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Partial Differential Equation — *A note on compactness properties of the singular Toda system*, by Luca Battaglia and Gabriele Mancini, communicated on 8 May 2015.

Abstract. — In this note, we consider blow-up for solutions of the $SU(3)$ Toda system on a compact surface Σ. In particular, we give a complete proof of the compactness result stated by Jost, Lin and Wang in [11] and we extend it to the case of singularities. This is a necessary tool to find solutions through variational methods.

Key words: Toda system, compactness of solutions, blow-up analysis, mass quantization

Mathematics Subject Classification: 35B44, 35J47, 35J60

1. Introduction

Let (Σ, g) be a smooth, compact Riemannian surface. We consider the $SU(3)$ Toda system on Σ:

$$(1) \qquad -\Delta u_i = \sum_{j=1}^{2} a_{ij}\rho_j\left(\frac{V_j e^{u_j}}{\int_\Sigma V_j e^{u_j}\,dv_g} - \frac{1}{|\Sigma|}\right) - 4\pi \sum_{j=1}^{l} \alpha_{ij}\left(\delta_{p_j} - \frac{1}{|\Sigma|}\right) \quad i = 1,2$$

with $\rho_i > 0$, $0 < V_i \in C^\infty(\Sigma)$, $\alpha_{ij} > -1$, $p_j \in \Sigma$ given and

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

is the $SU(3)$ Cartan matrix.

The Toda system is widely studied in both geometry (description of holomorphic curves in $\mathbb{CP}^N$, see e.g. [4, 6, 8]) and mathematical physics (non-abelian Chern-Simons vortices theory, see [10, 18, 19]).

In the regular case, Jost, Lin and Wang [11] proved the following important mass-quantization result for sequences of solutions of (1).

Theorem 1.1. *Suppose $\alpha_{ij} = 0$ for any i, j and let $u_n = (u_{1,n}, u_{2,n})$ be a sequence of solutions of* (1) *with $\rho_i = \rho_{i,n}$. Define, for $x \in \Sigma$, $\sigma_1(x)$, $\sigma_2(x)$ as*

$$(2) \qquad \sigma_i(x) := \lim_{r\to 0} \lim_{n\to+\infty} \rho_{i,n} \frac{\int_{B_r(x)} V_i e^{u_{i,n}}\,dv_g}{\int_\Sigma V_i e^{u_{i,n}}\,dv_g}.$$

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Then,

$$(\sigma_1(x), \sigma_2(x)) \in \{(0,0), (0,4\pi), (4\pi,0), (4\pi,8\pi), (8\pi,4\pi), (8\pi,8\pi)\}. \tag{3}$$

In the same paper, the authors state that Theorem 1.1 immediately implies the following compactness result.

THEOREM 1.2. *Suppose $\alpha_{ij} = 0$ for any i, j and let K_1, K_2 be compact subsets of $\mathbb{R}^+ \setminus 4\pi\mathbb{N}$. Then, the space of solutions of* (1) *with $\rho_i \in K_i$ satisfying $\int_\Sigma u_i \, dv_g = 0$ is compact in $H^1(\Sigma)$.*

Theorem 1.2 is a necessary step to find solutions of (1) by variational methods, as was done in [2, 16, 17].

Although Theorem 1.2 has been widely used, it was not explicitly proved how it follows from Theorem 1.1. Recently, in [13], a proof was given in the case $\rho_1 < 8\pi$.

The purpose of this note is to give a complete proof of Theorem 1.2, extending it to the singular case as well. Actually, the proof follows quite directly from [7].

In the presence of singularities, that is when we allow the α_{ij} to be non-zero, it is convenient to write the system (1) in an equivalent form through the following change of variables:

$$u_i \to u_i + 4\pi \sum_{j=1}^{l} \alpha_{ij} G_{p_j} \quad \text{where } G_p \text{ solves} \begin{cases} -\Delta G_p = \delta_p - \frac{1}{|\Sigma|} \\ \int_\Sigma G_p \, dv_g = 0 \end{cases}.$$

The new u_i's solve

$$-\Delta u_i = \sum_{j=1}^{2} a_{ij}\rho_j \left(\frac{\tilde{V}_j e^{u_j}}{\int_\Sigma \tilde{V}_j e^{u_j} \, dv_g} - \frac{1}{|\Sigma|} \right) \quad i = 1, 2. \tag{4}$$

with

$$\tilde{V}_i = \Pi_{j=1}^{l} e^{-4\pi\alpha_{ij} G_{p_j}} V_i \quad \Rightarrow \quad \tilde{V}_i \sim d(\cdot, p_j)^{2\alpha_{ij}} \quad \text{near } p_j.$$

In this case, we still have an analogue of Theorem 1.1 for the newly defined u_i. The finiteness of the local blow-up values has been proved in [14].

