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Partial Differential Equations — *L^p theory for fractional gradient PDE with VMO coefficients*, by ARMIN SCHIKORRA[1], TIEN-TSAN SHIEH[2] and DANIEL SPECTOR[3], communicated on 11 June 2015.[4]

ABSTRACT. — In this paper, we prove L^p estimates for the fractional derivatives of solutions to elliptic fractional partial differential equations whose coefficients are *VMO*. In particular, our work extends the optimal regularity known in the second order elliptic setting to a spectrum of fractional order elliptic equations.

KEY WORDS: Fractional elliptic PDE, VMO coefficients, commutators

MATHEMATICS SUBJECT CLASSIFICATION: 35R11

1. INTRODUCTION

In his 1959 paper on some composition formulas for vector-valued potentials, J. Horváth introduced [9, p. 434] the differential object

$$D^s u := D I_{1-s} u. \tag{1.1}$$

Here, $s \in (0,1)$ and I_{1-s} is the Riesz potential of order $1-s$.

This object was subsequently termed the *Riesz fractional gradient* by the second and third author in [15], where it was utilized to generalize divergence form elliptic partial differential equations from the second order setting to that of differential order $2s \in (0,2)$. In particular, assuming that A is uniformly elliptic, i.e.

$$\lambda|\xi|^2 \le A(x)\xi \cdot \xi \le \Lambda|\xi|^2, \tag{1.2}$$

for all $x, \xi \in \mathbb{R}^N$ and some $0 < \lambda \le \Lambda < +\infty$, the authors showed that given $\varphi \in H^s(\mathbb{R}^N)$ and $g \in L^2(\Omega)$ there exists $u \in H^s(\mathbb{R}^N)$ that satisfies

$$\int_{\mathbb{R}^N} A(x) D^s u(x) \cdot D^s v(x)\, dx = \int_{\mathbb{R}^N} gv \tag{1.3}$$

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for all $v \in C_c^\infty(\mathbb{R}^N)$ and $u = \varphi$ in $\mathbb{R}^N \backslash \Omega$. Here, $\Omega \subset \mathbb{R}^N$ is open and bounded, $N \geq 2$, and

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : D^s u \in L^2(\mathbb{R}^N; \mathbb{R}^N)\},$$

which coincides with any standard definition of the fractional Sobolev space (see, for example, [7, p. 524, 532]).

One observes that when $s = 1$ and the boundary of Ω is sufficiently nice, the equation (1.3) agrees with the weak formulation of a divergence form elliptic PDE, since prescribing u on the complement gives rise to a trace that would be a more standard way to frame the existence. Meanwhile for $s \in (0, 1)$ one obtains a family of fractional partial differential equations with analogous structure. The interest in generalizing partial differential equations via (1.1) is two-fold. Firstly, that one should be concerned with non-integer order differential objects can be simply explained by quoting Sobolev and Nikol'skiĭ's 1963 paper (who even implicitly consider (1.1), see [14, p. 148]) where they note that "an imbedding theory containing only derivatives of integral order is incomplete and imperfect." Secondly, the structure of (1.1) closely resembles the gradient and therefore such a generalization preserves the structural properties of the equation, a point which we will return to later. This aspect has been important in the development of L^1 fractional Sobolev inequalities in terms of (1.1) in [13], as such inequalities are known to be false for the fractional Laplacian.

In this paper we continue to develop this perspective of classical equations as a part of a continuous spectrum. In particular, we take the first step in addressing for this class of equations a question of fundamental importance in the second order case, that of regularity. As there are a number of possible assumptions one can make to investigate the question of regularity of u that satisfies (1.3), let us further describe the hypothesis of interest to us. In addition to the ellipticity condition (1.2), we will assume A is of vanishing mean oscillation.

Definition 1.1. We define the semi-norm (on the space of functions of bounded mean oscillation)

$$[\varphi]_{BMO} := \sup_Q \int_Q \left| \varphi - \fint_Q \varphi \right|,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^N$. Then we define the space of functions of *vanishing mean oscillation* by

$$VMO(\mathbb{R}^N) := \overline{\{C_c^\infty(\mathbb{R}^N)\}}^{[\cdot]_{BMO}}.$$

The main result of this paper is the following theorem on the regularity of such equations with VMO coefficients.

