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**Mathematical Analysis** — *On the measure of Lagrangian invariant tori in nearly-integrable mechanical systems*, by Luca Biasco and Luigi Chierchia, communicated on 11 June 2015.[1]

Abstract. — Consider an $n$-degrees-of-freedom real-analytic mechanical system with potential $\varepsilon f = \varepsilon f(x)$, $x$ being a $n$-dimensional angle variable. Then, for "general" potentials $f$'s and $\varepsilon$ small enough, the Liouville measure of the complementary of invariant tori is smaller than $\varepsilon|\ln\varepsilon|^a$ (for a suitable $a > 0$).

Key words: Nearly-integrable Hamiltonian systems, normal forms, KAM theory

Mathematics Subject Classification: 37J40

## 1. Introduction and results

The main result of last century in the theory of nearly-integrable Hamiltonian systems is that, *under suitable non-degeneracy and regularity assumptions, "most" of the regular solutions of the integrable regime, which span Lagrangian tori in phase space, persist under small perturbations*. This celebrated result is due to N. N. Kolmogorov (1954, [4]) and is the core of the so-called KAM (Kolmogorov-Arnold-Moser) theory (compare [1] and references therein). "Most", as it was later clarified in [5] (in dimension 2) and [7], [8] (in any dimension), means that the union of Lagrangian invariant tori on which the flow is conjugated to a linear flow, has relative measure of order (at least) $1 - \sqrt{\varepsilon}$, if $\varepsilon$ is the perturbation parameter.

The latter statement is sometimes rephrased by saying that the "non-torus (invariant) set" is of measure less than $\sqrt{\varepsilon}$. Indeed, it is easy to see that this result is optimal if one consider only Lagrangian tori which are graphs over the angle variables (hence, are homotopically non-trivial). In fact, just consider the simple pendulum $\frac{1}{2}y^2 + \varepsilon\cos x$ with $y \in \mathbb{R}$ and $x \in \mathbb{T}^1$: the phase region inside the separatrix (corresponding to oscillations of the pendulum) does not contain any invariant torus (circle) which is a global graph over the angle on $\mathbb{T}^1$ and this region has measure $4\sqrt{2\varepsilon}$. Of course, in this trivial integrable example, the full phase space is covered by Lagrangian invariant tori with the exception of

[1] The purpose of this paper is to announce and present results which are to appear (see reference [2] in the paper).

the stable equilibrium and the unstable (hyperbolic) equilibrium together with the separatrix (which coincides with the stable and unstable one-dimensional manifold of the hyperbolic equilibrium). The point is that invariant tori may have different topologies: in the pendulum case, the tori corresponding to librations (circles above or below the separatrix) are homoptopically non trivial, while the tori corresponding to oscillations ("secondary tori"), are contractible circles enclosed by the separatrix. The tori obtained by standard applications of KAM theory in mechanical systems are homotopically non-trivial and are deformations of the integrable tori $y = \text{const}$.

In general, when $\varepsilon \neq 0$ "secondary" (homotopically different) invariant Lagrangian tori arise near "resonances", i.e., regions of phase space where $\omega \cdot k = 0$, where $\omega$ is the unperturbed frequency, "$\cdot$" the standard inner product and $k$ an integer non vanishing vector (in the pendulum example $\omega = y = 0$ and $k = 1$). It is therefore natural to expect that the non-torus region is of measure, in general, *smaller* than $\sqrt{\varepsilon}$. On the other hand a simple argument (see Remark (ii) below) suggests that the non-torus region is, in general, of measure larger than $\varepsilon$. Indeed, Arnold, Kozlov and Neishtadt conjecture that such region has measure of order $\varepsilon$ [1, Remark 6.18].

*We can prove that Arnold, Kozlov and Neishtadt's conjecture is "essentially" true in the case of mechanical systems.*

The word "essentially" means that the result holds up to logarithmic corrections in $\varepsilon$ and for a suitable class of real-analytic potentials, which is of full measure (on a natural probability function space), contains an open dense set and is prevalent; we recall that a Borel set $P$ of a Banach space $X$ is called *prevalent* if there exists a compactly supported probability measure $v$ such that $v(x + P) = 1$ for all $x \in X$, compare, e.g., [3].

