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Partial Differential Equations — *Existence, uniqueness and behaviour of solutions for a nonlinear diffusion equation with third type boundary value condition*, by Fatma Gamze Duzgun and Kamal Soltanov, communicated on 11 June 2015.[1]

Abstract. — In this work, we investigate a mixed problem with boundary condition of third type for a nonlinear diffusion equation having nonlocal term. Existence and uniqueness of a solution of the posed problem are proved under fairly general conditions. Moreover, we obtain some results on the behaviour of the solution and the existence of an absorbing set for the problem under consideration.

Key words: Nonlinear diffusion equations, Robin boundary condition, non-local effect, existence and uniqueness, behavior of solution, absorbing set

Mathematics Subject Classification: 35D30, 35K58, 35M12

1. Introduction

We consider the problem

$$\frac{\partial u}{\partial t} - \Delta u + g(x,t,u) + e(x,t)\|u\|_{L_2(\Omega)}(t) = h(x,t), \qquad (x,t) \in Q_T \equiv \Omega \times (0,T) \tag{1.1}$$

$$u(x,0) = u_0, \quad x \in \Omega \subset \mathbb{R}^n,\ n \geq 3 \tag{1.2}$$

$$\left(\frac{\partial u}{\partial \eta} + a(x',t)u\right)\bigg|_{\Sigma_T} = \varphi(x',t), \quad (x',t) \in \Sigma_T \equiv \partial\Omega \times [0,T],\ T > 0 \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$; Δ denotes the Laplace operator with n-dimension ($\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$); $g : Q_T \times \mathbb{R}^1 \to \mathbb{R}^1$, $e : Q_T \to \mathbb{R}^1$ and $a : \Sigma_T \to \mathbb{R}^1$ and u_0 are given functions; h, φ are given generalized functions.

In this article we investigate nonhomogenous nonlinear diffusion type equation (1.1) with initial value (1.2) and nonhomogenous third type boundary (Robin boundary) value (1.3).

Equation (1.1) has a usually nonlinear mapping g in general form and a nonlocal nonlinear term $e(x,t)\|u\|_{L_2(\Omega)}(t)$ considering function e is different from zero.

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Above model generally arises from many areas like nuclear sciences, population dynamics and biological sciences. According to the model of a physical or chemical reaction events, term of $g(x, t, u)$ can represent the growth of a quantity like growth of temperature or growth of a certain cell while the other nonlinear term represents the effect of a material put into the system (see for instance [6]). As it is noted in [5]: "In ecological context, there is no real justification for assuming that the interactions are local. There are many (hypothetical) examples where such an assumption is clearly untenable, such as: (a) a population in which individua complete for a shared rapidly equilibrated (e.g. by convection) resource; (b) a population in which individua communicate either visually or by chemical means. For example, the total biomass should play a role in a model that incorporates group defense or visual communication, which if the single-species model is a large-diffusion approximation of a system, averages appear naturally."

In [4], we investigate the existence and uniqueness of such type problem without of nonlocal term and when the initial date is zero. But here we study problem (1.1)–(1.3) that is more general than mentioned before. We should also note that in this paper non-local nonlinear term $e(x, t)\|u\|_{L_2(\Omega)}(t)$ added some difficulties for investigation of posed problem. For instance, this nonlinearity is independent from the local nonlinearity which makes difficulties and diversities for the studies of the questions on uniqueness and on behavior of the solution.

It should be noted that equation (1.1) has been studied mostly in homogeneous form by taking mapping g in special cases and taking e as zero function with Dirichlet or Neumann boundary conditions, see for instance [1], [2], [8], [12], [15], [16].

More similarly to equation (1.1), we could see the following studies: M. Jazar and R. Kiwan [7] consider

$$\frac{\partial u}{\partial t} - \Delta u - |u|^p + \frac{1}{|\Omega|}\int_\Omega |u|^p\,dx = 0,$$

W. Gao and Y. Han [6] consider

$$\frac{\partial u}{\partial t} - \Delta u - |u|^{p-1}u + \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx = 0$$

and C. P. Niculescu, I. Roventa [13] consider

$$\frac{\partial u}{\partial t} - \Delta u - f(|u|) + \frac{1}{|\Omega|}\int_\Omega f(|u|)\,dx = 0$$

equations with homogenous Neumann condition and they investigated the behaviour of the solutions.

As a different from the previous studies, we investigate nonhomogenous equation as taking mapping g in general form and having another globally nonlinear term with nonhomogenous third type boundary (Robin boudary) value condition.