Theorem 1.2. *Suppose that* $A \in VMO(\mathbb{R}^N; \mathbb{R}^{N \times N})$ *satisfies* (1.2), *that* $G \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ *for some* $1 < p < +\infty$ *and* $u \in H^s(\mathbb{R}^N)$ *satisfies*

$$\int_{\mathbb{R}^N} A(x) D^s u(x) \cdot D^s v(x)\, dx = \int_{\mathbb{R}^N} G \cdot D^s v \tag{1.4}$$

for all $v \in C_c^\infty(\Omega)$. *Then* $D^s u \in L^p_{loc}(\Omega)$ *and for any* $K \subset\subset \Omega$ *there exists a constant* $C = C(K, \Omega, A, s, p) > 0$ *such that*

$$\|D^s u\|_{L^p(K;\mathbb{R}^N)} \leq C(\|G\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}).$$

Here, $(-\Delta)^{\frac{s}{2}} u$ denotes the fractional Laplacian of u of order s, which can be defined as a Fourier multiplier with symbol $(2\pi|\xi|)^s$, see [16, p. 117] or [7, p. 528, 530]. The fractional Laplacian is related to the fractional gradient via the identity

$$D^s u \equiv R(-\Delta)^{\frac{s}{2}} u, \tag{1.5}$$

for $s \in (0, 1)$ and u with sufficient smoothness and integrability, and where $R = DI_1$ is the vector-valued Riesz transform. In what follows we take (1.5) as our definition of $D^s u$, which enables us to include the classical case $s = 1$ (and more generally $s > 1$ though one loses the interpretation of a fractional gradient in this range).

Our proof is based on the beautiful technique of Iwaniec and Sbordone, introduced in [10] for u satisfying (1.4) with $v \in C_c^\infty(\mathbb{R}^N)$ and $s = 1$. We recall that in this setting they had shown [10, p. 186] that (1.4) has exactly one (up to a constant) solution with the estimate

$$\|Du\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \leq C\|G\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

Comparing this with our result, one sees that the preservation of structure in the equation results in regularity that is completely analogous to the well-studied elliptic theory.

As a consequence of this result we can return to the question of regularity of solutions to (1.3). In particular, one can transform equation (1.3) into (1.4) by defining $G = I_s Rg$ (where one extends g by zero outside Ω), since one has

$$\begin{aligned}\int_{\mathbb{R}^N} gv\,dx &= \int_{\mathbb{R}^N} I_s Rg \cdot R(-\Delta)^{\frac{s}{2}} v\,dx \\ &= \int_{\mathbb{R}^N} G \cdot D^s v\,dx\end{aligned}$$

for $v \in C_c^\infty(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$. The assumption $g \in L^2(\Omega)$ then implies that $G \in L^{2N/(N-2s)}(\mathbb{R}^N;\mathbb{R}^N)$, and so our result allows us to conclude that for the solution to (1.3) we have for every $K \subset\subset \Omega$ the estimate

$$\|D^s u\|_{L^{2N/(N-2s)}(K;\mathbb{R}^N)} \leq C(\|g\|_{L^2(\Omega)} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}).$$

When $s = 1$ this localizes the result of Iwaniec and Sbordone and can be compared with a result of Di Fazio in [6] (who in fact obtains regularity up to the boundary).

2. Estimates and proof of the main result

The main tool we utilize is the following result of Iwaniec and Sbordone [10, see p. 187, 201–206].

Theorem 2.1 (Iwaniec, Sbordone). *Let* $A \in VMO \cap L^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$ *satisfy* (1.2). *Then for all* $1 < q < +\infty$, *the operator*

$$T := R_i A_{ij} R_j : L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$$

is invertible, and moreover, there exists $C = C(A, q) > 0$ *such that*

$$\|f\|_{L^q(\mathbb{R}^N)} \leq C \|Tf\|_{L^q(\mathbb{R}^N)} \tag{2.1}$$

for all $f \in L^q(\mathbb{R}^N)$.

