Partial Differential Equation — *Linear equation with data in non standard spaces*, by J. M. Rakotoson, communicated on 13 March 2015.

Abstract. — Given a finite family of Banach function spaces V_α over a bounded set Ω, $V = \prod_\alpha V_\alpha$, and let T be an element of the dual of the Sobolev space $W^2 V$. We discuss the existence, uniqueness and regularity of the solution of the linear equation $Lu = T$ under the Dirichlet or Neumann condition on the boundary of Ω.

Our results extend recent works on very weak solution with data in weighted distance space or Lorentz space.

1. Introduction

Recently, Merker J. and the author extended the study of the so called very weak solution to the Neumann problem whose right hand side belongs to the dual space of Lorentz-Sobolev space $W^2 L^{p,q}(\Omega)$ (see below for the exact definition). We have shown in particular the

Theorem 1. *Let Ω be a bounded open set of class C^2, and let $T \in (W^2 L^{p,q}(\Omega))^*$, with $T(1) = 0$, $1 < p < +\infty$, $1 \leq q < \infty$.*

Then there exists a unique function $u \in L^{p',\infty}(\Omega)$, p' is the conjugate of p, such that

1. $\displaystyle \int_\Omega u(x)\,dx = 0.$
2. $\displaystyle -\int_\Omega u\Delta\varphi\,dx = \langle T, \varphi\rangle$, $\forall \varphi \in W^2 L^{p,1}(\Omega)$, $\dfrac{\partial \varphi}{\partial n} = 0$ on $\partial\Omega$ and

$$\|u\|_{L^{p',\infty}(\Omega)} \leq c(\Omega)\|T\|_{(W^2 L^{p,q}(\Omega))^*}.$$

A first aim of this paper is to extend such results replacing the Lorentz space $L^{p,q}$ by a family of Banach functions spaces $L(\rho_\alpha; \Omega)$ (see below for the precise definition).

We shall distinguish the Dirichlet and Neumann cases (although the proofs of those cases are similar). We shall consider Ω a bounded open set of class C^2 and

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the space of bounded mean oscillation functions $\mathrm{bmo}_r(\Omega)$ which coincides to the Campanato space $\mathscr{L}^{2,N}(\Omega)$ (see below for the definition). The main result that we shall prove is the

THEOREM 2 (existence of v.w.s. in $L^1(\Omega)$ for data in Banach function spaces). *Let* $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ *with* $|\alpha| = \alpha_1 + \cdots + \alpha_N \leq 2$, Ω *a bounded open set of class* C^2 *and let* $L(\rho_\alpha; \Omega)$ *be a reflexive Banach function space satisfying* $(H_{Fund})_1$ *i.e.* $\mathrm{bmo}_r(\Omega) \subset_> L(\rho_\alpha; \Omega)$. *We set:* $V = \prod\limits_{|\alpha| \leq 2} L(\rho_\alpha; \Omega)$ *and*

$$W^2 V = \{v \in L^1_{loc}(\Omega) : D^\alpha v \in L(\rho_\alpha; \Omega), |\alpha| \leq 2\}.$$

Then for $T \in (W^2 V)^*$, *there exists a unique* $u \in L^1(\Omega)$ *satisfying*

$$-\int_\Omega u \Delta\varphi \, dx = T(\varphi), \quad \forall \varphi \in W^2 L^\infty(\Omega),$$

with $\varphi = 0$ *on* $\partial\Omega$ *(resp.* $\dfrac{\partial\varphi}{\partial n} = 0$ *on* $\partial\Omega$ *for Neumann problem with the additional conditions* $T(1) = 0$ *and* $\displaystyle\int_\Omega u(x) \, dx = 0$*).*

Moreover, there exists a constant $c(\Omega) > 0$ *such that*

$$|u|_{L^1(\Omega)} \leq c(\Omega)\|T\|_{(W^2 V)*}.$$

REMARK 1. We can remove the condition on the reflexivity of $L(\rho_\alpha; \Omega)$ but in that case, we replace the hypothesis $(H_{Fund})_1$ by a different condition, for instance a stronger inclusion say, for some $p > 1$, for all α,

$$L^{p,1}(\Omega) \subset_> L(\rho_\alpha; \Omega) \quad (H_{Fund})_2$$

In this case

$$(W^2 V)^* \subset (W^2 L^{p,1}(\Omega))^*.$$

Therefore, the existence and uniqueness of the function u are a consequence of Theorem 1.

A particular Banach function space $L(\rho_\alpha; \Omega)$ that we shall consider is the so called Generalized Gamma spaces $G\Gamma(p, m, w(\cdot))$ that we have introduced in previous papers, [11, 12].

These spaces give a unified formulation of different spaces among other thing the Lorentz spaces $L^{p,q}(\Omega)$, the small Lebesgue spaces $L^{(p}(\Omega)$ [9, 10, 16], the Orlicz spaces $(L^p \operatorname{Log} L)(\Omega)$.

There are many applications of the notion of very weak solution. Here is an application of Theorem 1 that we shall prove in the last paragraph:

LEMMA 1 (of Density). *Let Ω be a bounded set of class $C^{1,1}$. Then, the set*

$$\left\{ \varphi \in C^2(\overline{\Omega}) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \Omega \right\} = \Delta_2$$

is dense in

$$\left\{ \varphi \in W^2 L^{p,q}(\Omega) : \frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial \Omega \right\} = V,$$

$1 < p < +\infty, 1 \leq q \leq +\infty.$

2. NOTATION AND PRELIMINARY RESULTS

Here are some spaces that we shall use

DEFINITION 1 (of bmo($\mathbb{R}^N$)) (see Goldberg in [2]). A locally integrable function f on $\mathbb{R}^N$ is said to be in bmo($\mathbb{R}^N$) if

$$\sup_{0<\text{diam}(Q)<1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + \sup_{\text{diam}(Q)\geq 1} \frac{1}{|Q|} \int_Q |f(x)| \, dx$$
$$\equiv \|f\|_{\text{bmo}(\mathbb{R}^N)} < +\infty,$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^N$ whose sides are parallel to the coordinates axis.

DEFINITION 2 (of bmo$_r(\Omega)$ and main property) [2, 3, 19]. A locally integrable function f on a Lipschitz bounded domain Ω is said to be in bmo$_r(\Omega)$ (r stands for restriction) if

$$\sup_{0<\text{diam}(Q)<1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx + \int_\Omega |f(x)| \, dx \equiv \|f\|_{\text{bmo}_r(\Omega)} < +\infty,$$

where the supremum is taken over all cube $Q \subset \Omega$ whose sides are parallel to the coordinates axis.

In this case, there exists a function $\tilde{f} \in \text{bmo}(\mathbb{R}^N)$ such that

$$\tilde{f}|_\Omega = f \quad \text{and} \quad \|\tilde{f}\|_{\text{bmo}(\mathbb{R}^N)} \leq c_\Omega \cdot \|f\|_{\text{bmo}_r(\Omega)}.$$

EXPLANATION. *The above definition is adapted to the case where the domain is bounded, and it is equivalent to the definition given in* [3, 2, 19].