We will also show how this quantization result implies compactness of solutions outside a closed, zero-measure set of $\mathbb{R}^{+2}$.

THEOREM 1.3. *There exist two discrete subset $\Lambda_1, \Lambda_2 \subset \mathbb{R}^+$, depending only on the α_{ij}'s, such that for any $K_i \Subset \mathbb{R}^+ \setminus \Lambda_i$, the space of solutions of* (1) *with $\rho_i \in K_i$ satisfying $\int_\Sigma u_i \, dv_g = 0$ is compact in $H^1(\Sigma)$.*

As in the regular case, Theorem 1.3 has an important application in the variational analysis of (1), see for instance [2, 1].

2. Proof of the main results

Let us consider a sequence u_n of solutions of (1) with $\rho_i = \rho_{i,n} \underset{n\to+\infty}{\to} \bar{\rho}_i$ and let us define

$$w_{i,n} := u_{i,n} - \log \int_\Sigma \tilde{V}_i e^{u_{i,n}}\, dv_g + \log \rho_{i,n}, \tag{5}$$

which solves

$$\begin{cases} -\Delta w_{i,n} = \sum_{j=1}^{2} a_{ij}\big(\tilde{V}_j e^{w_{j,n}} - \frac{\rho_{j,n}}{|\Sigma|}\big); \\ \int_\Sigma \tilde{V}_i e^{w_{i,n}}\, dv_g = \rho_{i,n} \end{cases} \tag{6}$$

moreover,

$$\sigma_i(x) = \lim_{r\to 0} \lim_{n\to+\infty} \int_{B_r(x)} \tilde{V}_i e^{w_{i,n}}\, dv_g.$$

Let us denote by S_i the blow-up set of $w_{i,n}$:

$$S_i := \{x \in \Sigma : \exists \{x_n\} \subset \Sigma, w_{i,n}(x_n) \underset{n\to+\infty}{\to} +\infty\}.$$

For $w_{i,n}$ we have a concentration-compactness result from [15, 3]:

Theorem 2.1. *Up to subsequences, one of the following alternatives holds:*

- (*Compactness*) *$w_{i,n}$ is bounded in $L^\infty(\Sigma)$ for $i = 1, 2$.*
- (*Blow-up*) *The blow-up set $S := S_1 \cup S_2$ is non-empty and finite and $\forall i \in \{1, 2\}$ either $w_{i,n}$ is bounded in $L^\infty_{loc}(\Sigma\backslash S)$ or $w_{i,n} \to -\infty$ locally uniformly in $\Sigma\backslash S$.*

  In addition, if $S_i\backslash(S_1 \cap S_2) \neq \emptyset$, then $w_{i,n} \to -\infty$ locally uniformly in $\Sigma\backslash S$.

Moreover, denoting by μ_i the weak limit of the sequence of measures $\tilde{V}_i e^{w_{i,n}}$, one has

$$\mu_i = r_i + \sum_{x\in S_i} \sigma_i(x)\delta_x$$

with $r_i \in L^1(\Sigma) \cap L^\infty_{loc}(\Sigma\backslash S_i)$ and $\sigma_i(x) \geq 2\pi \min\{1, 1 + \alpha_i(x)\}\ \forall x \in S_i,\ i = 1, 2,$ where

$$\alpha_i(x) = \begin{cases} 0 & \text{if } x \neq p_j\ j = 1, \dots, l \\ \alpha_{ij} & \text{if } x = p_j. \end{cases}$$

Here we want to show that one has $r_i \equiv 0$ for at least one $i \in \{1, 2\}$. It may actually occur that only one of the r_i's is zero, as shown in [9]. Anyway, to prove Theorems 1.2 and 1.3 we only need one between r_1 and r_2 to be identically zero.