From this we obtain the localization:

Proposition 2.2. *Let* A, T *as in Theorem* 2.1. *Then for any* Ω_1, Ω_2 *open and bounded with* $\Omega_1 \subset\subset \Omega_2$, $2 < q < +\infty$, *there exists* $C = C(A, q, \Omega_1, \Omega_2) > 0$ *such that*

$$\|f\|_{L^q(\Omega_1)} \leq C(\|Tf\|_{L^q(\Omega_2)} + \|f\|_{L^2(\mathbb{R}^N)})$$

for all $f \in L^2(\mathbb{R}^N)$.

One of the important ideas underlying Theorem 2.1 (see [10, p. 202]) is Uchiyama's compactness result for certain commutators involving Riesz transforms and VMO functions [17]—itself an extension of the celebrated commutator estimates by Coifman, Rochberg, and Weiss [2]. Here we additionally will make use of the following more elementary commutator estimate, whose proof we provide for the convenience of the reader.

Proposition 2.3. *Let* $b, f : \mathbb{R}^N \to \mathbb{R}$ *and define the commutator* $\mathscr{C}(b, R_i)[f]$ *by*

$$\mathscr{C}(b, R_i)[f] := bR_i[f] - R_i[bf],$$

where R_i *is the i-th Riesz transform. If b is Lipschitz, then*

$$\|\mathscr{C}(b, R_i)[f]\|_{L^p(\mathbb{R}^N)} \leq C[b]_{\mathrm{Lip}(\mathbb{R}^N)} \|I_1|f|\,\|_{L^p(\mathbb{R}^N)}.$$

Proof. Since

$$R_i g(x) = c_N \int_{\mathbb{R}^N} \frac{x_i - z_i}{|x - z|^{N+1}} g(z)\, dz,$$

we have

$$\mathscr{C}(b, R_i)[f](x) = c_N \int_{\mathbb{R}^N} \frac{x_i - z_i}{|x-z|^{N+1}} (b(x) - b(z)) f(z)\, dz,$$

and consequently,

$$|\mathscr{C}(b, R_i)[f](x)| \le c_N [b]_{\mathrm{Lip}(\mathbb{R}^N)} \int_{\mathbb{R}^N} |x-z|^{-N+1} |f|(z)\, dz = C[b]_{\mathrm{Lip}(\mathbb{R}^N)} I_1 |f|(x). \qquad \square$$

Proof of Proposition 2.2. Let $\eta \in C_0^\infty(\Omega_2)$ be a usual cutoff function, i.e. $\eta \ge 0$ and $\eta \equiv 1$ on a neighbourhood of Ω_1. From (2.1) we have

$$\|f\|_{L^q(\Omega_1)} \le \|\eta f\|_{L^q(\mathbb{R}^N)} \le C \|T(\eta f)\|_{L^q(\mathbb{R}^N)}.$$

Let us now recall the definition of the commutator of an operator T and two functions b, f (which can be thought of as the error term to a product rule). We have

$$\mathscr{C}(b, T)[f] := bT[f] - T[bf].$$

Then we continue the preceding estimate as follows. For $\operatorname{supp} \eta \subset\subset K_0 \subset\subset L_1 \subset\subset \Omega_2$ and denoting χ_{L_1} the characteristic function of L_1, we estimate

$$\begin{aligned}
\|T(\eta f)\|_{L^q(\mathbb{R}^N)} &= \|T(\eta \chi_{L_1} f)\|_{L^q(\mathbb{R}^N)} \\
&\le \|\eta T(\chi_{L_1} f)\|_{L^q(\mathbb{R}^N)} + \|\mathscr{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} \\
&\le \|T(\chi_{L_1} f)\|_{L^q(K_0)} + \|\mathscr{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} \\
&\le \|T(f)\|_{L^q(K_0)} + \|T(\chi_{L_1^c} f)\|_{L^q(K_0)} + \|\mathscr{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} \\
&=: \|T(f)\|_{L^q(K_0)} + I + II.
\end{aligned}$$