The main property is due to P. W Jones [17], *this extension result implies that* bmo$_r(\Omega)$ *embeds continuously into* $L_{\exp}(\Omega)$.

DEFINITION 3 (of the Campanato space $\mathscr{L}^{2,N}(\Omega)$). A function $u \in \mathscr{L}^{2,N}(\Omega)$ if

$$\|u\|_{L^2(\Omega)} + \sup_{x_0 \in \Omega, r>0} \left[r^{-N} \int_{Q(x_0,r) \cap \Omega} |u - u_r|^2 \, dx \right]^{\frac{1}{2}} = \|u\|_{\mathscr{L}^{2,N}(\Omega)} < +\infty.$$

Here

$$u_r = \frac{1}{|Q(x_0;r) \cap \Omega|} \int_{Q(x_0;r) \cap \Omega} u(x) \, dx.$$

THEOREM 3 (Equivalence of the two definitions). *For a Lipschitz bounded domain Ω one has:*

$$\mathscr{L}^{2,N}(\Omega) = \mathrm{bmo}_r(\Omega), \quad \textit{with equivalent norms.}$$

This theorem is not essential for our purpose, we refer to [20] for the proof.

DEFINITION 4 (Banach function norm). Let Ω be a bounded domain in $\mathbb{R}^N$,

$$L^0(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable}\}, \quad L^0_+(\Omega) = \{f \geq 0, f \in L^0(\Omega)\}.$$

A mapping $\rho : L^0_+(\Omega) \to [0, +\infty]$ is called a Banach function norm if it satisfies the following properties: $\forall f, g, f_n \in L^0_+(\Omega)$

1. ρ is a norm i.e.
$$\begin{cases} \rho(f) = 0 \text{ if and only if } f = 0, \\ \rho(\lambda f) = \lambda \rho(f) \; \forall \lambda \in \mathbb{R}_+, \\ \rho(f + g) \leq \rho(f) + \rho(g). \end{cases}$$

2. $0 \leq g \leq f$ a.e. in Ω then $\rho(g) \leq \rho(f)$ (monotonicity).
3. $0 \leq f_n \nearrow f$ a.e. in Ω, then $\rho(f_n) \nearrow \rho(f)$ (Beppo-Levi property).
4. $\rho(1) < +\infty$.
5. There exists a constant $c_\Omega > 0$ such that $\forall f \in L^0_+(\Omega)$
$$\int_\Omega f \, dx \leq c_\Omega \rho(f).$$

DEFINITION 5 (Banach function space). Let ρ be a function norm. Then the linear space

$$L(\rho; \Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable such that } \rho(|f|) < +\infty\}$$

is called a *Banach function space* (BFS), it is a Banach space endowed with the norm $\|f\| = \rho(|f|) = \rho(f)$.

The associate norm ρ' is defined on $L^0_+(\Omega)$ by

$$\rho'(g) = \sup \left\{ \int_\Omega fg \, dx : f \in L^0_+(\Omega), \rho(f) \leq 1 \right\}$$

ρ' is a Banach function norm and the BFS $L(\rho';\Omega) = L(\rho;\Omega)'$ is called the associated space of $L(\rho;\Omega)$.

Setting $\rho'' = (\rho')'$. Then one has

$$L(\rho'';\Omega) = L(\rho;\Omega).$$

One fundamental property that we shall need is

DEFINITION 6 (Absolute continuity of the norm of function f). Let $f \in L(\rho;\Omega)$. We shall say that it has absolutely continuous norm if, for any sequence f_n such that $0 \le f_n \le |f|$, $f_n(x) \to 0$ a.e. one has $\rho(f_n) \xrightarrow[n \to +\infty]{} 0$.

In other words, the dominate Lebesgue theorem is true for any sequence pointwise convergent and dominated by $|f|$:

$$\text{If } g_n \to g \text{ a.e.,} \quad |g_n| \le |f| \text{ then } \rho(g_n - g) \xrightarrow[n \to +\infty]{} 0.$$

The link of this definition with the reflexivity is

THEOREM 4 (Reflexivity). *Let $L(\rho;\Omega)^*$ be the Banach dual space of $L(\rho;\Omega)$.*

$L(\rho;\Omega)^*$ is canonically isometrically isomorphic to the associate space $L(\rho';\Omega)$ if and only if $L(\rho;\Omega)$ has absolutely continuous norm.

In particular, $L(\rho;\Omega)$ is reflexive if and only if $L(\rho;\Omega)$ and $L(\rho';\Omega)$ have absolutely continuous norms.

DEFINITION 7 (Rearrangement invariant space). The space $L(\rho;\Omega)$ is said to be rearrangement invariant if for all f and g in $L(\rho;\Omega)$ and $\forall t \in \mathbb{R}$

$$\text{measure}\{x \in \Omega, f(x) > t\} = \text{measure}\{x \in \Omega, g(x) > t\} \Rightarrow \rho(f) = \rho(g).$$

REMARK 2. For convenience, we shall denote by $|E|$ the measure of a set E. If ρ is rearrangement invariant, we can associate to $X = L(\rho;\Omega)$ the so called fundamental function $\varphi_X(t) = \rho(\chi_E)$ whenever $|E| = t$. This function does not depend on E. Since ρ' is also rearrangement invariant, we can associate $\varphi_{X'} = \rho'(\chi)$ to $X' = L(\rho';\Omega)$.

One has

PROPOSITION 1 (see Bennett-Sharpley [1] p. 66).

1. $\varphi_X(t)\varphi_{X'}(t) = t$, $\forall t \in \,]0, |\Omega|[$.
2. φ_X *is increasing,* $\varphi_X(t) = 0$ *iff* $t = 0$.
3. $\dfrac{\varphi_X(t)}{t}$ *is decreasing.*
4. φ_X *is continuous except perhaps at the origin.*

EXAMPLE 1. *The most well-known Banach function spaces are the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le +\infty$, and the Lorentz spaces $L^{p,q}(\Omega)$. We give a quick definition of these last spaces.*

DEFINITION 8 (of the Distribution function and monotone rearrangement). Let $u \in L^0(\Omega)$. The distribution function of u is the decreasing function $m = m_u : \mathbb{R} \mapsto [0, |\Omega|]$

$$m_u(t) = \text{measure}\{x : u(x) > t\} = |\{u > t\}|.$$

The generalized inverse u_* of m is defined by

$$u_*(s) = \inf\{t : |\{u > t\}| \le s\}, \quad s \in [0, |\Omega|[$$

and is called the decreasing rearrangement of u. We shall set $\Omega_* =]0, |\Omega|[$.

We recall

DEFINITION 9. Let $1 \le p \le +\infty$, $0 < q \le +\infty$.