As a first thing, we can show that the profile near blow-up points resembles a combination of Green's functions:

Lemma 2.1. *$w_{i,n} - \overline{w}_{i,n} \to \sum_{j=1}^{2}\sum_{x \in S_j} a_{ij}\sigma_j(x)G_x + s_i$ in $L^\infty_{loc}(\Sigma \backslash S)$ and weakly in $W^{1,q}(\Sigma)$ for any $q \in (1,2)$ with $e^{s_i} \in L^p(\Sigma)$ $\forall p \geq 1$.*

Proof. If $q \in (1,2)$

$$\int_\Sigma \nabla w_{i,n} \cdot \nabla \varphi \, dv_g \leq \|\Delta w_{i,n}\|_{L^1(\Sigma)} \|\varphi\|_\infty \leq C\|\varphi\|_{W^{1,q'}(\Sigma)}$$

$\forall \varphi \in W^{1,q'}(\Sigma)$ with $\int_\Sigma \varphi = 0$, hence one has $\|\nabla w_{i,n}\|_{L^q(\Sigma)} \leq C$. In particular $w_{i,n} - \overline{w}_{i,n}$ converges to a function $w_i \in W^{1,q}(\Sigma)$ weakly in $W^{1,q}(\Sigma)$ $\forall q \in (1,2)$ and, thanks to standard elliptic estimates, we get convergence in $L^\infty_{loc}(\Sigma \backslash S)$.

The limit functions w_i are distributional solutions of

$$-\Delta w_i = \sum_{j=1}^{2} a_{ij}\Big(r_j + \sum_{x \in S_j} \sigma_j(x)\delta_x - \frac{\bar{\rho}_j}{|\Sigma|}\Big).$$

In particular $s_i := w_i - \sum_{j=1}^{2}\sum_{x \in S_j} a_{ij}\sigma_j(x)G_x$ solves

$$-\Delta s_i = \sum_{j=1}^{2} a_{ij}\Big(r_j + \frac{1}{|\Sigma|}\sum_{x \in S_j} \sigma_j(x) - \frac{\bar{\rho}_j}{|\Sigma|}\Big).$$

Since $-\Delta s_i \in L^1(\Sigma)$ we can exploit Remark 2 in [5] to prove that $e^{s_i} \in L^p(\Sigma)$ $\forall p \geq 1$. □

The following Lemma shows the main difference between the case of vanishing and non-vanishing residual.

Lemma 2.2.

- *$r_i \equiv 0 \Rightarrow \overline{w}_{i,n} \to -\infty$.*
- *$r_i \not\equiv 0 \Rightarrow \overline{w}_{i,n}$ is bounded.*

Proof. First of all, $\overline{w}_{i,n}$ is bounded from above due to Jensen's inequality.

Now, take any non-empty open set $\Omega \Subset \Sigma \backslash S$.

$$\int_\Omega \tilde{V}_i e^{w_{i,n}} \, dv_g = e^{\overline{w}_{i,n}} \int_\Omega \tilde{V}_i e^{w_{i,n} - \overline{w}_{i,n}} \, dv_g$$

and by Lemma 2.1

$$\int_\Omega \tilde{V}_i e^{w_{i,n} - \overline{w}_{i,n}} \, dv_g \underset{n \to +\infty}{\longrightarrow} \int_\Omega \tilde{V}_i e^{\sum_{j=1}^{2}\sum_{x \in S_j} a_{ij}\sigma_j(x)G_x + s_i} \, dv_g \in (0, +\infty).$$

On the other hand,

$$\int_\Omega \tilde{V}_i e^{w_{i,n}}\, dv_g \underset{n\to+\infty}{\longrightarrow} \mu_i(\Omega) = \int_\Omega r_i(x)\, dv_g(x).$$

If $r_i \equiv 0$ one has $\overline{w}_{i,n} \to -\infty$. If instead $r_i \not\equiv 0$, choosing Ω such that $\int_\Omega r_i(x)\, dv_g > 0$ we must have $\overline{w}_{i,n}$ necessarily bounded. □

REMARK 2.1. From the previous two lemmas, we can write $r_i = \hat{V}_i e^{s_i}$, where

$$\hat{V}_i := \tilde{V}_i e^{\lim_{n\to+\infty} \overline{w}_{i,n}} e^{\sum_{j=1}^2 \sum_{x\in S_j} a_{ij}\sigma_j(x) G_x}$$

satisfies $\hat{V}_i \sim d(\cdot, x)^{2\alpha_i(x) - \frac{\sum_{j=1}^2 a_{ij}\sigma_j(x)}{2\pi}}$ around each $x \in S_i$, provided $r_i \not\equiv 0$.

The key Lemma is an extension of Chae-Ohtsuka-Suzuki [7] to the singular case. Basically, it gives necessary conditions on the σ_i's to have non-vanishing residual.