Note that in the display above with our T we have

$$\mathscr{C}(\eta, T)[\chi_{L_1} f] = R_i A_{ij} [\mathscr{C}(\eta, R_j)[\chi_{L_1} f]] + \mathscr{C}(\eta, R_i)[A_{ij} R_j(\chi_{L_1} f)].$$

As for I, since the supports of L_1^c and K_0 are disjoint, we have the estimate

$$\|T(\chi_{L_1^c} f)\|_{L^q(K_0)} \le ||A||_\infty C_{K_0, L_1} \|f\|_{L^2(\mathbb{R}^N)} \tag{2.2}$$

Indeed, let $\tilde{K}$ be so that $K_0 \subset\subset \tilde{K} \subset\subset L_1$. Then by the boundedness of the Riesz transform on $L^q(\mathbb{R}^N)$,

$$\begin{aligned}
\|T(\chi_{L_1^c} f)\|_{L^q(K_0)} &\le \|R_i(\chi_{\tilde{K}} A_{ij} R_j((\chi_{L_1^c} f))\|_{L^q(K_0)} + \|R_i(\chi_{\tilde{K}^c} A_{ij} R_j((\chi_{L_1^c} f))\|_{L^q(K_0)} \\
&\le \|A\|_{L^\infty(\mathbb{R}^N)} \|R_j(\chi_{L_1^c} f)\|_{L^q(\tilde{K})} + \|R_i(\chi_{\tilde{K}^c} A_{ij} R_j((\chi_{L_1^c} f))\|_{L^q(K_0)}
\end{aligned}$$

We now apply the Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
\|R_j(\chi_{L_1^c} f)\|_{L^q(\tilde{K})} &= \left(\int_{K_0} \left| \int_{\mathbb{R}^N \setminus L_1} f(y) \frac{x_j - y_j}{|x-y|^{N+1}} \, dy \right|^q dx \right)^{\frac{1}{q}} \\
&\le \left(\int_{K_0} \|f\|^q_{L^2(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N \setminus L_1} \frac{1}{|x-y|^{2N}} \, dy \right)^{q/2} dx \right)^{\frac{1}{q}} \\
&\le C |K_0|^{1/q} \|f\|_{L^2(\mathbb{R}^N)} \left(\int_c^\infty \frac{1}{t^{2N}} t^{N-1} \, dt \right)^{\frac{1}{2}} \\
&\le C_{K_0, L_1, q} \|f\|_{L^2(\mathbb{R}^N)},
\end{aligned}
$$

where we have used the disjointness of K_0 and L_1^c (in particular that $\operatorname{dist}(K_0, L_1^c) = c > 0$). A similar argument shows that

$$
\|R_i(\chi_{\tilde{K}^c} A_{ij} R_j((\chi_{L_1^c} f))\|_{L^q(K_0)} \le C_{\tilde{K}, L_1, q} \|A_{ij} R_j((\chi_{L_1^c} f))\|_{L^2(\mathbb{R}^N)},
$$

and so using the boundedness of the Riesz transform on $L^2(\mathbb{R}^N)$, we conclude that

$$
\|R_i(\chi_{\tilde{K}^c} A_{ij} R_j((\chi_{L_1^c} f))\|_{L^q(K_0)} \le C_{\tilde{K}, K_0, q} \|A\|_{L^\infty(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}.
$$

It thus remains to estimate II. Let us begin by observing that the commutator estimates with a Lipschitz continuous function (see Proposition 2.3) imply that

$$
\begin{aligned}
II &= \|\mathscr{C}(\eta, T)[\chi_{L_1} f]\|_{L^q(\mathbb{R}^N)} \\
&\le C_\eta (\|I_1 |\chi_{L_1} f|]\|_{L^q(\mathbb{R}^N)} + \|I_1 |A_{ij} R_j(\chi_{L_1} f)| \, \|_{L^q(\mathbb{R}^N)}).
\end{aligned}
$$