• If $q < +\infty$, one defines the following norm for $u \in L^0(\Omega)$

$$\|u\|_{p,q} = \|u\|_{L^{p,q}} = \left[\int_{\Omega_*} [t^{\frac{1}{p}} |u|_{**}(t)]^q \frac{dt}{t} \right]^{\frac{1}{q}}.$$

• If $q = +\infty$,

$$\|u\|_{p,\infty} = \sup_{0 < t \le |\Omega|} t^{\frac{1}{q}} |u|_{**}(t), \quad |u|_{**}(t) = \frac{1}{t} \int_0^t |u|_*(\sigma) \, d\sigma.$$

The space $L^{p,q}(\Omega) = \{u \in L^0(\Omega) : \|u\|_{p,q} < +\infty\}$ is called a *Lorentz space*.
  If $p = q = +\infty$, $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$.
  The dual of $L^{1,1}(\Omega)$ is called $L_{\exp}(\Omega)$

The Lorentz spaces are particular cases of the Generalized Gamma spaces $G\Gamma(p, m, w)$.

DEFINITION 10 (Gamma weight). Let $1 \le p < +\infty$, $1 \le m < +\infty$, and assume for simplicity that $|\Omega| = 1$, and let w be a measurable nonnegative function such that:

1. $t \to w(t) t^{\frac{m}{p}}$ is integrable near zero,
2. w is in $L^1_{loc}(]0, 1])$

$$\left(\int_a^1 w(\sigma) \, d\sigma < +\infty, \forall a > 0 \right),$$

3. $\exists a > 0 : 0 < a < 1$, $\underset{(0,a)}{\text{essinf }} w > 0$.

Note that 1. and 2. are equivalent to $t^{\frac{m}{p}} w \in L^1(]0, 1[)$, and the last condition avoid w to be a trivial function.

The spaces $G\Gamma(p, m, w)$ are defined by

$$G\Gamma(p, m, w)$$

$$= \left\{ f \in L^0(\Omega) : \|f\|_{G\Gamma(p,m,w)} = \left[\int_0^1 w(t) \left(\int_0^t |f|_*^p(s) \, ds \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}} \text{ is finite} \right\}$$

is called Generalized Γ-spaces

for $1 \le p < +\infty$, $1 \le m < +\infty$.

A striking fact that we have shown in those papers [11, 12], is that, the $G\Gamma$-Sobolev spaces,

$$W^1 G\Gamma(p, m, w(\cdot)) = \{v \in L^1(\Omega) : \nabla v \in G\Gamma(p, m, w(\cdot))^N\}$$

is compactly embedded in $L^{p^*}(\Omega)$, $p^* = \dfrac{Np}{N-p}$, $1 \le p < N$ if and only $w \notin L^1(0, 1)$.

REMARK 3. More generally, if V is a Banach space contained in $L^1(\Omega)$, we set:

$$W^1 V = \{v \in L^1(\Omega) : \nabla v \in V^N\}$$

and

$$W_0^1 V = W^1 V \cap W_0^{1,1}(\Omega).$$

REMARK 4 (on some non compactness). Applying this result to the small Lebesgue-Sobolev space $W^1 L^{(p}(\Omega)$, we have for $1 < p < N$ the following compact embedding $W^1 L^{(p}(\Omega) \underset{c}{\hookrightarrow} L^{p^*}(\Omega)$. If we replace the small Lebesgue space by the closest Lorentz space $L^{p,1}(\Omega)$:

$$L^{(p}(\Omega) \subset L^{p,1}(\Omega) \subset L^p(\Omega),$$

The following inclusion is not compact, anymore,

$$W^1 L^{p,1}(\Omega) = \{v \in L^1(\Omega) : \nabla v \in L^{p,1}(\Omega)^N\} \subset L^{p^*}(\Omega).$$

The proof relies on a straight forward computation using an explicit counterexample (see also [6]). We drop it since it is beyond the scope of this paper.

The resolution of Theorem 2 needs the following decomposition whose proof will be given in the last paragraph.

THEOREM 5. *Let $m \in \mathbb{N}$, $m \ge 1$, for an $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| \le m$, we consider the Banach function space $L(\rho_\alpha; \Omega)$ with absolutely continuous norm.*
We set: $V = \prod\limits_{|\alpha| \le m} L(\rho_\alpha; \Omega)$ and

$$W^m V = \{v \in L^1_{loc}(\Omega) : D^\alpha v \in L(\rho_\alpha; \Omega), |\alpha| \le m\}.$$

Let $T \in (W^m V)^*$. Then there exists a family $f_\alpha \in L(\rho_\alpha'; \Omega)$ such that

$$T(u) = \sum_{|\alpha| \le m} \int_\Omega f_\alpha D^\alpha u \, dx \quad \forall u \in W^m V.$$

Moreover, we have:

$$\|T\|_* \le \operatorname*{Max}_{|\alpha| \le m} \|f_\alpha\|_{L(\rho_\alpha'; \Omega)}$$

1. *If $\beta \in \mathbb{N}^N$ is such that ρ_β is rearrangement invariant, then*

$$\operatorname*{Max}_{t \le |\Omega|} \left[\varphi_{X_\beta'}(t) |f_\beta|_{**}(t) \right] \le \|T\|_*,$$

    with $X_\beta = L(\rho_\beta; \Omega)$, $\varphi_{X_\beta'}(\cdot)$ being the fundamental function of X_β'.
2. *If $L(\rho_\alpha; \Omega)$ is reflexive for all α then*

$$\|T\|_* = \operatorname*{Max}_{|\alpha| \le m} \|f_\alpha\|_{L(\rho_\alpha'; \Omega)}.$$

In the above theorem, in statement 2, the reflexivity assumption on $L(\rho_\alpha; \Omega)$ is not necessarily useful to obtain the equality as we suggest in the following proposition whose proof is given in the last paragraph.

PROPOSITION 2 (Decomposition of $T \in (W^1 L^{p,1}(\Omega))^*$, $1 \le p < +\infty$, ($L^{p,1}$ can be replaced by L^1)). *There exists*

$$(g_0, \ldots, g_N) \in \begin{cases} L^{p',\infty}(\Omega)^{N+1} & \text{if } 1 < p < +\infty, \\ L_{\exp}(\Omega)^{N+1} & \text{if } p = 1, \\ L^\infty(\Omega)^{N+1} & \text{if } T \in W^1 L^1(\Omega)^*, \end{cases}$$

such that for all $\varphi \in W^1 L^{p,1}(\Omega)$ (resp $\varphi \in W^1 L^1(\Omega)$):

1. $T(\varphi) = \displaystyle\int_\Omega g_0 \varphi \, dx + \sum_{j=1}^N \int_\Omega g_j \frac{\partial \varphi}{\partial x_j} dx.$
2. $\|T\|_* = \operatorname*{Max}_{j \le N} \|g_j\|_{L^{p',\infty}(\Omega)}$ *if $p > 1$ and $\|T\|_* = \operatorname{Max}_{j \le N} \|g_j\|_{L_{\exp}}$ if $p = 1$. (resp*
    $\|T\|_* = \operatorname*{Max}_{j \le N} \|g_j\|_{L^\infty(\Omega)}$ *for $W^1 L^1(\Omega)^*$)*

3. Proof of the Main Results and Regularity Theorem

We begin this paragraph by the proof of the main theorem 2. Following the decomposition in Theorem 5, we can write

$$T(\varphi) = \sum_{|\alpha| \le 2} \int_\Omega f_\alpha D^\alpha \varphi, \ \forall \varphi \in W^2 V, \quad \|T\|_* = \operatorname*{Max}_{|\alpha| \le m} \|f_\alpha\|_{L(\rho_\alpha'; \Omega)}.$$

Let $k \in \mathbb{N}$, $k \geq 1$ and define

(1) $$f_{\alpha k}(x) = \min(|f_\alpha(x)|; k) \operatorname{sign} f_\alpha(x), \quad x \in \Omega,$$

$$T_k(\varphi) = \sum_{|\alpha| \leq 2} \int_\Omega f_{\alpha k} D^\alpha \varphi \, dx, \quad \forall \varphi \in W^2 L^1(\Omega). \qquad \Box$$

One has

LEMMA 2 (density and approximation).

