_\Omega |\nabla u| \right\}.$$

Clearly, the functional in (ROF) is strictly convex. It follows from standard Functional Analysis that, given $f \in L^2(\Omega)$, there exists a unique minimizer $u_0 \in BV(\Omega) \cap L^2(\Omega)$ of (ROF).

The second filter has been proposed by G. Aubert and P. Kornprobst in [AK]. Set

$$F(u) = \Phi_\varepsilon(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|) \, dx \, dy,$$

where ρ_ε is a sequence of radial mollifiers as in Section 2.1. The corresponding minimization problem is

$$(AK_\varepsilon) \qquad \min_u \left\{ \lambda \int_\Omega |u - f|^2 + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon(|x - y|) \, dx \, dy \right\}.$$

As above, (AK_ε) admits a unique minimizer u_ε. In [AK] it is established (using the same strategy as in the proof of Theorem 2.2), that as $\varepsilon \to 0$, (u_ε) converges to the solution of (ROF_{K_N}) where K_N is the constant defined in (2.8).

The third filter, due to G. Gilboa and S. Osher [GO1] (see also [GO2]), has the form

$$F(u) = \int_\Omega \left(\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^2} w(x, y) \, dy \right)^{1/2} dx.$$

and the corresponding minimization problem is

$$(GO) \qquad \min_u \left\{ \lambda \int_\Omega |u - f|^2 + \int_\Omega \left(\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^2} w(x, y) \, dy \right)^{1/2} dx \right\}.$$

The functional in (GO) is strictly convex. Again by standard Functional Analysis, there exists a unique minimiser u of (GO). Using Theorem 2.3 and a few additional ingredients, one can show (see [BN2]) that if $w(x, y) = \rho_\varepsilon(|x - y|)$, where ρ_ε is any sequence of radial mollifiers, then the corresponding minimizers (u_ε) of (GO_ε) (i.e., (GO) with $w(x, y) = \rho_\varepsilon(|x - y|)$) converge, as $\varepsilon \to 0$, to the unique solution of the (ROF_k) problem

$$(ROF_k) \qquad \qquad \min_u \left\{ \lambda \int_\Omega |u - f|^2 + k \int_\Omega |\nabla u| \right\},$$

where $k = K_{N,2}$ is defined above.

3. A NON-CONVEX NON-LOCAL APPROXIMATION OF THE TOTAL VARIATION

Throughout this section we asume that $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the following properties

(3.1) $\qquad\qquad\qquad \varphi$ is non-decreasing,

(3.2) $\qquad \varphi(t) \le at^2 \quad \forall t \in [0, 1],$ for some positive constant a,

(3.3) $\qquad \varphi(t) \le b \quad \forall t \ge 0,$ for some positive constant b,

(3.4) $\quad \varphi$ is continuous on $[0, +\infty)$ except at a finite number of points in $(0, +\infty)$,

(3.5) $\qquad\qquad\qquad \varphi(t) = \varphi(t-) \quad \forall t > 0,$

and the normalization condition

$$(3.6) \qquad\qquad\qquad K_N \int_0^\infty \varphi(t) t^{-2} \, dt = 1$$

where K_N has been defined in (2.8).

Here are three particular functions φ of interest. Take $\varphi = \varphi_i = c_i \tilde{\varphi}_i$, $i = 1, 2, 3$, where $\tilde{\varphi}_i$ is one of the functions defined below and c_i is chosen so that the normalization condition (3.6) holds.

EXAMPLE 1.

$$\tilde{\varphi}_1(t) = \begin{cases} 0 & \text{if } t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

EXAMPLE 2.

$$\tilde{\varphi}_2(t) = \begin{cases} t^2 & \text{if } t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

EXAMPLE 3.

$$\tilde{\varphi}_3(t) = 1 - e^{-t^2} \quad \forall t \geq 0.$$

Given a measurable function u on Ω and a small parameter $\delta > 0$ set

$$(3.7) \qquad \Lambda(u) = \int_\Omega \int_\Omega \frac{\varphi(|u(x) - u(y)|)}{|x - y|^{N+1}} \, dx \, dy \quad \text{and} \quad \Lambda_\delta(u) = \delta \Lambda(u/\delta).$$

Note that $\Lambda_\delta(u) < \infty$, e.g. if $u \in C_c^1(\Omega)$. Our goal is to show that $\Lambda_\delta(u)$ converges as $\delta \to 0$ to a multiple of $\int_\Omega |\nabla u|$. But the mode of convergence is extremely delicate.