In particular, $q > 2$ implies that $Nq/(N+q) > 1$ and so $I_1 : L^{Nq/(N+q)}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is bounded. Moreover, $R_j : L^r(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ is bounded for $1 < r < +\infty$, which combined with the fact that $A \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N \times N})$ (recall that $N \ge 2$) implies that

$$
II \le C \|f\|_{L^{Nq/(N+q)}(L_1)}.
$$

If we let $L_0 := \Omega_1$, then our estimates show that

$$
\|f\|_{L^{q_0}(L_0)} \le C(\|T(f)\|_{L^{q_0}(K_0)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_1}(L_1)})
$$

for $q_i := Nq/(N+iq)$. Now, if $q_1 \le 2$ then an application of Hölder's inequality implies the desired result. Otherwise we iterate the previous argument by finding

$$
K_0 \subset\subset L_1 \subset\subset K_1 \subset\subset L_2 \subset\subset \cdots K_i \subset\subset L_{i+1} \subset\subset \Omega_2
$$

to obtain the estimate

$$\|f\|_{L^{q_i}(L_i)} \leq C(\|T(f)\|_{L^{q_i}(K_i)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_{i+1}}(L_{i+1})}),$$

provided $q_{i+1} > 1$ (in order that $I_1 : L^{q_{i+1}}(\mathbb{R}^N) \to L^{q_i}(\mathbb{R}^N)$). However, $q_i > 2$ implies $q_{i+1} > 1$, and so we continue the iteration a finite number of times until we obtain that $q_j \leq 2$ for some $j \in \mathbb{N}$. Then collecting the terms our estimate reads

$$\|f\|_{L^q(\Omega_1)} \leq C\Big(\sum_{i=0}^{j-1} \|T(f)\|_{L^{q_i}(K_i)} + \|f\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^{q_j}(L_j)}\Big),$$

from which the inequality (2.2) is a simple consequence of Hölder's inequality, and thus the proposition is established. □

Finally, we require the following result.

PROPOSITION 2.4. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $s \in [0, N)$, and $2 \leq p < +\infty$. Assume that for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int f(-\Delta)^{\frac{s}{2}}\varphi = \int h(-\Delta)^{\frac{s}{2}}\varphi.$$

Then for $\Omega_1 \subset\subset \Omega$, there exists a constant $C = C(\Omega_1)$ such that

$$\|f\|_{L^p(\Omega_1)} \leq C(\|h\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^2(\mathbb{R}^N)}).$$

PROOF. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega_2)$ be such that

$$\|f\|_{L^p(\Omega_1)} \leq 2\int f\varphi$$

and $\|\varphi\|_{L^{p'}(\mathbb{R}^N)} \leq 1$.

We argue by first reducing to the case where the support of φ is a ball. We can accomplish this by covering Ω_2 with finitely many balls $B(x_j, r_j)$ of controlled overlap such that $B(x_j, 4r_j) \subset\subset \Omega$, where the number of balls can be taken to depends only on the distance of Ω_1 to Ω^c. Then by subordinating a partition of unity to balls $B(x_j, r_j)$ we can write

$$\varphi = \sum_{j=1}^{l} \varphi_j$$

with $\operatorname{supp}\varphi_j \subset B(x_j, r_j)$ for each j and $|\varphi_j| \leq |\varphi|$. Then for j fixed we have

$$\begin{aligned}\int f\varphi_j &= 2\int f(-\Delta)^{\frac{s}{2}} I_s\varphi_j \\ &= 2\int f(-\Delta)^{\frac{s}{2}}(\eta_j I_s\varphi) + 2\int f(-\Delta)^{\frac{s}{2}}((1-\eta_j)I_s\varphi_j)\end{aligned}$$