LEMMA 2.3. *For both $i = 1, 2$ we have $s_i \in W^{2,p}(\Sigma)$ for some $p > 1$. Moreover, if $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0))$ for some $x_0 \in S_i$, then $r_i \equiv 0$.*

PROOF. If both r_1 and r_2 are identically zero, then also s_1 and s_2 are both identically zero, so there is nothing to prove.

Suppose now $r_1 \not\equiv 0$ and $r_2 \equiv 0$. In this case,

$$\begin{cases} -\Delta s_1 = 2\left(r_1 + \frac{1}{|\Sigma|}\sum_{x_0\in S_1} \sigma_1(x_0) - \frac{\bar{\rho}_1}{|\Sigma|}\right) \\ -\Delta s_2 = -\left(r_1 + \frac{1}{|\Sigma|}\sum_{x_0\in S_1} \sigma_1(x_0) - \frac{\bar{\rho}_1}{|\Sigma|}\right) \end{cases}.$$

Then, being $G_x(y) \geq -C$ for all $x, y \in \Sigma$ with $x \neq y$, we get

$$s_1(x) = \int_\Sigma G_x(y) 2r_1(y)\, dv_g(y) \geq -2C \int_\Sigma r_1\, dv_g \geq -C'.$$

Therefore, from the previous remark, around each $x_0 \in S_1$ we get

$$r_1(y) \geq C d(x_0, y)^{2\alpha_1(x_0) - \frac{\sum_{j=1}^2 a_{1j}\sigma_j(x_0)}{2\pi}},$$

so being $r_1 \in L^1(\Sigma)$, it must be $\sum_{j=1}^2 a_{1j}\sigma_j(x_0) < 4\pi(1 + \alpha_1(x_0))$.

Moreover, being $e^{qs_1} \in L^1(\Sigma)$ for any $q \geq 1$, from Holder's inequality we get $r_1 \in L^p(\Sigma)$ for some $p > 1$; therefore, standard estimates yield $s_i \in W^{2,p}(\Sigma)$ for both $i = 1, 2$.

Consider now the case of both non-vanishing residuals, which means by Theorem 2.1 $S_1 = S_2 = S$. In this case,

$$-\Delta\left(\frac{2s_1 + s_2}{3}\right) = \left(r_1 + \frac{1}{|\Sigma|}\sum_{x_0\in S_1} \sigma_1(x_0) - \frac{\bar{\rho}_1}{|\Sigma|}\right)$$

hence, arguing as before, $\frac{2s_1+s_2}{3} \geq -C$. Therefore, using the convexity of $t \to e^t$ we get

$$\begin{aligned} C\int_\Sigma \min\{\hat{V}_1, \hat{V}_2\}\, dv_g &\leq \int_\Sigma \min\{\hat{V}_1, \hat{V}_2\} e^{\frac{2s_1+s_2}{3}}\, dv_g \\ &\leq \frac{2}{3}\int_\Sigma \hat{V}_1 e^{s_1}\, dv_g + \frac{1}{3}\int_\Sigma \hat{V}_2 e^{s_2}\, dv_g \\ &= \frac{2}{3}\int_\Sigma r_1\, dv_g + \frac{1}{3}\int_\Sigma r_2\, dv_g < +\infty. \end{aligned}$$

Therefore, for any $x_0 \in S$ there exists $i \in \{1, 2\}$ such that $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) < 4\pi(1 + \alpha_i(x_0))$. Fix x_0 and suppose, without loss of generality, that this is true for $i = 1$. This implies that $r_1 \in L^p(B_r(x_0))$ for small r, so for $x \in B_{\frac{r}{2}}(x_0)$ we have

$$\begin{aligned} s_2(x) &= \int_\Sigma G_x(y) 2r_2(y)\, dv_g(y) - \int_{B_r(x_0)} G_x(y) r_1(y)\, dv_g(y) \\ &\quad - \int_{\Sigma\setminus B_r(x_0)} G_x(y) r_1(y)\, dv_g(y) \\ &\geq -C - \sup_{z\in\Sigma} \|G_z\|_{L^{p'}(\Sigma)} \|r_1\|_{L^p(B_r(x_0))} \\ &\quad - \sup_{z\in B_{\frac{r}{2}}(x_0)} \|G_z\|_{L^\infty(\Sigma\setminus B_r(x_0))} \|r_1\|_{L^1(\Sigma)} \\ &\geq -C'. \end{aligned}$$

Therefore, arguing as before, we must have $\sum_{j=1}^2 a_{2j}\sigma_j(x_0) < 4\pi(1 + \alpha_2(x_0))$ and $r_2 \in L^p(B_{\frac{r}{2}}(x_0))$. This implies $-\Delta s_i \in L^p(B_{\frac{r}{2}}(x_0))$ for both i's. Hence, being x_0 arbitrary and $-\Delta s_i \in L^p_{loc}(\Sigma\setminus S)$, by elliptic estimates the proof is complete. □

From Lemmas 2.1 and 2.3 we can deduce, through a Pohozaev identity, the following information about the local blow-up values. This was explicitly done in [12, 14].