3.1. Another suggestive computation

We start with the simple case where $u \in C_c^1(\Omega)$.

THEOREM 3.1. *We have*

$$\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \forall u \in C_c^1(\Omega).$$

SKETCH OF PROOF. For simplicity we take now $\Omega = \mathbb{R}^N$. Write

$$\Lambda_\delta(u) = A_\delta + B_\delta$$

where

$$A_\delta = \delta \iint_{|x-y| < \delta^\alpha} \cdots \quad \text{and} \quad B_\delta = \delta \iint_{|x-y| \geq \delta^\alpha} \cdots,$$

with $0 < \alpha < 1$.

Let $M > 0$ be such that $\operatorname{supp} u \subset B_M$. Clearly

$$B_\delta = \delta \iint_{\substack{|y| \leq M \\ |x-y| \geq \delta^\alpha}} \cdots + \delta \iint_{\substack{|y| \geq M \\ |x| \leq M \\ |x-y| \geq \delta^\alpha}} \cdots \leq 2\delta \iint_{\substack{|y| \leq M \\ |x-y| \geq \delta^\alpha}} \cdots$$

$$\leq C\delta \int_{|y| \leq M} dy \int_{|x-y| \geq \delta^\alpha} \frac{dx}{|x - y|^{N+1}}.$$

(Here we have used assumption (3.3)). Therefore $B_\delta \leq C\delta^{1-\alpha}$ and it remains to prove that $\lim_{\delta \to 0} A_\delta = \int_\Omega |\nabla u|$.

Rewrite A_δ as

$$A_\delta = \delta \int_{|x| \leq M+1} dx \int_{|h| < \delta^\alpha} \varphi\left(\frac{|u(x+h) - u(x)|}{\delta}\right) \frac{1}{|h|^{N+1}} dh.$$

Taylor's expansion gives

$$|u(x+h) - u(x)| = |h \cdot \nabla u(x)| + O(|h|^2)$$

Next, we assume that φ is Lipschitz (the general case is slightly more complicated, see [BN1]) and we deduce that

$$(3.8) \qquad A_\delta = \delta \int_{|x| \leq M+1} dx \int_{|h| < \delta^\alpha} \varphi\left(\frac{|h \cdot \nabla u(x)|}{\delta}\right) \frac{1}{|h|^{N+1}} dh + O(\delta^\alpha).$$

As in Section 2.2 we compute the integral on the RHS of (3.8) using polar coordinates:

$$(3.9) \quad \int_{|h| < \delta^\alpha} \varphi\left(\frac{|h \cdot \nabla u(x)|}{\delta}\right) \frac{1}{|h|^{N+1}} dh = \int_{S^{N-1}} d\sigma \int_0^{\delta^\alpha} \varphi\left(\frac{r|\sigma \cdot \nabla u(x)|}{\delta}\right) \frac{1}{r^2} dr.$$

Making the change of variable $s = \frac{r|\sigma \cdot \nabla u(x)|}{\delta}$ we obtain

$$(3.10) \qquad \int_0^{\delta^\alpha} \varphi\left(\frac{r|\sigma \cdot \nabla u(x)|}{\delta}\right) \frac{1}{r^2} dr$$

$$= \frac{|\sigma \cdot \nabla u(x)|}{\delta} \int_0^{\delta^{\alpha-1}|\sigma \cdot \nabla u(x)|} \frac{\varphi(s)}{s^2} ds$$

$$= \frac{|\sigma \cdot \nabla u(x)|}{\delta} \left[\int_0^\infty \frac{\varphi(s)}{s^2} ds + O\left(\frac{1}{\delta^{\alpha-1}|\sigma \cdot \nabla u(x)|}\right) \right].$$

Therefore

$$(3.11) \quad \int_0^{\delta^\alpha} \varphi\left(\frac{r|\sigma \cdot \nabla u(x)|}{\delta}\right) \frac{1}{r^2} dr = \frac{|\sigma \cdot \nabla u(x)|}{\delta} \int_0^\infty \frac{\varphi(s)}{s^2} ds + O\left(\frac{1}{\delta^\alpha}\right).$$