$$
\begin{aligned}
&= 2\int h(-\Delta)^{\frac{s}{2}}(\eta_j I_s\varphi_j) + 2\int f(-\Delta)^{\frac{s}{2}}((1-\eta_j)I_s\varphi_j)\\
&\le 2(\|h\|_{L^p(\mathbb{R}^N)}\|(-\Delta)^{\frac{s}{2}}(\eta_j I_s\varphi)\|_{L^{p'}(\mathbb{R}^N)}\\
&\quad + \|f\|_{L^2(\mathbb{R}^N)}\|(-\Delta)^{\frac{s}{2}}((1-\eta_j)I_s\varphi)\|_{L^2(\mathbb{R}^N)}),
\end{aligned}
$$

where $\eta_j \in C_c^\infty(\Omega)$ with $\eta \equiv 1$ on $B(x_j, 4r_j)$. Then if we can establish the estimates

$$\|(-\Delta)^{\frac{s}{2}}(\eta_j I_s\varphi)\|_{L^{p'}(\mathbb{R}^N)} \le C\|\varphi_j\|_{L^{p'}(\mathbb{R}^N)} \tag{2.3}$$

$$\|(-\Delta)^{\frac{s}{2}}((1-\eta_j)I_s\varphi)\|_{L^2(\mathbb{R}^N)} \le C\|\varphi_j\|_{L^{p'}(\mathbb{R}^N)}, \tag{2.4}$$

the result will follow by summing in j and using the pointwise inequality $|\varphi_j| \le |\varphi|$.

Let us therefore first examine (2.3), and to save notation we drop the dependence in j. If we take the three term commutator H_s introduced by Da Lio and Rivière [4]

$$H_s(\eta, I_s\varphi) := (-\Delta)^{\frac{s}{2}}(\eta I_s\varphi) - (-\Delta)^{\frac{s}{2}}\eta I_s\varphi - \eta\varphi,$$

we can use

$$\|H_s(\eta, I_s\varphi)\|_{L^{p'}(\mathbb{R}^N)} \le C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.$$

This estimate follows via the Littlewood-Paley decomposition in [4] or using the *pointwise* estimates in [12] (see [5, Theorem 1.2] for a precise version that can be applied here and also [1, 3] for various extensions). Thus, it suffices to show that

$$\|(-\Delta)^{\frac{s}{2}}\eta I_s\varphi\|_{L^{p'}(\mathbb{R}^N)} + \|\eta\varphi\|_{L^{p'}(\mathbb{R}^N)} \le C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.$$

The second term can be estimated in terms of the right hand side trivially since $|\eta| \le 1$, while for the first term one applies Hölder's inequality with exponent $Np'/(N-sp')$ and its Hölder conjugate r when $N - sp' > 0$ (Note that from $\eta \in C_c^\infty(\mathbb{R}^n)$ we know that $(-\Delta)^{\frac{s}{2}}\eta \in L^r(\mathbb{R}^n)$ for any $r \in (1,\infty)$, e.g. by interpolation.), which yields

$$
\begin{aligned}
\|(-\Delta)^{\frac{s}{2}}\eta I_s\varphi\|_{L^{p'}(\mathbb{R}^N)} &\le \|(-\Delta)^{\frac{s}{2}}\eta\|_{L^r(\mathbb{R}^N)}\|I_s\varphi\|_{L^{Np'/(N-sp')}(\mathbb{R}^N)}\\
&\le C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.
\end{aligned}
$$

If $N - sp' < 0$, then

$$
\begin{aligned}
\|(-\Delta)^{\frac{s}{2}}\eta I_s\varphi\|_{L^{p'}(\mathbb{R}^N)} &\le \|(-\Delta)^{\frac{s}{2}}\eta\|_{L^{p'}(\mathbb{R}^N)}\|I_s\varphi\|_{L^\infty(\mathbb{R}^N)}\\
&\le C\|\varphi\|_{L^{p'}(\mathbb{R}^N)}
\end{aligned}
$$

follows from the fact that φ has compact support. When $N - sp' = 0$, we take $\tilde{p}' < p'$ and set $\frac{1}{r} := \frac{1}{p'} - \frac{1}{\tilde{p}'}$, then

$$\|(-\Delta)^{\frac{s}{2}}\eta I_s\varphi\|_{L^{p'}(\mathbb{R}^N)} \leq \|(-\Delta)^{\frac{s}{2}}\eta\|_{L^{\tilde{r}}(\mathbb{R}^N)}\|I_s\varphi\|_{L^{N\tilde{p}'/(N-s\tilde{p}')}(\mathbb{R}^N)}$$
$$\leq C\|\varphi\|_{L^{\tilde{p}'}(\mathbb{R}^N)}.$$

The estimate follows again in this case by the fact that φ has compact support.