1. *For all k, $T_k \in (W^2 V)^*$, $\lim_{k \to +\infty} T_k(\varphi) = T(\varphi)$, $\forall \varphi \in W^2 V$.*
2. $\|T_k - T\|_* \leq \operatorname*{Max}_{|\alpha| \leq 2} \|f_\alpha - f_{\alpha k}\|_{L(\rho'_\alpha; \Omega)} \xrightarrow[k \to +\infty]{} 0.$
3. $C_c^\infty(\Omega)$ *is dense in* $(W^2 V)^*$.

PROOF.

1. Let us note that $W^2 L^\infty(\Omega) \subset W^2 V$ since for $\varphi \in W^2 L^\infty(\Omega)$

$$\rho_\alpha(D^\alpha \varphi) \leq \|D^\alpha \varphi\|_\infty \rho_\alpha(\chi_\Omega) < +\infty.$$

2. If $\varphi \in W^2 V$, then

$$|T_k(\varphi) - T(\varphi)| \leq \sum_{|\alpha| \leq 2} \int_\Omega |f_{\alpha k} - f_\alpha|(x)|D^\alpha \varphi|(x) \, dx$$

$$\leq \operatorname*{Max}_{|\alpha| \leq 2} \|f_{\alpha k} - f_\alpha\|_{L(\rho'_\alpha; \Omega)} \cdot \sum_{|\alpha| \leq 2} \rho_\alpha(D^\alpha \varphi),$$

so that we derive the inequality. The convergence to zero is a consequence of the fact that $L(\rho'_\alpha; \Omega)$ has absolutely continuous norm.
3. For k (fixed), we consider $f_{\alpha k j} \in C_c^\infty(\Omega)$ such that
   (a) $f_{\alpha k j}(x) \to f_{\alpha k}(x)$ a.e. in Ω
   (b) $|f_{\alpha k j}(x)| \leq k$ $\forall j$ and a.e. in Ω.
   We then have

$$\rho'_\alpha(|f_{\alpha k j} - f_{\alpha k}|) \xrightarrow[j \to +\infty]{} 0.$$

The same argument as it is given in statement 2. shows that

$$\|T_{kj} - T_k\|_* \leq \operatorname*{Max}_{|\alpha| \leq 2} \|f_{\alpha k j} - f_{\alpha k}\|_{L(\rho'_\alpha; \Omega)} \xrightarrow[j \to +\infty]{} 0,$$

where $T_{kj}(\varphi) = \sum_{|\alpha| \leq 2} \int_\Omega f_{\alpha k j}(x) D^\alpha \varphi(x) \, dx.$ $\qquad \Box$

We apply now these approximations to prove the existence of the very weak solution.

3.1. 1st case: Dirichlet condition

LEMMA 3. *Let $(T_k)_k$ be the sequence defined in relation* (1). *Then, there exists a unique element $u_k \in \bigcap_{p' < +\infty} L^{p', \infty}(\Omega)$ satisfying*

$$(2) \qquad\qquad -\int_\Omega u_k \Delta\varphi \, dx = T_k(\varphi),$$

$\forall \varphi \in W^2 L^\infty(\Omega) \cap W_0^1 L^{p,q}(\Omega),$ *for all p such that $1 \leq p < +\infty$, $1 \leq q < +\infty$.*

PROOF. Let k be fixed, $T_k(\varphi) = \displaystyle\sum_{|\alpha| \leq 2} \int_\Omega f_{\alpha k} D^\alpha \varphi \, dx \quad \forall \varphi \in W^2 L^\infty(\Omega)$.

Let $f_{\alpha k j} \ C_c^\infty(\Omega)$ such that $f_{\alpha k j}(x) \xrightarrow[j \to +\infty]{} f_{\alpha k}(x)$ a.e. and $\|f_{\alpha k j}\|_\infty \leq k$. Therefore, for all $1 < p < +\infty$, $\forall q \in [1, +\infty[$

$$\|f_{\alpha k j} - f_{\alpha k}\|_{L^{p', q'}(\Omega)} \xrightarrow[j \to +\infty]{} 0.$$

The number p' (resp. q') is the conjugate of p (resp. of q). The function

$$(3) \qquad\qquad T_{kj} = -\sum_{|\alpha| \leq 2} (-1)^{|\alpha|} D^\alpha f_{\alpha k j} \in C_c^\infty(\Omega)$$

and then there exists $u_{kj} \in W^2 \, \mathrm{bmo}_r(\Omega) \cap W_0^1 \, \mathrm{bmo}_r(\Omega)$ for all $p < +\infty$ solution of

$$(\mathscr{L}_{kj}) \begin{cases} -\Delta u_{kj} = T_{kj} & in \ \Omega \\ u_{kj} = 0 & on \ \partial\Omega. \end{cases}$$

This is equivalent to

$$-\int_\Omega u_{kj} \Delta\varphi \, dx = \int_\Omega T_{kj} \varphi \, dx = \sum_{|\alpha| \leq 2} \int_\Omega f_{\alpha k j}(x) D^\alpha \varphi(x) \, dx$$

$\varphi \in W^2 L^1(\Omega) \cap W_0^1 L^1(\Omega)$.

Arguing as in the proof of Theorem 1 of [18] (see also [7, 8]), we have for $1 < p < +\infty$

$$\sup_{t \leq |\Omega|} [t^{\frac{1}{p'}} |u_{kj}|_{**}(t)] \leq c \sum_{|\alpha| \leq 2} \|f_{\alpha k j}\|_{L^{p', q'}(\Omega)},$$

and then, $(u_{kj})_j$ is a Cauchy sequence in $L^{p', \infty}(\Omega)$ and we derive the existence of $u_k \in L^{p', \infty}(\Omega)$

$$\|u_{kj} - u_k\|_{L^{p', \infty}(\Omega)} \xrightarrow[j \to +\infty]{} +\infty.$$

And u_k satisfies the equation (2). Such equation has a unique solution, since if $v \in L^1(\Omega)$;

$$-\int_\Omega v\Delta\varphi\,dx = 0 \quad \forall\varphi \in C^2(\overline{\Omega}); \quad \varphi = 0 \text{ on } \partial\Omega.$$

Then $v = 0$. In particular $u_k \in \bigcap_{p'<+\infty} L^{p',\infty}(\Omega)$. $\qquad\qquad\square$