Combining (3.9), (3.11), (2.12) and (3.6) yields

$$(3.12) \qquad \int_{|h| < \delta^\alpha} \varphi\left(\frac{|h \cdot \nabla u(x)|}{\delta}\right) \frac{1}{|h|^{N+1}} dh = \frac{1}{\delta} |\nabla u(x)| + O\left(\frac{1}{\delta^\alpha}\right).$$

Inserting (3.12) in (3.8) we find

$$A_\delta = \int_\Omega |\nabla u(x)| \, dx + O(\delta^{1-\alpha}),$$

which completes the proof of Theorem 3.1.

3.2. *Pointwise convergence of* Λ_δ

In view of Theorem 3.1, and by analogy with Theorem 2.1, one might have expected that

$$(3.13) \qquad \lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega).$$

Assertion (3.13) is definitely wrong. In fact, the study of the asymptotic behavior of Λ_δ as $\delta \to 0$ is *extremely delicate*. Two basic properties satisfied by Φ_ε are *not* fulfilled by Λ_δ:

- there is *no* constant C such that $\Lambda_\delta(u) \leq C \int_\Omega |\nabla u| \ \forall \delta > 0, \ \forall u \in C_c^\infty(\Omega)$, despite the fact that $\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u| \ \forall u \in C_c^\infty(\Omega)$,
- Λ_δ is *not* a convex functional.

The following result summarizes what is known about the pointwise convergence of Λ_δ.

THEOREM 3.2. *One has*

$$(3.14) \qquad \liminf_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u| \quad \forall u \in W^{1,1}(\Omega).$$

and

$$(3.15) \qquad \limsup_{\delta \to 0} \Lambda_\delta(u) \geq \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega).$$

There is a constant $k \in (0,1]$ *depending on* φ *such that*

$$(3.16) \qquad \liminf_{\delta \to 0} \Lambda_\delta(u) \geq k \int_\Omega |\nabla u| \quad \forall u \in L^1(\Omega).$$

The proofs of (3.14) and (3.15) are presented in [BN1]. The proof of (3.16) is delicate; this assertion is basically due to J. Bourgain and H.-M. Nguyen [BoNg]. Alternatively, (3.16) can also be viewed as a special case of the (deep) Γ-convergence result described in Section 3.3.

REMARK 3.1. Many pathologies may occur:

(1) As already mentioned, one can construct a function $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) = +\infty.$$

This example, originally discovered by A. Ponce, is presented in [Ng1] for $\varphi = \varphi_1$; see also [BN1]. Theorem 3.2 raises many open problems. For example, can one characterize the functions u for which $\lim_{\delta \to 0} \Lambda_\delta(u) = \int_\Omega |\nabla u|$, resp. $\liminf_{\delta \to 0} \Lambda_\delta(u)$ is finite?

(2) One can construct functions φ and $u \in BV(\Omega) \cap C(\overline{\Omega})$ such that

$$\liminf_{\delta \to 0} \Lambda_\delta(u) < \int_\Omega |\nabla u|,$$

see [BN1].

3.3. Where Γ-convergence saves the situation

It turns out that Γ-convergence is perfectly suited to analyze the asymptotic behavior of Λ_δ as $\delta \to 0$. The principal result is the following.

THEOREM 3.3 ([BN1]). *There exists a constant $k \in (0,1]$ such that, as $\delta \to 0$,*

$$\Lambda_\delta \to \Lambda_0 \text{ in the sense of Γ-convergence in } L^1(\Omega),$$

where

$$\Lambda_0(u) = \begin{cases} k \int_\Omega |\nabla u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover k depends only on φ and N.

REMARK 3.2. The proof of Theorem 3.3 is quite complicated (see [BN1]) and relies on ingredients developed by H.-M. Nguyen ([Ng2], [Ng3]), who established the same conclusion for $\varphi = c_1 \tilde{\varphi}_1$ (see Example 1 above) with a constant $k < 1$. This result provides an interesting situation where the pointwise limit and the Γ-limit are quite different. I must admit that the appearance of the constant k, with possibly $k < 1$, remains mysterious. It is not known whether $k < 1$ when $\varphi = c_2 \tilde{\varphi}_2$ or $\varphi = c_3 \tilde{\varphi}_3$. In fact, it is a *challenging open problem* to decide whether $k < 1$ for *every* φ.

