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Functional Analysis — *A Sobolev non embedding*, by Petru Mironescu and Winfried Sickel, communicated on 13 March 2015.

Abstract. — If Ω is a bounded domain in $\mathbb{R}^n$, $1 \le q < p \le \infty$ and $s = 0, 1, 2, \ldots$, then we clearly have $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$. We prove that this property does not hold when s in not an integer.[1]

Key words: Sobolev and Slobodeskii spaces, embeddings, lacunary series, wavelets

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1. A non embedding

In connection with his work on distributional Jacobians [3], H. Brezis asked us whether

$$\text{(1)} \qquad \text{the inclusion } W^{1/2,3}((0,1)) \subset W^{1/2,2}((0,1)) \text{ holds.}$$

The answer is *negative*. This is counterintuitive at first sight, since $L^3((0,1)) \subset L^2((0,1))$ and $W^{1,3}((0,1)) \subset W^{1,2}((0,1))$; thus, by "1/2 interpolation", we would expect (1) to hold.

Below we shall formulate our main result in a little bit greater generality. The class of *fractional* Sobolev spaces we have in mind is defined as follows. Let Ω be a nontrivial open subset of $\mathbb{R}^n$. Let $1 \le p \le \infty$. With $s = m + \sigma$, $m \in \mathbb{N}_0$ (the natural numbers including 0), and $0 < \sigma < 1$, the fractional Sobolev space $W^{s,p}(\Omega)$ is the collection of all $f \in L^p(\Omega)$ such that its distributional derivatives $D^\alpha f$, $\alpha \le m$, are regular and

$$\text{(2)} \qquad \max_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+\sigma p}} \, dx \, dy < \infty.$$

In this note, we give several proofs of the following

Theorem 1.1. *Let $s > 0$ be a non integer, and let $1 \le q < p \le \infty$. Then there exists some in Ω compactly supported function f such that $f \in W^{s,p}(\Omega)$ but $f \notin W^{s,q}(\Omega)$.*

[1] Presented by H. Brezis.

The same result was obtained independently by J. Van Schaftingen [12], using a proof similar to our second one.

Below we shall discuss three examples, all having their own advantages and disadvantages. In two examples we shall work with a periodic background, in the remaining with a non-periodic one. In the first example we shall work with the Gagliardo semi-norm itself (see (2)). In the other cases our computations will rely on norm equivalences whose proofs are sometimes delicate.

2. The first example

We shall work with the Gagliardo semi-norm. In some sense the first example is elementary.

Before proceeding, let us note that it suffices to establish the following fact: with s, p, q as above and with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ the standard torus,

$$\text{there exists some } g \in W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T}). \tag{3}$$

Proof of "(3) implies Theorem 1.1". Let $g \in W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T})$. Using a partition of unity on $\mathbb{T}$, we find that for some $\varphi \in C^\infty$ supported in some interval of length $< 2\pi$, the function $h := \varphi f$ is in $W^{s,p}(\mathbb{T}) \setminus W^{s,q}(\mathbb{T})$. By the choice of φ, h can be identified with a compactly supported function in $W^{s,p}(\mathbb{R}) \setminus W^{s,q}(\mathbb{R})$.

Consider next some function $\psi \in C_c^\infty(\mathbb{R}^{n-1})$, $\psi \not\equiv 0$. Then clearly $f := \psi \otimes h$ is compactly supported, and belongs to $W^{s,p}(\mathbb{R}^n) \setminus W^{s,q}(\mathbb{R}^n)$.

For all $\lambda > 0$ and all $x_0 \in \mathbb{R}^n$, the mapping $f \mapsto f(\lambda(\cdot - x_0))$ leaves the space $W^{s,p}(\mathbb{R}^n)$ invariant. Applying this argument our construction yields a function supported in a ball whose radius and centre are at our disposal. □

For $s = m + \sigma$, $m \in \mathbb{N}_0$ and $0 < \sigma < 1$, the periodic fractional order Sobolev space $W^{s,p}(\mathbb{T})$ can be normed with

$$\|f\|_{W^{s,p}(\mathbb{T})} := \|f\|_{L^p} + \Big(\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\Delta_h f^{(m)}(x)|^p}{|h|^{\sigma p+1}} \, dh \, dx \Big)^{1/p} \tag{4}$$

(obvious modification when $p = \infty$). Here, $\Delta_h g(x) := g(x+h) - g(x)$.