To obtain an uniform estimate with respect to k in $L^1(\Omega)$, we shall need the following hypothesis; for all α, $|\alpha| \le 2$

$$(H_{Fund})_1 \quad \text{bmo}_r(\Omega) \hookrightarrow L(\rho_\alpha; \Omega).$$

We want to show that $(u_k)_k$ is a Cauchy sequence in $L^1(\Omega)$. Indeed, consider u_k, u_m two functions satisfying Lemma 3. By the result of Chang-Dafni-Stein [3, 4] on the regularity in $W^2\text{bmo}_r(\Omega)$, we have $\varphi_{km} \in W^2\text{bmo}_r(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta\varphi_{km} = \text{sign}(u_k - u_m) & \text{in } \Omega, \\ \varphi_{km} = 0 & \text{on } \partial\Omega. \end{cases}$$

and $\underset{|\alpha|\le 2}{\text{Max}} \|D^\alpha\varphi_{km}\|_{\text{bmo}_r} \le c(\Omega)$. By the hypothesis $(H_{Fund})_1$ one has

$$\rho_\alpha(|D^\alpha\varphi_{km}|) \le c\|D^\alpha\varphi_{km}\|_{\text{bmo}_r} \le c(\Omega).$$

Since

$$-\int_\Omega (u_k - u_m)\Delta\varphi_{km}\,dx = \sum_{|\alpha|\le 2} \int_\Omega (f_{\alpha k} - f_{\alpha m})D^\alpha\varphi_{km}\,dx$$

$$\int_\Omega |u_k - u_m|\,dx \le \sum_{|\alpha|\le 2} \rho'_\alpha(|f_{\alpha k} - f_{\alpha m}|)\rho_\alpha(|D^\alpha\varphi_{km}|)$$

$$\int_\Omega |u_k - u_m|\,dx \le c(\Omega) \sum_{|\alpha|\le 2} \|f_{\alpha k} - f_{\alpha m}\|_{L(\rho'_\alpha;\Omega)} \xrightarrow[(k,m)\to+\infty]{} 0,$$

we conclude that there is an element $u \in L^1(\Omega)$ such that

$$\lim_{k\to+\infty} \int_\Omega |u_k - u|\,dx = 0.$$

We pass to the limit in Lemma 3 , to get

$$-\int_\Omega u\Delta\varphi = \sum_{|\alpha|\le 2} \int_\Omega f_\alpha D^\alpha\varphi\,dx \quad \forall\varphi \in W^2L^\infty(\Omega),$$

with $\varphi = 0$ on Ω.

Such u is unique since two solutions u_1, u_2 will satisfy

$$- \int_\Omega (u_1 - u_2)\Delta\varphi = 0 \quad \forall \varphi \in C^2(\Omega), \quad \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega$$

so that

$$u_1 = u_2.$$

Finally for the continuity of the mapping $T \to u$, we have

$$\int_\Omega |u_k|\, dx \leq c(\Omega) \sum_{|\alpha| \leq 2} \|f_{\alpha k}\|_{L(\rho'_\alpha;\Omega)} \leq c(\Omega) \operatorname{Max}\|f_\alpha\|_{L(\rho'_\alpha;\Omega)}$$

$$\leq c(\Omega)\|T\|_*,$$

from which we derive

$$|u|_{L^1(\Omega)} \leq c(\Omega)\|T\|_*.$$

3.2. 2nd case Neumann condition

The same argument holds for this 2nd case with Neumann condition replacing the set of test functions by

$$\left\{ \varphi \in W^2 V : \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

We only emphasize on the main changes in the proof.

LEMMA 4. *There exists a unique element* $u_k \in \bigcap_{p'<+\infty} L^{p',\infty}(\Omega)$ *satisfying*

$$\int_\Omega u_k\, dy = 0$$

and

$$(4) \qquad\qquad - \int_\Omega u_k \Delta\varphi\, dx = T_k(\varphi) - T_k(1)\left(\fint_\Omega \varphi\, dx \right),$$

$\forall \varphi \in W^2 L^\infty(\Omega)$ *and* $\dfrac{\partial\varphi}{\partial n} = 0$ *on* $\partial\Omega$.

SKETCH OF PROOF. There exists a unique $u_{kj} \in W^2\, \mathrm{bmo}_r(\Omega)$ such that

$$- \int_\Omega u_{kj}\Delta\varphi\, dx = \int_\Omega \varphi(x) T^0_{kj}(x)\, dx, \quad \forall \varphi \in W^2 L^\infty(\Omega) \text{ with } \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega.$$

$\int_\Omega u_{kj}\,dx = 0$ where T_{kj} is given in relation (3) and is the approximation of T_k given in (1) and $T_{kj}^0 = T_{kj} - \fint_\Omega T_{kj}(y)\,dy$.

Arguing as in the proof of Theorem 1 [18] (see also [7]), we have

$$\sup_{t \le |\Omega|}\left[t^{\frac{1}{p'}}|u_{kj}|_{**}(t)\right] \le c \sum_{|\alpha|\le 2} \|f_{\alpha kj}\|_{L^{p',q'}(\Omega)}$$

and we derive the existence of (u_k) satisfying (4). Since

$$\lim_{j\to+\infty} T_{kj}(\varphi) = T_k(\varphi), \quad \forall \varphi \in W^2 L^\infty(\Omega).$$

In particular, this convergence is true for $\varphi = 1$. $\qquad\square$

We can also introduce

$$\begin{cases} -\Delta\phi_{km} = \operatorname{sign}(u_k - u_m) - \fint_\Omega \operatorname{sign}(u_k - u_m)(y)\,dy, & in\ \Omega, \\[2mm] \dfrac{\partial\phi_{km}}{\partial n} = 0 & on\ \partial\Omega, \\[2mm] \displaystyle\int_\Omega \phi_{km}(y)\,dy = 0. \end{cases}$$

According to Theorem 5.9 of [3] (see also [4]), we know that $\phi_{km} \in W^2\,\mathrm{bmo}_r(\Omega)$ and

$$\|\phi_{km}\|_{W^2\,\mathrm{bmo}_r} \le c(\Omega) \quad \text{(independent of } k \text{ and } m\text{)}.$$

Since $W^2\,\mathrm{bmo}_r \hookrightarrow W^2 V$, we can use ϕ_{km} as a test function to derive

$$\int_\Omega |u_k - u_m|\,dx = -\int_\Omega (u_k - u_m)(x)\Delta\phi_{km}(x)\,dx$$

$$= (T_k - T_m)(\phi_{km})$$

$$\left(\text{since } \int_\Omega u_k(x)\,dx = \int_\Omega u_m(x)\,dx = \int_\Omega \phi_{km}(x)\,dx = 0\right)$$

$$= \sum_{|\alpha|\le 2} \int_\Omega (f_{\alpha k}(x) - f_{\alpha m}(x))D^\alpha\phi_{km}(x)\,dx.$$

$$\int_\Omega |u_k - u_m|\,dx \le \sum_{|\alpha|\le 2} \rho_\alpha'(|f_{\alpha k} - f_{\alpha m}|)\rho_\alpha(|D^\alpha\phi_{km}|)$$

$$\le c(\Omega) \sum_{|\alpha|\le 2} \rho_\alpha'(|f_{\alpha k} - f_{\alpha m}|) \xrightarrow[(k,m)\to+\infty]{} 0.$$