A precise statement and a sketch of proof will be given below (full proofs will appear elsewhere [2]).

**Functional setting, probability measures and Fourier-projections**

We proceed to describe the "good" set of potentials $f$, for which the result holds. Roughly speaking, such a set consists of real-analytic functions, whose Fourier-projections on one-dimensional resonance vectors have a derivative which is a Morse functions. Incidentally, let us mention that it would be easier to consider larger function spaces of smooth functions; however, the natural (both from the theoretical and applicative point of view) and most challenging setting is definitively that of real-analytic potentials.

Let $s > 0$ and consider the real-analytic functions on $\mathbb{T}^n$ having zero average and finite norm

$$|f|_s := \sup_{k \in \mathbb{Z}^n} |f_k| e^{|k|s} < \infty$$

where $f_k$ denotes Fourier coefficients and, as usual, $|k|$, for integer vectors, denotes the 1-norm $\sum |k_j|$. Denote by $\mathscr{B}_s^n$ the Banach space of such functions.

Now, let $\mathbb{Z}^n_\#$ denote the set of integer vectors $k \neq 0$ such that the first non-null component is positive, and denote $\ell^n_\infty$ the Banach space of complex sequences $z = \{z_k\}_{k\in\mathbb{Z}^n_\#}$ with finite sup-norm $|z|_\infty := \sup_{k\in\mathbb{Z}^n_\#}|z_k|$. The map

$$j : f \in \mathscr{B}^n_s \to \{f_k e^{|k|s}\}_{k\in\mathbb{Z}^n_\#} \in \ell^n_\infty \tag{1}$$

is an isomorphism of Banach spaces, which allows to identify functions in $\mathscr{B}^n_s$ with points in $\ell^n_\infty$ and the Borellians of $\mathscr{B}^n_s$ with those of $\ell^n_\infty$; (recall that since the functions in $\mathscr{B}^n_s$ are *real*-analytic one has the reality condition $f_k = \bar{f}_{-k}$). Now, consider the standard normalized Lebesgue-product measure on the unit closed ball of $\ell^n_\infty$, namely, the unique probability measure $\mu$ on the Borellians of $\{z \in \ell^n_\infty : |z|_\infty \leq 1\}$ such that, given Lebesgue measurable sets in the unit complex disk $E_k \subseteq D := \{w \in \mathbb{C} : |w| \leq 1\}$ with $E_k \neq D$ only for finitely many $k$, one has

$$\mu\Big(\prod_{k\in\mathbb{Z}^n_\#} E_k\Big) = \prod_{\{k\in\mathbb{Z}^n_\# : E_k \neq D\}} \mathrm{meas}(E_k)$$

where "meas" denotes the normalized Lebesgue measure on the unit complex disk $D$. *The isometry $j$ in* (1) *naturally induces a measure $\mu_s$ on the unit ball of $\mathscr{B}^n_s$.*

Now, denote by $\mathbb{Z}^n_*$ the set of vectors $k \in \mathbb{Z}^n_\#$ such that the greater common divisors of their components is 1. Then, any function $f \in \mathscr{B}^n_s$ can be uniquely decomposed, in Fourier space, as sum of real-analytic functions of one-variable, which are the projection of $f$ onto the one-dimensional Fourier modulus $\{jk : j \in \mathbb{Z}\}$, as follows:

$$f(x) = \sum_{k\in\mathbb{Z}^n_*} F_k(k, x), \quad \text{where } F_k(\xi) := \sum_{j\in\mathbb{Z}\setminus\{0\}} f_{jk} e^{ij\xi} \tag{2}$$

$f_{jk}$ being the Fourier coefficient of $f$ with Fourier index $jk \in \mathbb{Z}^n$; beware of notation: $\xi \to F_k(\xi)$ is a periodic function of one variable, whose Fourier coefficients, for $j \in \mathbb{Z}$, are given by $(F_k)_j = f_{jk}$.

Notice that, since $f \in \mathscr{B}^n_s$, the functions $F_k$ belong to $\mathscr{B}^1_{|k|s}$.