3.4. Connection to the Yaroslavsky filter in Image Processing

We now return to the setting of Section 2.6 and discuss another type of filter originally introduced by Yaroslavsky in [Y1], [Y2] and subsequently revisited by many authors (see e.g. [BCM], [KOJ]) under the name *neighborhood filters*. Such filters are of the form

$$F(u) = \int_\Omega \int_\Omega \varphi\left(\frac{|u(x) - u(y)|}{\delta}\right) w(|x - y|) \, dx \, dy$$

where $\varphi = \tilde{\varphi}_3$, w is an appropriate weight function, and δ is a small parameter. As in Section 2.6 we consider the minimization problem

$$(3.17) \qquad \inf_u \left\{ \lambda \int_\Omega |u - f|^2 + F(u) \right\}$$

Since F is *not* convex, uniqueness may fail and existence is problematic. Indeed the standard approach for existence relies either on convexity or on some form of compactness which is not transparent since F involves no derivative. In what follows we choose $F = \Lambda_\delta$ defined by (3.7), and we consider the minimization problem

$$(3.18_\delta) \qquad \inf_u \left\{ \lambda \int_\Omega |u - f|^2 + \Lambda_\delta(u) \right\}.$$

To the best of our knowledge there is no result in the literature concerning the existence of a minimizer for (3.18_δ). Our main contribution is twofold:

(a) We establish the existence of a minimizer u_δ in (3.18_δ) for every $\delta > 0$ under the additional assumption

$$(3.19) \qquad \varphi(t) > 0 \quad \forall t > 0.$$

(b) We establish the "convergence" as $\delta \to 0$, of the Yaroslavsky-type filter Λ_δ to the ROF filter—a fact which seems to have been overlooked by the experts of Imaging. For practical purposes it may be useful to keep $\delta > 0$ not too small. But it is gratifying to be aware of the underlying "hierarchy" in the models— as in the Euler-Boltzmann equations.

Our main result is the following.

THEOREM 3.4 ([BN1]). *Assume that φ satisfies (3.18) (in addition to the standard assumptions (3.1)–(3.6)). Then for every $\delta > 0$ and every $f \in L^2(\Omega)$ there exists a minimizer of (3.18_δ). Let u_δ be any such minimizer. Then $u_\delta \to u_0$ in $L^2(\Omega)$ as $\delta \to 0$, where u_0 is the unique minimizer of*

$$(ROF_k) \qquad \min_u \left\{ \lambda \int_\Omega |u - f|^2 + k \int_\Omega |\nabla u| \right\}$$

and $k > 0$ is the constant introduced in Theorem 3.3.

The proof of the *convergence* of u_δ relies heavily on Theorem 3.3 combined with a few additional ingredients. While the proof of *existence* of a minimizer is derived from the following new compactness result. This result is valid for any fixed $\delta > 0$ and we may as well take $\delta = 1$, i.e., $\Lambda_\delta = \Lambda$.

THEOREM 3.5 ([BN1]). *Assume that Ω is smooth and bounded, and that φ satisfies (3.1) and (3.18). Let (u_n) be a bounded sequence in $L^1(\Omega)$ such that $\sup_n \Lambda(u_n) <$*

$+\infty$. *Then there exists a subsequence* (u_{n_k}) *of* (u_n) *and* $u \in L^1(\Omega)$ *such that* (u_{n_k}) *converges to* u *in* $L^1(\Omega)$.

The proof of Theorem 3.5 rests on an intriguing inequality due to H.-M. Nguyen (with roots in [BoNg]):

LEMMA 3.6 ([Ng4]).   *Let* B_1 *be the unit ball (or cube) of* $\mathbb{R}^N$, $N \geq 1$. *There exists a positive constant* C_N, *depending only on* N, *such that, for every function* u *in* $L^1(B_1)$, *one has*

$$\int_{B_1} \int_{B_1} |u(x) - u(y)|\, dx\, dy \leq C_N \Big(\int_{\substack{B_1 \\ |u(x)-u(y)|>1}} \int_{B_1} \frac{1}{|x-y|^{N+1}}\, dx\, dy + 1 \Big).$$

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