We conclude as in the Dirichlet case. $\qquad\square$

REMARK 5. The hypothesis $(H_{Fund})_1$ is satisfied if $L_{\exp}(\Omega) \subset_> L(\rho_\alpha; \Omega)$ (this is a consequence of the John-Nirenberg inequality stating that $\text{bmo}(\mathbb{R}^N) \subset_> L_{\exp}(Q)$ for all cube $Q \subset \mathbb{R}^N$ [2, 20, 22] and with the Jones's extension theorems we then deduce

$$\text{bmo}_r(\Omega) = \mathscr{L}^{2,N}(\Omega) \subset_> L_{\exp}(\Omega)$$

(see for instance [1, 20])). In [1], page 382, this continuous embedding is already used with $\Omega = Q_0$ a cube. This hypothesis is equivalent to

$$L(\rho_\alpha'; \Omega) \subset_> L(\log L) = L^{1,1}(\Omega).$$

For instance, one has

PROPOSITION 3.

$$L_{\exp}(\Omega) \subset_> G\Gamma(p,m,w) \text{ if and only if } \int_0^1 w(t) \left(\int_0^t (1 - \log s)^p \, ds \right)^{\frac{m}{p}} dt < +\infty,$$

with $|\Omega| = 1$.

PROOF. Assume first that the integral is finite and let v be in $L_{\exp}(\Omega)$. Then, one has

$$\|v\|_{L_{\exp}} = \sup_{s \le |\Omega| = 1} \frac{|v|_*(s)}{1 - \log s} \text{ is finite}$$

and by the definition of $G\Gamma(p,m,w)$ we have

$$(5) \qquad \|v\|_{G\Gamma(p,m,w)} \le \|v\|_{L_{\exp}} \left[\int_0^1 w(t) \left(\int_0^t (1 - \log s)^p \, ds \right)^{\frac{m}{p}} \right]^{\frac{1}{m}} < +\infty.$$

Conversely, if $L_{\exp}(\Omega) \subset G\Gamma(p,m,w)$ then the integral is finite. Indeed, the function $s \to 1 - \log s$ is decreasing and continuous on $]0,1]$. Therefore, from the Lyapunov's Theorem (see Chong and Rice or Benoit Simon's thesis [5, 21]) we have a measurable function $v : \Omega \to \mathbb{R}$ such that $v_*(s) = 1 - \log(s)$. Thus, $v \in L_{\exp}(\Omega)$ and $\|v\|_{L_{\exp}} = 1$, we conclude that

$$v \in G\Gamma(p,m,w)$$
$$\|v\|_{G\Gamma(p,m,w)} = \left[\int_0^1 w(t) \left(\int_0^t (1 - \log s)^p \, ds \right)^{\frac{m}{p}} \right]^{\frac{1}{m}} \qquad \square$$

The condition that $G\Gamma(p,m,w)$ to be reflexive is proven in [15, 13].

PROPOSITION 4. *Assume that $p > 1$, $m > 1$. Then $G(p,m,w)$ is reflexive.*

In particular, we have:

COROLLARY 1 (of Theorem 2). *Let* $G\Gamma(p,m,w)$ *be the generalized Gamma space. If*

$$\int_0^1 w(t) \left(\int_0^t (1 - \log s)^p \, ds \right)^{\frac{m}{p}} dt < +\infty, \quad (p,m) \in (1, +\infty)^2,$$

then for any $T \in (W^2 G\Gamma(p,m,w))^*$, *there exists a unique* $u \in L^1(\Omega)$ *satisfies*

$$-\int_\Omega u \Delta\varphi \, dx = T(\varphi), \quad \forall \varphi \in W^2 L^\infty(\Omega) \cap H_0^1(\Omega).$$

Here $W^2 G\Gamma(p,m,w) = \{v \in L^1_{loc}(\Omega) : D^\alpha v \in G\Gamma(p,m,w); |\alpha| \leq 2\}$. *(resp.* $\forall \varphi \in W^2 L^\infty(\Omega)$ *with* $\dfrac{\partial \varphi}{\partial n} = 0$ $T(1) = 0$ *and* $\displaystyle\int_\Omega u \, dx = 0$ *for the Neumann problem.)*
    The mapping $T \to u$ *is continuous.*

THEOREM 6 (Integrability in $L_{\exp}(\Omega)$). *Assume that* $h_z^1(\Omega) \subset_> L(\rho_\alpha; \Omega)$, $\forall \alpha$. *Here* $h_z^1(\Omega)$ *is the Hardy space satisfying*

$$h_z^1(\Omega)^* = \mathrm{bmo}_r(\Omega)$$

Then, the solution u found in Theorem 2 under the hypothesis $(H_{Fund})_1$ *or* $(H_{Fund})_2$ *belongs to* $L_{\exp}(\Omega)$ *and there exists* $c(\Omega) > 0$:

$$\|u\|_{L_{\exp}} \leq c(\Omega)\|T\|_{(W^2 V)^*}.$$

PROOF. We only emphasize the main modification starting with the Dirichlet case. We assume that $|\Omega| = 1$ for simplicity. Let E be a measurable subset of Ω, and φ_{km} in $W^2 \mathrm{bmo}_r(\Omega)$ satisfying

$$\begin{cases} -\Delta\varphi_{km} = \chi_E \, \mathrm{sign}(u_k - u_m) & in \ \Omega, \\ \varphi_{km} = 0 & on \ \partial\Omega. \end{cases}$$

Then, one has from regularity result in [3, 4]:

$$\|D^2\varphi_{km}\|_{h_z^1(\Omega)} \leq c(\Omega)\|\chi_E \, \mathrm{sign}(u_k - u_m)\|_{L(Log\,L)}$$

$$\leq c(\Omega)|E|(1 - \log|E|)$$

$$\int_E |u_k - u_m| \, dx = -\int_\Omega (u_k - u_m)\Delta\varphi_{km} \, dx$$

$$= \sum_{|\alpha| \leq 2} \int_\Omega (f_{\alpha k} - f_{\alpha m}) D^\alpha \varphi_{km} \, dx$$

$$\leq \sum_{|a| \leq 2} \rho'_\alpha(f_{\alpha k} - f_{\alpha m}) \rho_\alpha(D^\alpha \varphi_{km})$$

$$\leq c \sum_{|\alpha| \leq 2} \rho'_\alpha(f_{\alpha k} - f_{\alpha m}) \|D^\alpha \varphi_{km}\|_{h^1_z}$$

$$\leq c(\Omega)|E|(1 - \log|E|) \sum_{|\alpha| \leq 2} \|f_{\alpha k} - f_{\alpha m}\|_{L(\rho'_\alpha;\Omega)}$$

we deduce

$$\|u_k - u_m\|_{L_{\exp}} = \sup_{t \leq 1} \frac{|u_k - u_m|_{**}(t)}{1 - \log t} \leq c(\Omega) \sum_{|\alpha| \leq 2} \|f_{\alpha k} - f_{\alpha m}\|_{L(\rho'_\alpha;\Omega)} \xrightarrow[k,m]{} 0.$$

And we conclude as before.