**The class $\mathscr{P}_s$ of "good" potentials**

We first define an auxiliary function $K_s(\delta) := \frac{2}{s}\ln\frac{c}{\delta}$ (where $c > 1$ depends only on $n$).

*Let $\mathscr{P}_s$ be the set of functions in the closed unit ball of $\mathscr{B}^n_s$ satisfying the following three properties. There exists $\delta > 0$ such that, for $k \in \mathbb{Z}^n_*$,*

(P1) $|f_k| \geq \delta|k|^{-\frac{n+3}{2}} e^{-|k|s}$, $\forall |k| > K_s(\delta)$;
(P2) $\min_{\xi\in\mathbb{R}}(|F'_k(\xi)| + |F''_k(\xi)|) > 0$, $\forall |k| \leq K_s(\delta)$;
(P3) $3F''_k(\bar{\xi})F''''_k(\bar{\xi}) \neq 5(F'''_k(\bar{\xi}))^2$, *for $\bar{\xi}$ minimum of $F_k$ and* $\forall |k| \leq K_s(\delta)$.

Consider a real-analytic mechanical system with Hamiltonian $H := \frac{1}{2}|y|^2 + \varepsilon f(x)$ defined on the phase space $B \times \mathbb{T}^n$, $B$ bounded domain in $\mathbb{R}^n$, endowed with standard symplectic form $dy \wedge dx$.

THEOREM

(a) *Let $f \in \mathscr{P}_s$ for some $s > 0$. Then, there exist $\varepsilon_0 > 0$ and $a > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, the measure of the set of $H$-trajectories in $B \times \mathbb{T}^n$, which do not lie on an invariant Lagrangian torus, is bounded by $\varepsilon |\ln \varepsilon|^a$.*
(b) *The set $\mathscr{P}_s$ has full $\mu_s$-measure, contains an open dense set (in the unit ball of $\mathscr{B}_s^n$) and is prevalent.*

**Comments and remarks**

(i) We stress that the set $\mathscr{P}_s$ is "large" in many ways: it is of full measure with respect to a quite natural product probability measure on a weighted Fourier space; it is generic in the topological sense (Baire), and actually is more than that, since it contains an open dense set; finally, it is prevalent, which is a measure-theoretic notion for subsets of infinite-dimensional spaces that is analogous to "full Lebesgue measure" in Euclidean spaces (compare [3]).

(ii) Let us give a simple heuristic argument suggesting that the measure of the non-torus set is, in general, at least $O(\varepsilon)$. Let $n = 2$, let $p = \sqrt{\varepsilon} y$, $q = x$ and divide the Hamiltonian $H$ by $\varepsilon$: we get a new Hamiltonian, $\tilde{H} = |p|^2/2 + f(q)$, which is parameter free; notice that the trajectories of $H$ and $\tilde{H}$ are in 1-1 correspondence (but with different times). In general, one expects an $O(1)$ torus-free region in the $(p, q)$ variables: this corresponds to a torus-free region of measure $O(\varepsilon)$ in the original variables. When $n \geq 2$ the same argument applies in neighborhoods of double resonances and lead to the same conclusion.

(iii) As far as we know, the only other general result about the measure of secondary tori nearby simple resonances is discussed by Medvedev, Neishtadt and Treschev in [6]. They proved that there is a set $\mathscr{D}_0$ inside the separatrix "eye" $\mathscr{D}$ arising near a simple resonance with $\text{meas}(\mathscr{D}_0) \sim \sqrt{\varepsilon}$ ($\sim \text{meas}(\mathscr{D})$), such that the non-torus set in $\mathscr{D}_0$ is of measure $O(\varepsilon)$. This, of course, does not imply that the non-torus set in bounded regions is $O(\varepsilon)$, which is what is stated in the Theorem above.

(iv) We are interested here in global statements, which apply to any region in phase space. On the other hand, it is well known that the density of invariant tori is non uniform in phase space and there are regions where such density is of order $1 - e^{-1/\varepsilon^c}$ for some $c > 0$; compare the Remark at p. 6.