LEMMA 2.4. *If $x_0 \in S$ then*

$$\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) = 4\pi(1 + \alpha_1(x_0))\sigma_1(x_0) + 4\pi(1 + \alpha_2(x_0))\sigma_2(x_0).$$

LEMMA 2.5. *If $x_0 \in S_1 \cap S_2$ then there exists i such that $\sum_{j=1}^2 a_{ij}\sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0))$.*

PROOF. Suppose the statement is not true. Then, by Lemmas 2.3 and 2.4, we would have

$$(7)\qquad \begin{cases} 2\sigma_1(x_0) - \sigma_2(x_0) < 4\pi(1+\alpha_1(x_0)) \\ 2\sigma_2(x_0) - \sigma_1(x_0) < 4\pi(1+\alpha_2(x_0)) \\ \sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) \\ \qquad = 4\pi(1+\alpha_1(x_0))\sigma_1(x_0) + 4\pi(1+\alpha_2(x_0))\sigma_2(x_0) \end{cases},$$

which has no solution between positive $\sigma_1(x_0)$, $\sigma_2(x_0)$.

In fact, by multiplying the first equation by $\frac{\sigma_1(x_0)}{2}$ and the second by $\frac{\sigma_2(x_0)}{2}$ and summing, we get

$$\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) < 2\pi(1+\alpha_1(x_0))\sigma_1(x_0) + 2\pi(1+\alpha_2(x_0))\sigma_2(x_0),$$

which contradicts the third equation.

The scenario is described by the picture.

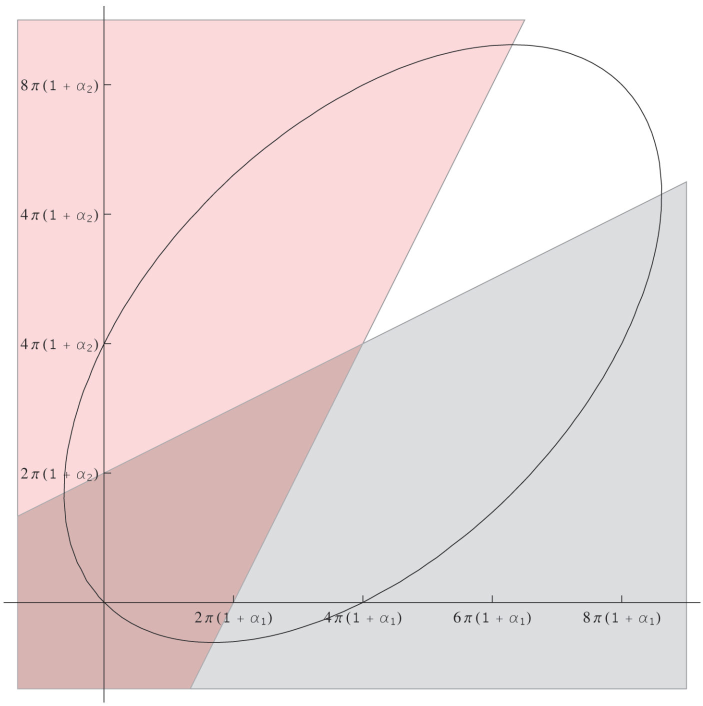

Figure 1. The algebraic conditions (7) satisfied by $\sigma_1(x_0)$, $\sigma_2(x_0)$

□

Corollary 2.1. *Let w_n be a sequence of solutions of* (6). *If $S \neq \emptyset$ then either $r_1 \equiv 0$ or $r_2 \equiv 0$. In particular there exists $i \in \{1,2\}$ such that $\bar{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$.*

PROOF OF THEOREMS 1.2 AND 1.3. Let u_n be a sequence of solutions of (1) with $\rho_i = \rho_{i,n} \underset{n\to+\infty}{\to} \bar{\rho}_i$ and $\int_\Sigma u_{1,n}\,dv_g = \int_\Sigma u_{2,n}\,dv_g = 0$ and let $w_{i,n}$ be defined by (5).