For the Neumann case, we consider as φ_{km}

$$\begin{cases} -\Delta\varphi_{km} = \operatorname{sign}(u_k - u_m)\chi_E - \fint_\Omega \operatorname{sign}(u_k - u_n)\chi_E \, dx \\[2mm] \dfrac{\partial\varphi_{km}}{\partial n} = 0 \ on \ \partial\Omega, \\[2mm] \displaystyle\int_\Omega \varphi_{km} \, dx = 0. \end{cases}$$

From Chang-Dafni-Stein's regularity in [3] (see also [4]) one has $\varphi_{km} \in W^2 \operatorname{bmo}_r(\Omega)$ and

$$\|D^\alpha \varphi_{km}\|_{h^1_z} \leq c|E|(1 - \log|E|).$$

And we conclude as in the Dirichlet case.   □

REMARK 6. The embedding assumption in Theorem 6 is satisfied for instance if $L(\rho_\alpha;\Omega) = L^1(\Omega)$.

More integrability result may be obtained if we replace $(H_{Fund})_1$ by $(H_{Fund})_2$.

THEOREM 7 (Integrability theorem in Lorentz spaces). *If $L^{p,q}(\Omega) \subset_> L(\rho_\alpha;\Omega)$, $\forall\alpha \in \mathbb{N}^N$, $|\alpha| \leq 2$, for some $1 < p < +\infty$, $1 \leq q < +\infty$, the v.w.s. solution of "$-\Delta u = T : in \ \Omega$" given in Theorem 2 satisfies*

$$u \in L^{p',\infty}(\Omega),$$

and

$$\|u\|_{L^{p',\infty}(\Omega)} \leq c(\Omega)\|T\|_{(W^2 V)^*}.$$

The proof is contained in Lemma 3, in Theorem 1 (see [18] for its proof). Besides the $G\Gamma$-spaces, one can also use variable exponent spaces.

Indeed, in a recent paper with A. Fiorenza and C. Sbordone [14] we have introduced the following Banach function space with absolutely continuous norm

$$L(\| \cdot \|_{p_*(\cdot)}; \Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable such that } f_{**} \in L^{p_*(\cdot)}(\Omega_*)\}$$

with the norm

$$\|f\|_{p_*(\cdot)} = \inf_{\lambda > 0} \lambda \left(1 + \int_0^1 \left| \frac{f_{**}(\sigma)}{\lambda} \right|^{p_*(\sigma)} d\sigma \right)$$

$p : \Omega \to]1, +\infty[$ measurable and bounded and p_* is the decreasing rearrangement

$$f_{**}(t) = t^{-1} \int_0^t |f|_*(\sigma) \, d\sigma.$$

Such spaces are suitable as $L(\rho_\alpha; \Omega)$ in Theorem 2.

4. PROOFS OF LEMMA 1, THEOREM 5 AND PROPOSITION 2

We start with the proof of Lemma 1 *Proof of Lemma* 1

Let us consider first the case $q = 1$.

Let $L \in V^*$, s.t. $\forall \varphi \in \Delta_2$, $L(\varphi) = 0$. According to Hahn-Banach Theorem, there is $\tilde{L} \in (W^2 L^{p,1}(\Omega))^* : \tilde{L}(\varphi) = L(\varphi)$, $\forall \varphi \in V$. According to Theorem 1, we have an unique function $u \in L^{p',\infty}(\Omega)$ with $\int_\Omega u(x) \, dx = 0$ satisfying

$$- \int_\Omega u \Delta \varphi \, dx = \tilde{L}(\varphi) = L(\varphi) \quad \forall \varphi \in V.$$

Therefore,

$$- \int_\Omega u \Delta \varphi \, dx = 0 \quad \forall \varphi \in \Delta_2.$$

Let $(u_\varepsilon)_\varepsilon$ be a sequence of $C_c^\infty(\Omega)$ such that

$$\int_\Omega u_\varepsilon(x) \, dx = 0, \quad u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u^k - \overline{u^k}, \quad \|u_\varepsilon\|_\infty \leq 3k$$

with

$$\overline{u^k} = \frac{1}{|\Omega|} \int_\Omega u^k(x) \, dx, \quad u^k(x) = \min(u(x); k) \, \text{sign}(u(x)).$$

Consider $\varphi_\varepsilon \in C^2(\overline{\Omega})$ such that $\begin{cases} -\Delta\varphi_\varepsilon = u_\varepsilon & \text{in } \Omega, \\ \dfrac{\partial\varphi_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$

Then $L(\varphi_\varepsilon) = 0$, since $\varphi_\varepsilon \in \Delta_2$, so that

$$\int_\Omega uu_\varepsilon \, dx = 0.$$

Letting ε goes to zero, we have

$$\int_\Omega u(u^k - \overline{u^k}) \, dx = 0 : \int_\Omega uu^k = 0.$$

But $uu^k(x) \geq 0$ so that $u(x) \cdot u^k(x) \equiv 0$ *a.e. in* $\Omega : u^2(x) = 0$ *a.e.* $u \equiv 0$ which implies

$$L(\varphi) = 0 \quad \forall \varphi \in V.$$

For the general case we observe that $W^2 L^{p,q}(\Omega) \subset_> W^2 L^{p-\varepsilon,1}(\Omega)$ for any $0 < \varepsilon < p - 1$. Since Δ_2 is dense in $\left\{ \varphi \in W^2 L^{p-\varepsilon,1}(\Omega), \dfrac{\partial\varphi}{\partial n} = 0 \right\}$ containing V from the above result, Δ_2 is dense in the smaller space V.   □

PROOF OF THEOREM 5. It is sufficient to prove the case $m = 1$. The general case is similar and the proof follows the same scheme as for the $(W^1 L^p)^*$ case.

We shall write

$$V = L(\rho_0, \Omega) \times \cdots \times L(\rho_N; \Omega), \quad D^0 v = v, \quad D^j v = \partial_j v,$$
$$W^1 V = W^1 L(\rho_0; \ldots \rho_N; \Omega).$$

1. Let J be the mapping

$$W^1 L(\rho_0; \ldots; \rho_N; \Omega) \to \prod_{j=0}^{N} L(\rho_j; \Omega)$$
$$u \mapsto (u; \partial_1 u; \ldots; \partial_N u).$$

It is an isomorphism so that its image $Im\, J$ is a closed subset.
Let us then consider the linear continuous form

$$T^* : \begin{array}{c} Im\, J \overset{J^{-1}}{\to} W^1 L(\rho_0; \ldots; \rho_N, \Omega) \overset{T}{\to} \mathbb{R} \\ w \mapsto J^{-1} w \mapsto T(J^{-1} w) \end{array}$$

According to the Hahn-Banach theorem, it can be extended to linear form on all the space $\prod_{j=0}^{N} L(\rho_j; \Omega)$ i.e. $\tilde{T}^*$ such that

$$\|\tilde{T}^*\| = \|T^*\| = \|T\|_* \quad \text{(since } J \text{ is an isometry)}.$$