(v) Even though the proof of the Theorem—briefly sketched below—exploits the particular structure of the Hamiltonian $H$ in mechanical systems, we believe that extending the proof to more general settings such as to Hamiltonians of the form $H = h(y) + \varepsilon f(y, x)$ with $h$ quasi-convex is an interesting but essentially technical matter.

A more challenging question would be to determine the most general class of unperturbed Hamiltonians $h$ for which the result holds.

(vi) Two naive questions: Can one take out the logarithms? Is the result true for *all* potentials?

(vii) We close this section with a technical comment on the definition of $\mathscr{P}_s$.

Condition (P1) insures that, for $|k| > K_s(\delta)$, the first Fourier coefficient in the expansion of $F_k$, namely $(F_k)_1 = f_k$, is large with respect to the other ones, since $|(F_k)_j| = |f_{jk}| \le e^{-|j|\,|k|s}$. Roughly speaking, this means that $F_k$ "behaves like a cosine".

(P2) says that $F_k'$ is a "Morse function" uniformly for $|k| \le K_s(\delta)$. In particular, all critical points of $F_k$ are non degenerate minima or maxima and the number of critical points of $F_k$ is bounded by a constant times $\beta^{-1}$, where $\beta$ is the strictly positive number in the left hand side of (P2).

(P3) is needed to prove a non degeneracy condition for the action-angle variables of one-dimensional systems of the type $\eta^2/2 + F_k(\xi)$.

## 2. Outline of the proof of the theorem

We begin by discussing *part* (a), whose proof may be divided in five steps.

*Step 1. Geometry of resonances*

Define

$$\mathbb{Z}^n_\varepsilon := \{k \in \mathbb{Z}^n_* \text{ s.t. } 0 < |k| \le |\ln \varepsilon|^2\}$$

and, for $k \in \mathbb{Z}^n_\varepsilon$,

$$R_k := \{y \in B : |y \cdot k| \le \sqrt{\varepsilon}|\ln \varepsilon|^{c_n}\},$$

with $c_n := 2n + 6$.

Decompose the phase space $\mathscr{M} = B \times \mathbb{T}^n$ as $B = B_0 \sqcup B_1 \sqcup B_2$ ("$\sqcup$" denoting disjoint union), where:

$$B_0 := B \setminus \bigcup_{k \in \mathbb{Z}^n_\varepsilon} R_k, \quad B_2 := \bigcup_{k \neq k' \in \mathbb{Z}^n_\varepsilon} R_k \cap R_{k'}, \quad B_1 := \bigcup_{k \in \mathbb{Z}^n_\varepsilon} R_k \setminus B_2.$$

Roughly speaking (and for $|k| \le |\ln \varepsilon|^2$), $B_0$ is the non-resonant set, $B_1$ contains simple resonances and $B_2$ contains double (or higher) resonances.

On the non-resonant set $B_0$, by an averaging procedure, we can remove the perturbation at order $\varepsilon^{|\ln \varepsilon|}$. Then, applying a standard KAM theorem we obtain that the measure of the non-torus set in $B_0$ is $O(\varepsilon^{|\ln \varepsilon|/2}) \ll \varepsilon$.

The set $B_2$ can be disregarded since it is of measure $O(\varepsilon|\ln \varepsilon|^{2c_n+2})$.

The main problem comes from the neighborhood $B_1$ of simple resonances (a set whose measure is not negligible since $\text{meas}(B_1) \ge \sqrt{\varepsilon} \gg \varepsilon$).

Remark. The geometry of resonances here is different from the geometry of resonances (in the convex case) as discussed, e.g., in [9]. In fact, in [9] more resonances are disregarded in the non-resonant set, namely, the resonances with $|k| \le (1/\varepsilon)^a$, $a > 0$; moreover the neighborhood of simple resonances has width $\varepsilon^b$, $0 < b < 1/2$, which is larger than $R_k$. As a consequence, the set of double resonances has measure greater than $\varepsilon^{2b}$, which is a set not negligible for our

purposes. On the other hand in Nekhoroshev's theorem one can average out the perturbation up to an exponentially high order $e^{-\text{const}(1/\varepsilon)^a}$, while we get only $\varepsilon^{|\ln\varepsilon|}$ (see (3) below).