We will rely on the Brezis-Lieb lemma [2] that we recall here: if $1 \le p < \infty$, $f_\ell \to f$ a.e. and $\|f_\ell\|_{L^p} \le C$, then

$$\|f_\ell\|_{L^p}^p = \|f\|_{L^p}^p + \|f_\ell - f\|_{L^p}^p + o(1) \quad \text{as } \ell \to \infty.$$

We also rely on the following straightforward.

Lemma 2.1. *We have*

$$\|x \mapsto e^{\imath \ell x}\|_{W^{s,p}} \sim \ell^s \quad \textit{as } \ell \to \infty. \tag{5}$$

PROOF. The case $p = \infty$ being left to the reader, we assume that $1 \le p < \infty$. Clearly, it is enough to consider $0 < s < 1$. Set $f_\ell(x) = e^{\imath \ell x}$. Since $\|f_\ell\|_{L^p} \sim 1$, in order to prove the lemma it suffices to prove that

$$I_\ell := \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\Delta_h f_\ell(x)|^p}{|h|^{sp+1}} \, dh \, dx \sim \ell^{sp}.$$

This follows from the identity

$$I_\ell = 2\pi \ell^{sp} \int_0^{2\ell\pi} \frac{|e^{\imath\xi} - 1|^p}{|\xi|^{sp+1}} \, d\xi \tag{6}$$

and the fact that the integral in (6) has a positive finite limit as $\ell \to \infty$. □

FIRST PROOF OF THEOREM 1.1. We let to the reader the case where $p = \infty$, which is obtained by a rather straightforward modification of the argument below. We thus assume that $p < \infty$.

We will construct by induction on j sequences λ_j and ℓ_j such that

$$x \mapsto g(x) := \sum_{j \ge 1} \lambda_j e^{\imath \ell_j x} \quad \text{belongs to } W^{s,p} \text{ but not to } W^{s,q}. \tag{7}$$

We pick $\lambda_1 = 1$, $\ell_1 = 1$. Assuming $\lambda_1, \ldots, \lambda_j$, $\ell_1, \ldots, \ell_j$ already constructed, let

$$f_\ell(x) := \frac{1}{j^{1/q} \ell^s} e^{\imath \ell x}.$$

By Lemma 5, we have

$$\|f_\ell\|_{W^{s,r}} \sim \frac{1}{j^{1/q}}, \quad \forall 1 \le r < \infty.$$

On the other hand, if we write $s = m + \sigma$ then we have $f_\ell \to 0$ and $f_\ell^{(m)} \to 0$ pointwise as $\ell \to \infty$. By the Brezis-Lieb lemma, for $1 \le r < \infty$ we have, as $\ell \to \infty$,

$$\left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,r}}^{r} = \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,r}}^{r} + \|f_\ell\|_{W^{s,r}}^{r} + o(1).$$

Thus, for large ℓ, we have

$$\left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,p}}^{p} \le \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,p}}^{p} + \frac{k_1}{j^{p/q}} \tag{8}$$

and

$$\left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} + f_\ell(x) \right\|_{W^{s,q}}^{q} \geq \left\| x \mapsto \sum_{k=1}^{j} \lambda_k e^{\imath \ell_k x} \right\|_{W^{s,q}}^{q} + \frac{k_2}{j}. \tag{9}$$

Using (8) and (9), we construct λ_j and ℓ_j such that

$$\|g\|_{W^{s,p}}^{p} \leq C_p + k_1 \sum_{j\geq 2} \frac{1}{j^{p/q}}$$

and

$$\|g\|_{W^{s,q}}^{q} \geq k_2 \sum_{j\geq 2} \frac{1}{j},$$

and thus g satisfies (7). □

3. The second example

We shall work with lacunary series and Fourier-analytical characterizations of $W^{s,p}(\mathbb{T})$.