Thus, $\tilde{T}^* \in \prod_{j=0}^{N} L(\rho_j; \Omega)^*$. But $L(\rho_j; \Omega)$ has absolutely continuous norm therefore $L(\rho_j; \Omega)^* = L(\rho_j'; \Omega)$ and then there exits $v_j \in L(\rho_j'; \Omega)$ such that

$$\tilde{T}^*(w_0, \ldots, w_n) = \sum_{j=0}^{N} \int_{\Omega} w_j v_j \, dx \quad \forall (w_0, \ldots, w_N) \in \prod_{j=0}^{N} L(\rho_j; \Omega).$$

In particular,

$$(6) \quad T(u) = T(J^{-1}(u; \nabla u)) = T^*(u, \nabla u) = \sum_{j=1}^{N} \int_{\Omega} \partial_j u v_j \, dx + \int_{\Omega} u v_0 \, dx$$

$$\forall u \in W^1 L(\rho_0; \ldots; \rho_N; \Omega).$$

$$|T(u)| \leq \operatorname*{Max}_{j \leq N} \|v_j\|_{L(\rho_j'; \Omega)} \left[\sum_{j=1}^{N} \|\partial_j u\|_{L(\rho_j; \Omega)} + \|u\|_{L(\rho_0; \Omega)} \right],$$

$$\|T\|_* \leq \operatorname*{Max}_{j \leq N} \|v_j\|_{L(\rho_j'; \Omega)}.$$

2. For the proof of the rearrangement invariant case, we introduce for a measurable set E the function $h(x) = \dfrac{\operatorname{sign}(g_k(x))}{\rho_k(\chi_E)} \chi_E(x)$, $x \in \Omega$, where k is such that $L(\rho_k; \Omega)$ is a rearrangement invariant space. Then

$$\rho_k(h) \leq 1 \text{ and } \int_{\Omega} h g_k(x) \, dx = \tilde{T}^*(0, \ldots 0, \underbrace{h}_{k^{th}}, 0, \ldots, 0), \quad \|\tilde{T}_*\| = \|T\|_*.$$

So that:

$$(7) \quad \int_{\Omega} h g_k(x) \, dx \leq \|T\|_* \|h\|_{L(\rho_k; \Omega)} \leq \|T\|_*$$

Taking $t \in]0, |\Omega|[$ and $|E| = t$, we derive from (7)

$$(8) \quad \varphi_{X'}(t) \frac{1}{t} \int_{E} |g_k| \, dx = \rho_k'(\chi_E) \frac{1}{|E|} \int_{E} |g_k| \, dx \leq \|T\|_*$$

Then

$$\sup_{t \le |\Omega|} [\varphi_{X'}(t) g_{k**}(t)] \le \|T\|_*.$$

Let us end with the proof of the last statement.

3. Let $k \in \{0, \dots, N\} : \|v_k\|_{L(\rho'_k; \Omega)} = \text{Max}_{j \le N} \|v_j\|_{L(\rho'_j; \Omega)}$, one has by definition

$$\|v_k\|_{L(\rho'_k; \Omega)} = \sup\left\{ \left| \int_\Omega v_k h \, dx \right| : \rho_k(h) \le 1 \right\}.$$

Therefore, there exists a sequence $(h^n)_n : \rho_k(h^n) \le 1$, s.t.

$$\lim_n \left| \int_\Omega v_k h^n \, dx \right| = \rho'_k(v_k).$$

Since $L(\rho_k; \Omega)$ is reflexive, there exist h and a subsequence such that

$$\lim_{n \to +\infty} \int_\Omega h^n g \, dx = \int_\Omega h g \, dx \quad \forall g \in L(\rho_k; \Omega)^* = L(\rho'_k; \Omega)$$

$$\Rightarrow \rho'_k(v_k) = \lim_{n \to +\infty} \left| \int_\Omega h^n v_k \, dx \right| = \left| \int_\Omega h v_k \, dx \right|.$$

Moreover,

$$\rho_k(h) \le \liminf_{n \to +\infty} \rho_k(h^n) \le 1.$$

So that the supremum is achieved at h

$$\rho'_k(v_k) = \max_{j \le N} \|v_j\|_{L(\rho'_j; \Omega)} = \|v_k\|_{L(\rho'_k; \Omega)} = \left| \int_\Omega h v_k \, dx \right|$$

$$= |\tilde{T}^*(0, \dots, h, 0, \dots)| \le \|\tilde{T}^*\| \rho_k(h) \le \|T\|_*, \quad h \text{ is the } k^{th} \text{ variable.}$$

Thus,

$$\text{Max}_{j \le N} \|v_j\|_{L(\rho'_j; \Omega)} = \|T\|_* \quad \text{(with the help of (6))}. \qquad \square$$

SKETCH OF PROOF OF PROPOSITION 2. We start with $L^{p,1}(\Omega)$, $1 < p < +\infty$. As in the proof of Theorem 5 we fix k such that:

$$\text{Max}_{j \le N} \|g_j\|_{L^{p',\infty}} = \|g_k\|_{L^{p',\infty}}.$$

Since the fundamental function for $X' = L^{p',\infty}(\Omega)$ satisfies $\varphi_{X'}(t) = t^{\frac{1}{p'}}$, applying Theorem 5, one has:

$$\|g_k\|_{L^{p',\infty}} = \sup_{t \le |\Omega|} [t^{\frac{1}{p'}} g_{k**}(t)] \le \|T\|_* \le \|g_k\|_{L^{p',\infty}(\Omega)}$$

which gives the result. $\qquad \square$

CASE OF $W^1L^{1,1}(\Omega)^*$. We choose $h(x) = \dfrac{\text{sign}(g_k(x))}{|E|\big(1 + \text{Log}\frac{|\Omega|}{|E|}\big)}\chi_E(x)$ for a measurable set E. Then

$$|h|_{**}(t) \leq \frac{1}{t}\,\frac{\min(t;|E|)}{|E|\big(1 + \text{Log}\frac{|\Omega|}{|E|}\big)}$$

$$\int_0^{|\Omega|} |h|_{**}(t)\,dt \leq 1 : \|h\|_{L^{1,1}} \leq 1$$

$$\Big(\frac{1}{|E|}\int_E |g_k|\,dx\Big)\frac{1}{1 + \text{Log}\frac{|\Omega|}{|E|}} = \int_\Omega hg_k\,dx \leq \|\tilde{T}^*\|\,\|h\|_{L^{1,1}} \leq \|T\|_*$$

$$\sup_{t \leq |\Omega|}\frac{|g_k|_{**}(t)}{1 + \text{Log}\frac{|\Omega|}{t}} \leq \|T\|_* \leq \text{Max}_{j \leq N}\|g_j\|_{L_{\exp}(\Omega)}$$

$$\text{Max}_{j \leq N}\|g_j\|_{L_{\exp}(\Omega)} = \|T\|_*.$$

CASE OF $W^1L^1(\Omega)^*$. We choose $h(x) = \dfrac{\text{sign}(g_k(x))\chi_E}{|E|}$ for a measurable set E. Then $\|h\|_{L^1} = 1$. We conclude as above. ☐

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