*Step 2. Averaging over a simple resonance*

Fix $R_k$ with $k \in \mathbb{Z}^n_\varepsilon$. After performing

– a linear change of variables putting the resonance in $\{y_n = 0\}$,
– an averaging over the "fast angles" $x_1, \dots, x_{n-1}$,
– a $1/\sqrt{\varepsilon}$-blow up of $y$ and an $\varepsilon$-time rescaling,

the Hamiltonian takes the form:

$$h(\hat{y}) + \frac{1}{2}y_n^2 + \tilde{F}_k(y, \|k\|^2 x_n) + \varepsilon^{|\ln\varepsilon|} g_k(y, x), \quad \hat{y} := (y_1, \dots, y_{n-1}), \tag{3}$$

where

$$\tilde{F}_k(y, \|k\|^2 x_n) = F_k(y_n, \|k\|^2 x_n) + O(|\ln\varepsilon|^{-c}), \tag{4}$$

$F_k$ being defined in (2) and satisfying, for some $\delta > 0$, (P1)–(P3), (and $\|k\|^2 := \sum_{j=1}^n k_j^2$). Notice that the Hessian of $h$ is non degenerate since $h$ is convex. Note also that the unperturbed Hamiltonian in (3), i.e., when $\varepsilon = 0$, is $2\pi/\|k\|^2$-periodic in $x_n$, while the complete Hamiltonian is only $2\pi$-periodic in $x_n$, a well known fact due to the presence of a $k$-resonance.

To simplify the exposition, in view of (4), henceforth we replace in (3) $\tilde{F}_k$ with $F_k$.

*Step 3. Action-angle variables*

We want to prove that

(∗) the measure of the non-torus set in a $O(|\ln\varepsilon|^{c_n})$-neighborhood of $\{y_n = 0\}$ for the Hamiltonian in (3) is $O(\sqrt{\varepsilon}|\ln\varepsilon|^c)$,

which, in view of the $1/\sqrt{\varepsilon}$-blow up of the action variables (Step 2), corresponds, in the original variables, to a measure of $O(\varepsilon|\ln\varepsilon|^c)$.

The idea is to use action-angle variables $(p_n, q_n) \mapsto (y_n, x_n)$ to integrate the one degree-of-freedom Hamiltonian

$$E_k(y_n, x_n) := \frac{1}{2}y_n^2 + F_k(y_n, \|k\|^2 x_n) = E_k(p_n).$$

Thus, in the new variables (completed with $\hat{p} := \hat{y}$, $\hat{q} := \hat{x}$) the full Hamiltonian (3) becomes

$$h(\hat{p}) + E_k(p_n) + \varepsilon^{|\ln\varepsilon|}\tilde{g}_k(p, q), \tag{5}$$

a form suitable (in principle) for applying a standard KAM theorem. However, there are two (quantitative) problems to overcome:

(S) the transformation putting $E_k$ in action-angle variables becomes singular as separatrices or elliptic equilibria are approached;
(K) Kolmogorov's non-degeneracy condition (namely that the Hessian of $h(\hat{p}) + E_k(p_n)$ is invertible) needs to be checked.

These two problems will be overcome in the next two steps.

*Step 4. On the singularities of action-angle variables*

This analysis will be based on the following lemma:

*Assume that in the one-dimensional system* $E(\eta, \xi) := \eta^2/2 + F(\xi)$ *the derivative of the potential* $F(\xi)$ *is a Morse function with* $\beta := \min_{\xi \in \mathbb{R}}(|F_k'(\xi)| + |F_k''(\xi)|) > 0$. *Then, there exists* $c > 0$, *depending only on* $\beta$, *such that for any "critical" energy* $E_0$ *the measure of the points* $(\eta, \xi)$ *for which* $E(\eta, \xi)$ *is* $\theta$*-close to* $E_0$ *is bounded by* $c\theta|\ln\theta|$; "critical" meaning that $E_0 := F(\xi_0)$ with $F'(\xi_0) = 0$ (so that the level set $E = E_0$ contains equilibria and/or separatrices).

We will discuss, here, only modes $k$ for which $|k| \le K_s(\delta)$, since the case $|k| > K_s(\delta)$ is easier (recall that assumption (P1) implies that $F_k$ "behaves like a cosine").