Therefore we recall the following characterization of $W^{s,p}(\mathbb{T})$ in terms of Fourier series, see [6, Theorem 3.5.3]. If $f(x) = \sum f_\ell e^{\imath \ell x}$, set

$$f^0 = f_0, \quad f^j(x) = \sum_{2^{j-1} < |\ell| \leq 2^j} f_\ell e^{\imath \ell x}, \quad \forall j \geq 1.$$

If $1 < p < \infty$, then

$$\|f\|_{W^{s,p}(\mathbb{T})} \sim \Big(\sum_{j\geq 0} 2^{sjp} \|f^j\|_{L^p}^{p} \Big)^{1/p}. \tag{10}$$

To incorporate the extremal cases $p = 1$ and $p = \infty$ we need the following a little bit more technical modification. Let ψ be an infinitely differentiable compactly supported function such that $\psi(x) = 1$ if $|x| \leq 1$. We define

$$\varphi_0(x) := \psi(x), \quad \varphi_j(x) := \psi(2^{-j}x) - \psi(2^{-j+1}x), \quad j = 1, 2, \ldots .$$

This results in a smooth dyadic decomposition of unity, i.e.,

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

If we assume in addition $\operatorname{supp} \psi \subset [-2, 2]$, then $\varphi_j(2^j) = 1$ and

$$\operatorname{supp} \varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \quad j = 1, 2, \ldots,$$

follow. Just from the Fourier-analytic definition used in [6, Chapt. 3] we derive

$$\|f\|_{W^{s,p}(\mathbb{T})} \sim \Big(\sum_{j \geq 0} 2^{sjp} \|\tilde{f}^j\|_{L^p}^p\Big)^{1/p}, \tag{11}$$

where

$$\tilde{f}^j(x) = \sum_{\ell=-\infty}^{\infty} f_\ell \varphi_j(\ell) e^{\imath \ell x}, \quad j = 0, 1, \ldots,$$

and (11) holds for all $p \in [1, \infty]$.

Second proof of Theorem 1.1. We choose

$$\lambda_j := \frac{1}{2^{sj} j^{1/q}}, \quad \forall j \geq 1,$$

and put

$$g(x) := \sum_{j \geq 1} \lambda_j e^{\imath 2^j x}.$$

Using either (10) (if $1 < p < \infty$) or (11) (for $p = 1$ or $p = \infty$), we clearly have $g \in W^{s,p}(\mathbb{T}) \backslash W^{s,q}(\mathbb{T})$. □

Note that the above yields an explicit version of our first example, in the sense that the λ_j's and the ℓ_j's are given by explicit formulas.

4. The third example

In this example we apply wavelets. We follow [11, Section 1.7], but see also Meyer [5].

In this perspective, it will be more convenient to construct some

$$g \text{ such that } g \in W_c^{s,p}(\mathbb{R}) \text{ but } g \notin W_c^{s,q}(\mathbb{R}), \tag{12}$$

i.e., we work in the non-periodic context from the very beginning.

Let $k > s + 2$ be an integer, and consider father and mother Daubechies wavelets ψ_F and ψ_M, compactly supported and of class C^k. Let, for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$,

$$\psi_m^j(x) = \begin{cases} \psi_F(x - m), & \text{if } j = 0 \text{ and } m \in \mathbb{Z} \\ 2^{(j-1)/2} \psi_M(2^{j-1} x - m), & \text{if } j \geq 1 \text{ and } m \in \mathbb{Z} \end{cases}$$

Set (assuming say $g \in L^1_{loc}$)

$$g_j^m = \int_{\mathbb{R}} \psi_j^m(x) g(x)\, dx.$$

Then

$$\|g\|_{W^{s,p}} \sim \Big(\sum_{j=0}^{\infty} 2^{j(sp+p/2-1)} \sum_{m \in \mathbb{Z}} |g_j^m|^p \Big)^{1/p}, \tag{13}$$

with the obvious modification when $p = \infty$.