If both $w_{1,n}$ and $w_{2,n}$ are bounded from above, then by standard estimates u_n is bounded in $W^{2,p}(\Sigma)$, hence is compact in $H^1(\Sigma)$.

Otherwise, from Corollary 2.1 we must have $\bar{\rho}_i = \sum_{x\in S_i}\sigma_i(x)$ for some $i \in \{1,2\}$. In the regular case, from Theorem 1.1 follows that ρ_i must be an integer multiple of 4π, hence the proof of Theorem 1.2 is complete.

In the singular case, local blow-up values at regular points are still defined by (3), whereas for any $j = 1,\dots,l$ there exists a finite Γ_j such that $(\sigma_1(p_j),\sigma_2(p_j)) \in \Gamma_j$. Therefore, it must hold

$$\rho_i \in \Lambda_i := \left\{4\pi k + \sum_{j=1}^{l} n_j\sigma_j,\ k\in\mathbb{N},\ n_j\in\{0,1\},\ \sigma_j\in\Pi_i(\Gamma_j)\right\},$$

where Π_i is the projection on the i^{th} component; being Λ_i discrete we can also conclude the proof of Theorem 1.3. □

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REFERENCES

[1] L. BATTAGLIA. *Existence and multiplicity result for the singular Toda system*, J. Math. Anal. Appl., 424(1):49–85, 2015.

[2] L. BATTAGLIA - A. JEVNIKAR - A. MALCHIODI - D. RUIZ. *A general existence result for the Toda System on compact surfaces*, arXiv 1306.5404, 2013.

[3] L. BATTAGLIA - A. MALCHIODI. *A Moser-Trudinger inequality for the singular Toda system*, Bull. Inst. Math. Acad. Sin. (N.S.), 9(1):1–23, 2014.

[4] J. BOLTON - L. M. WOODWARD. *Some geometrical aspects of the 2-dimensional Toda equations*, In Geometry, topology and physics (Campinas, 1996), pages 69–81. de Gruyter, Berlin, 1997.

[5] H. BREZIS - F. MERLE. *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations, 16(8–9):1223–1253, 1991.

[6] E. CALABI. *Isometric imbedding of complex manifolds*, Ann. of Math. (2), 58:1–23, 1953.

[7] D. CHAE - H. OHTSUKA - T. SUZUKI. *Some existence results for solutions to $SU(3)$ Toda system*, Calc. Var. Partial Differential Equations, 24(4):403–429, 2005.

[8] S. S. CHERN - J. G. WOLFSON. *Harmonic maps of the two-sphere into a complex Grassmann manifold. II*, Ann. of Math. (2), 125(2):301–335, 1987.

[9] T. D'APRILE - A. PISTOIA - D. RUIZ. *Asymmetric blow-up for the $SU(3)$ Toda System*, arXiv 1411.3482, 2014.

[10] G. DUNNE. *Self-dual Chern-Simons Theories*, Lecture notes in physics. New series m: Monographs. Springer, 1995.

[11] J. Jost - C. Lin - G. Wang. *Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions*, Comm. Pure Appl. Math., 59(4):526–558, 2006.

[12] J. Jost - G. Wang. *Analytic aspects of the Toda system. I. A Moser-Trudinger inequality*, Comm. Pure Appl. Math., 54(11):1289–1319, 2001.

[13] C. Lin - J. Wei - W. Yang. *Degree counting and shadow system for $SU(3)$ Toda system: one bubbling*, arXiv 1408.5802, 2014.

[14] C. Lin - J. Wei - L. Zhang. *Classification of blowup limits for $SU(3)$ singular Toda systems*, arXiv 1303.4167, 2014.

[15] M. Lucia - M. Nolasco. *$SU(3)$ Chern-Simons vortex theory and Toda systems*, J. Differential Equations, 184(2):443–474, 2002.

[16] A. Malchiodi - C. B. Ndiaye. *Some existence results for the Toda system on closed surfaces*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 18(4):391–412, 2007.

[17] A. Malchiodi - D. Ruiz. *A variational analysis of the Toda system on compact surfaces*, Comm. Pure Appl. Math., 66(3):332–371, 2013.

[18] G. Tarantello. *Selfdual gauge field vortices*, Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston Inc., Boston, MA, 2008. An analytical approach.

[19] Y. Yang. *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.

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