Now, we can use the above result thanks to (P2) with

$$\theta = \varepsilon^{a|\ln\varepsilon|}, \quad 0 < a < a_0 \tag{6}$$

for a suitable $a_0 < 1$. Thus, up to a small set of measure $\theta|\ln\theta|$, we can use "non-singular" action-angle variables $(p_n, q_n)$. In particular, the $p_n$-radius of analyticity, the width of the complex $q_n$-strip and the $C^2$-norm of $E_k$ are of order, respectively, $\theta$, $1/|\ln\theta|$ and $1/\theta$.

*Step 5. On Kolmogorov's condition*

Since $h$ is (strictly) convex, Kolmogorov's non-degeneracy condition amounts to say that $|E_k''(p_n)|$ is bounded away from zero and up to a small measure set of $p_n$.

We also notice that for "high energies", namely for energies larger than then the maximal critical energy, $E_k(p_n)$ is strictly convex and the Kolmogorov's condition is easily satisfied. Therefore, we need only to discuss the case inside separatrices where $E_k''$ may become negative or null.

As above, by (P1), the modes $|k| > K_s(\delta)$ can be easily handled directly.

The case $|k| \le K_s(\delta)$ is more difficult: although it involves only a finite number of $k$'s, the structure of the Hamiltonian $E_k(p_n)$ is rather "arbitrary" and it is not at all obvious how to handle it in a direct way. To overcome this problem we will check the non degeneracy condition in an *indirect way*, using the *analyticity* of the function $E_k$.

$F_k$ has a finite number of critical points, which are non-degenerate local maxima or minima (recall the comment (vii) on (P2)). The critical energy levels determine a finite number of open connected components, where one can define

analytic action-angle variables. Recall that we are discussing the *bounded* components and let us fix one of such components. Let the action variable $p_n$, correspondingly, be defined on an interval $(a, b)$ on which $E_k(p_n)$ will be a real-analytic strictly increasing function. By construction $e_a := \lim_{p_n \to a^+} E_k(p_n)$ and $e_b := \lim_{p_n \to b^-} E_k(p_n)$ correspond to critical values of $F_k$. It is simple to see that $e_b$ corresponds to a (local) maximum value of $F_k$, while $e_a$ corresponds to a (local) maximum or minimum value of $F_k$. It is also simple to see that $\lim_{p_n \to b^-} |E_k''(p_n)| = \infty$ (and the same holds for $\lim_{p_n \to a^+} |E_k''(p_n)|$ if also $e_a$ corresponds to a maximum). Moreover, when $e_a$ corresponds to a minimum, we can prove that $\lim_{p_n \to a^+} E_k''(p_n) \neq 0$ using assumption (P3). In any case we have $\lim_{p_n \to a^+} E_k''(p_n) \neq 0$, $\lim_{p_n \to b^-} E_k''(p_n) \neq 0$. Now, as it is not difficult to prove

*if $f$ be a non identically zero (real) analytic function in an open interval containing the closed interval $[x_1, x_2]$, then there exists a constant $c \geq 1$ such that for every $0 < \theta < 1/c$*

$$\text{meas}\{x \in [x_1, x_2] : |f(x)| \leq \theta\} \leq c\theta^{1/c}.$$

Thanks to such basic property of level sets of analytic functions, we may conclude that there exists $c > 1$ such that, up to a set of $p_n$'s of measure $\theta$ (as in (6)), the estimate $|E_k''| \geq \theta^c$ holds. Since the size of the perturbation in (5) is $O(\varepsilon^{|\ln \varepsilon|})$ this is enough to apply a KAM theorem (recall the choice of parameters given at the end of Step 4 and choose $a$ small enough in (6)). This concludes the sketch of the proof of part (a).

Let us now outline the proof of part (b).

To check that $\mu_s(\mathscr{P}_s) = 1$ we shall prove that, for every $\delta > 0$, the measure of the sets of potentials $f$ that do not satisfy, respectively, (P1), (P2), (P3) is, respectively, $O(\delta^2), 0, 0$; the result will follow letting $\delta \to 0$.