Third proof of Theorem 1.1. The generators of the wavelet basis are compactly supported. Without loss of generality we may assume

$$\operatorname{supp} \psi_M \subset [0, N] \tag{14}$$

for some $N = N(s)$ sufficiently large. We put

$$\lambda_j := \frac{1}{2^{j(s+1/2)} j^{1/q}}, \quad j = 1, 2, \ldots \tag{15}$$

Define

$$g := \sum_{j=1}^{\infty} \lambda_j \sum_{m=0}^{2^j - 1} \psi_j^m. \tag{16}$$

By (15) and the fact that the ψ_j^m's define an orthonormal basis in $L^2(\mathbb{R})$, we find that $g \in L^2(\mathbb{R})$, and in particular we have

$$g_j^m = \begin{cases} \lambda_j, & \text{if } j \geq 1 \text{ and } 0 \leq m \leq 2^j - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

By (14) and (16) we have $\operatorname{supp} g \subset [0, N+1]$. Finally, by (13), (15) and (17) we find that g satisfies (12). □

5. Besov spaces and the interpolation argument

Unlike the first proof, the second and the third one are suited to the scale of Besov or Triebel-Lizorkin spaces. This goes beyond the scope of this note. However, we would like to mention that in Example 2 and 3 we already used the identification of our fractional Sobolev spaces as special cases of Besov spaces. More exactly

$$W^{s,p}(\mathbb{T}) = B^s_{p,p}(\mathbb{T}) \quad \text{and} \quad W^{s,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n),$$

$s > 0$, $s \notin \mathbb{N}$, $1 \leq p \leq \infty$, see [6, 3.5.4] and [10, 2.5.12].

In the framework of Besov spaces a straightforward adaptation of the second proof lead to the following improvement of (3):

$$W^{s,p}(\mathbb{T}) \not\hookrightarrow B^s_{q,r}(\mathbb{T}) \quad \text{if } p \geq q \text{ and } r < p. \tag{18}$$

Completely analoguous, Example 3 yields the following counterpart for non-periodic spaces

$$W^{s,p}(\Omega) \not\hookrightarrow B^s_{q,r}(\Omega) \quad \text{if } p \geq q \text{ and } r < p. \tag{19}$$

Here the Besov space on the domain Ω is defined by restriction, i.e., $f \in L^q(\Omega)$ belongs to $B^s_{q,r}(\Omega)$ if there exists some $g \in B^s_{q,r}(\mathbb{R}^n)$ such that

$$f = g \quad \text{on } \Omega.$$

Some comments to the literature. Necessary and sufficient conditions for embeddings of one Besov space into another can be found in Taibleson [8], S., Triebel [7] and Haroske, Skrzypczak [4]. Whereas in [7] the authors were dealing with the situation on $\mathbb{R}^n$, Taibleson [8] also considered the periodic case. E.g., (18) can be found in [8, Thm. 19(b)]. For smooth domains Ω Haroske and Skrzypczak [4] have proved (19) in the much more general context of Besov-Morrey spaces.

Finally, for convenience of the reader, we will comment on the "interpolation argument" from page 1. We restrict ourselves to real and complex interpolation. It is known that

$$(L^u(0,1), W^{1,u}(0,1))_{1/2,r} = B^{1/2}_{u,r}(0,1), \quad 1 \leq r \leq \infty.$$

Now, choosing $u = r = 3$ we conclude

$$\begin{aligned} W^{1/2,3}(0,1) &= (L^3(0,1), W^{1,3}(0,1))_{1/2,3} \\ &\hookrightarrow (L^2(0,1), W^{1,2}(0,1))_{1/2,3} = B^{1/2}_{2,3}(0,1). \end{aligned}$$

The Besov space $B^{1/2}_{2,3}(0,1)$ does not belong to the scale of fractional Sobolev spaces under consideration, it is just a space containing $W^{1/2,2}(0,1) = B^{1/2}_{2,2}(0,1)$. Similarly for the complex method we obtain that

$$[L^u(0,1), W^{1,u}(0,1)]_{1/2} = F^{1/2}_{u,2}(0,1), \quad 1 < u < \infty.$$

Here $F^{1/2}_{u,2}(0,1)$ denotes a Lizorkin-Triebel space. Again choosing $u = 3$ we conclude

$$\begin{aligned} F^{1/2}_{3,2}(0,1) &= [L^3(0,1), W^{1,3}(0,1)]_{1/2} \\ &\hookrightarrow [L^2(0,1), W^{1,2}(0,1)]_{1/2} = W^{1/2,2}(0,1). \end{aligned}$$

The Lizorkin-Triebel space $F^{1/2}_{3,2}(0,1)$ does also not belong to the scale of fractional Sobolev spaces, it is just a space embedded into $W^{1/2,2}(0,1)$. For all this we refer to [1, 6.4] and [9, 2.4].

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