Finally, to establish (2.4) we write

$$(1-\eta) = \sum_{k=2}^{\infty} \theta_{A_{2^k r}},$$

where each $\theta_{A_{2^k r}}$ is supported on an annulus of width $2^k r$. Then disjoint support arguments (see, for example, Lemma 3.7 in [11]) imply the estimate

$$\|(-\Delta)^{\frac{s}{2}}(\theta_{A_{2^k r}} I_s\varphi)\|_{L^2(\mathbb{R}^N)} \leq C(2^k r)^{-N/2} r^{N/p}\|\varphi\|_{L^{p'}(\mathbb{R}^N)},$$

from which we obtain

$$\|(-\Delta)^{\frac{s}{2}}((1-\eta)I_s\varphi)\|_{L^2(\mathbb{R}^N)} \leq \sum_{k=2}^{\infty}\|(-\Delta)^{\frac{s}{2}}(\theta_{A_{2^k r}} I_s\varphi)\|_{L^2(\mathbb{R}^N)}$$
$$\leq \left(C\sum_{k=2}^{\infty}(2^k r)^{-N/2} r^{N/p}\right)\|\varphi\|_{L^{p'}(\mathbb{R}^N)}.$$

As the series is summable we have established the desired inequality and therefore the theorem is proved. □

We are now ready to prove the main result.

PROOF OF THEOREM 1.2. Suppose $G \in L^p(\mathbb{R}^N;\mathbb{R}^N)$ and $u \in H^s(\mathbb{R}^N)$ satisfies the equation (1.4). The claim of this theorem is that for any $K \subset\subset \Omega$, one has the estimate

$$\|D^s u\|_{L^p(K;\mathbb{R}^N)} \leq C(\|G\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}).$$

We will see that the result is a consequence of a combination of Propositions 2.2 and 2.4, and we argue as follows. Define $g := R^*G = -\sum_{j=1}^N R_j G_j$, so that $g \in L^p(\mathbb{R}^N)$ and u satisfies

$$\int_\Omega T(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi = \int g(-\Delta)^{\frac{s}{2}}\varphi \quad \forall\varphi \in C_c^\infty(\Omega),$$

where T is as in Theorem 2.1. Moreover, a cutoff argument similar to those previously employed implies that if $K \subset\subset \Omega_1$, then one has

$$\|D^s u\|_{L^p(K;\mathbb{R}^N)} = \|R(-\Delta)^{\frac{s}{2}}u\|_{L^p(K;\mathbb{R}^N)}$$
$$\leq C(\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_1)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}),$$

and so this and boundedness of the Riesz transforms (to obtain bounds on g in terms of G in L^p) imply that it suffices to show the estimate

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_1)} \leq C(\|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}).$$

for $\Omega_1 \subset\subset \Omega$.

We first apply Proposition 2.2 with $f = (-\Delta)^{\frac{s}{2}}u$ and for $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ yielding

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_1)} \leq C(\|T(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_2)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}).$$

Now Proposition 2.4 and boundedness of $T : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ gives

$$\begin{aligned}\|T(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_2)} &\leq C(\|g\|_{L^p(\mathbb{R}^N)} + \|T(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}) \\ &\leq C(\|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}).\end{aligned}$$

Therefore, we find

$$\|(-\Delta)^{\frac{s}{2}}u\|_{L^p(\Omega_1)} \leq C(\|g\|_{L^p(\mathbb{R}^N)} + \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}),$$

which is the thesis. □

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