First, by the identification (1), the measure of the set of potentials $f$ that do not satisfy (P1) with a given $\delta$ is bounded by $\sum_{k \in \mathbb{Z}^n} \delta^2 |k|^{-n-3} \leq c_n \delta^2$.

Next, recall that properties (P2) and (P3) concern only a *finite* number of $k$, i.e., $k \in \mathbb{Z}^n_*$, $|k| \leq K_s(\delta)$.

To show that the set of potentials that do not satisfy (P2) has $\mu_s$-measure zero it is enough to check that, for every $k \in \mathbb{Z}^n_*$, $|k| \leq K_s(\delta)$, the set $\mathscr{E}^{(k)}$ of $f$'s for which $F_k$ has a degenerate critical point has zero $\mu_s$-measure; (recall the definition of $F_k$ in (2)).

Fix $k \in \mathbb{Z}^n_*$, $|k| \leq K_s(\delta)$ and denote points in $\mathscr{E}^{(k)}$ by $(\zeta, \varphi)$, where $\zeta = f_k$ and $\varphi = \{f_h\}_{h \neq k}$. Write

$$F_k(\xi) = \zeta e^{\mathrm{i}\xi} + \bar{\zeta} e^{-\mathrm{i}\xi} + G(\xi), \quad \text{where } \zeta := f_k \text{ and } G(\xi) := \sum_{|j| \geq 2} f_{jk} e^{\mathrm{i}j\xi}.$$

Now, one checks immediately that $F_k'(\xi_0) = 0 = F_k''(\xi_0)$ is equivalent to $\zeta = \zeta(\xi_0; \varphi) = \frac{1}{2} e^{-\mathrm{i}\xi_0}(\mathrm{i}G'(\xi_0) + G''(\xi_0))$, which, as $\xi_0$ varies in $\mathbb{T}$, describes a smooth closed "critical" curve in $\mathbb{C}$; as a side remark, notice that $\zeta$ depends on $\varphi$ only through the Fourier coefficients $f_{jk}$ with $|j| \geq 2$. Thus the section $\mathscr{E}^{(k)}_\varphi = \{\zeta \in D : (\zeta, \varphi) \in \mathscr{E}^{(k)}\}$ is (a piece of) a smooth curve in $D = \{z \in \mathbb{C} : |z| \leq 1\}$;

hence $\mathrm{meas}(\mathscr{E}_\varphi^{(k)}) = 0$ for every $\varphi$ and by Fubini's theorem $\mu_s(\mathscr{E}^{(k)}) = 0$, as claimed.

An analogous result holds true for (P3); in this case the critical curve is given by $\{\zeta = (-b(\xi) \pm \sqrt{b^2(\xi) - c(\xi)} + \mathrm{i}G'(\xi))e^{-\mathrm{i}\xi}/2, \xi \in \mathbb{R}, b^2(\xi) \geq c(\xi)\}$, where $b(\xi) := (G''''(\xi) - G''(\xi))/2$ and $c(\xi) := -G''(\xi)G''''(\xi) + 5(G'(\xi) + G'''(\xi))^2/3$.

We now show that $\mathscr{P}_s$ contains an open subset $\mathscr{P}_s'$ which is dense in the unit ball of $\mathscr{B}_s^n$.

Let us define $\mathscr{P}_s'$ as $\mathscr{P}_s$ but with the difference that (P1) is replaced by the *stronger condition*

(P1′) $\exists \delta > 0$ s.t. $|f_k| \geq \delta e^{-|k|s}$, $\forall k \in \mathbb{Z}_*^n$, $|k| > K_s(\delta)$; note that $\mu_s(\mathscr{P}_s') = 0$.

Let us first prove that $\mathscr{P}_s'$ is open. Let $f \in \mathscr{P}_s'$. We have to show that there exists $\rho > 0$ such that if $|g|_s < \rho$, then $f + g \in \mathscr{P}_s'$. Fix $\delta > 0$ such that (P1′) holds and choose $\rho < \delta$ small enough such that $[K_s(\delta)] > K_s(\delta') - 1$, where $\delta' := \delta - \rho$ and $[\cdot]$ denotes integer part. Then, it is immediate to verify that $|k| > K_s(\delta) \Leftrightarrow |k| > K_s(\delta')$. Moreover

$$|f_k + g_k|e^{|k|s} \geq |f_k|e^{|k|s} - |g|_s \geq \delta - \rho = \delta', \quad \forall k \in \mathbb{Z}_*^n, |k| > K_s(\delta'),$$

namely $f + g$ satisfies (P1′) (with $\delta'$ instead of $\delta$). Since (P2) and (P3) are "open" conditions and regard only a finite number of $k$ it is simple to see that they are satisfied also by $f + g$ for $\rho$ small enough. Then $f + g \in \mathscr{P}_s'$ for $\rho$ small enough.

Let us now show that $\mathscr{P}_s'$ is dense in the unit ball of $\mathscr{B}_s^n$. Take $f$ in the unit ball of $\mathscr{B}_s^n$ and $0 < \theta < 1$. We have to find $\tilde{f} \in \mathscr{P}_s'$ with $|\tilde{f} - f|_s \leq \theta$. Let $\delta := \theta/4$ and denote by $f_k$ and $\tilde{f}_k$ (to be defined) be the Fourier coefficients of, respectively, $f$ and $\tilde{f}$. We, then, let $\tilde{f}_k = f_k$ unless one of the following two cases occurs:

- $k \in \mathbb{Z}_*^n$, $|k| > K_s(\delta)$ and $|f_k|e^{|k|s} < \delta$, in which case, $\tilde{f}_k = \delta e^{-|k|s}$;
- $k \in \mathbb{Z}_*^n$, $|k| \leq K_s(\delta)$ and $F_k$ (defined as in (2)) does not satisfy either (P2) or (P3), in which case, $\tilde{f}_k$ is chosen at a distance less than $\theta e^{-|k|s}$ from $f_k$ but outside the critical curves defined above.

At this point, it is easy to check that $\tilde{f} \in \mathscr{P}_s'$ and is $\theta$-close to $f$.

We finally prove that $\mathscr{P}_s$ is prevalent. Consider the following compact subset of $\ell_\infty^n$: let $\mathscr{K} := \{z = \{z_k\}_{k \in \mathbb{Z}_\#^n} : z_k \in D_{1/|k|}\}$, where $D_{1/|k|} := \{w \in \mathbb{C} : |w| \leq 1/|k|\}$, and let $\nu$ be the unique probability measure supported on $\mathscr{K}$ such that, given Lebesgue measurable sets $E_k \subseteq D_{1/|k|}$, with $E_k \neq D_{1/|k|}$ only for finitely many $k$, one has

$$\nu\Big(\prod_{k \in \mathbb{Z}_\#^n} E_k\Big) := \prod_{\{k \in \mathbb{Z}_\#^n : E_k \neq D_{1/|k|}\}} \frac{|k|^2}{\pi} \mathrm{meas}(E_k).$$

The isometry $j_s$ in (1) naturally induces a probability measure $\nu_s$ on $\mathscr{B}_s^n$ with support in the compact set $\mathscr{K}_s := j_s^{-1}\mathscr{K}$. Now, for $\delta > 0$, let $\mathscr{P}_{s,\delta}$ be the set of $f$'s in the unit ball of $\mathscr{B}_s^n$ satisfying (P1)–(P3), so that $\mathscr{P}_s = \bigcup_{\delta > 0} \mathscr{P}_{s,\delta}$. Reasoning as in the proof of $\mu_s(\mathscr{P}_s) = 1$, one can show that $\nu_s(\mathscr{P}_{s,\delta}) \geq 1 - \mathrm{const}\,\delta^2$. It is

also easy to check that, for every $g \in \mathscr{B}_s^n$, the translated set $\mathscr{P}_{s,\delta} + g$ satisfies $\nu_s(\mathscr{P}_{s,\delta} + g) \geq \nu_s(\mathscr{P}_{s,\delta})$. Thus, one gets $\nu_s(\mathscr{P}_s + g) = \nu_s(\mathscr{P}_s) = 1$, $\forall g \in \mathscr{B}_s^n$, which means that $\mathscr{P}_s$ is prevalent.

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