Here we proved the solvability and uniqueness theorems, moreover we obtained some results about the behaviour of the solution in corresponding spaces for posed problem (1.1)–(1.3).

2. Formulation and the main conditions of problem (1.1)–(1.3)

For problem (1.1)–(1.3), we assume $h \in L_2(0,T;(W_2^1(\Omega))^*) + L_q(Q_T)$, $\varphi \in L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$. We consider the following conditions:

(1) $g : Q_T \times R^1 \to R^1$ is a Caratheodory function and there exist a number $\alpha \geq 0$ and functions $c_1 \in L_{s_1}(0,T;L_{r_1}(\Omega))$, $c_0 \in L_{s_2}(0,T;L_{r_2}(\Omega))$ such that g satisfies the following inequality for a.e. $(x,t) \in Q_T$ and for any $\xi \in \mathbb{R}^1$:

$$|g(x,t,\xi)| \leq c_1(x,t)|\xi|^\alpha + c_0(x,t)$$

$(r_1, r_2, s_1, s_2 > 1$ will be defined later according to $\alpha)$.

(2) $a \in L_\infty(0,T;L_{n-1}(\partial\Omega))$

(3) $e \in L_\infty(0,T;L_{\tilde{q}}(\Omega))$, $\tilde{q} := \begin{cases} q_0, & \alpha \leq 1 \\ \frac{\alpha+1}{\alpha}, & \alpha > 1 \end{cases}$

Here, $p_0 := \frac{2n}{n-2}$, $q_0 := (p_0)'$ and α is coming from condition (1).

We understand the solution of considered problem in the following sense: Let $P_0 := L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = u_0\}$.

Definition 2.1. A function $u \in P_0$ is called generalized solution of problem (1.1)–(1.3) if it satisfies the equality

$$\begin{aligned} -\int_0^T \int_\Omega u\frac{\partial v}{\partial t}\,dx\,dt + \int_\Omega u(x,T)v(x,T)\,dx - \int_\Omega u(x,0)v(x,0)\,dx \\ + \int_0^T \int_\Omega Du.Dv\,dx\,dt + \int_0^T \int_\Omega g(x,t,u)v\,dx\,dt \\ + \int_0^T \int_\Omega e(x,t)\|u\|_{L_2(\Omega)}v\,dx\,dt + \int_0^T \int_{\partial\Omega} a(x',t)uv\,dx'\,dt \\ = \int_0^T \int_\Omega hv\,dx\,dt + \int_0^T \int_{\partial\Omega} \varphi v\,dx'\,dt \end{aligned}$$

for all $v \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*)$.

Since different sufficient conditions are obtained for the solvability according to the values of α in condition (1), we investigate problem (1.1)–(1.3) in three different sections: Solvability in case of $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

3. Solvability in case of $\alpha < 1$

Let $0 \le \alpha < 1$. This case is the sublinear case for mapping g and since $L_2(0, T; W_2^1(\Omega)) \subset L_{\alpha+1}(Q_T)$, then $P_0 \equiv L_2(0, T; W_2^1(\Omega)) \cap W_2^1(0, T; (W_2^1(\Omega))^*) \cap \{u : u(x, 0) = u_0\}$. Consider the following conditions:

(1)′ Condition (1) is satisfied with $0 \le \alpha < 1$ and parameters: $s_1 := \frac{2}{1-\alpha}$, $r_1 := \frac{p_0 q_0}{p_0 - \alpha q_0}$, $s_2 := 2$, $r_2 := q_0$, where $p_0 := \frac{2n}{n-2}$, $q_0 := (p_0)'$.

(4) There exists a number $a_0 > 0$ such that $a(x', t) \ge a_0$ for a.e. $(x', t) \in \Sigma_T$.

(5) $\|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} \le \frac{\theta_0 c_2}{c_3 c_6}$, here $\theta_0 < \min\{\widetilde{b}_0, a_0\}$ where $0 \ll \widetilde{b}_0 < 1$ (c_2 is constant[2] [19]; c_3 and c_6 are constants[3] [1], here and in the following, c_i are constants generally coming from of Imbedding inequalities of Sobolev type.)

Theorem 3.1 (Existence Theorem). *Let conditions* (1)′, (2), (3), (4) *and* (5) *be fulfilled. Then problem* (1.1)–(1.3) *is solvable in* P_0 *for any* $(h, \varphi) \in L_2(0, T; (W_2^1(\Omega))^*) \times L_2(0, T; W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega)$.

For the proof of this theorem, we make use a general result [17] that is given below:

Theorem 3.2. *Let X and Y be Banach spaces with duals X^* and Y^* respectively, Y be a reflexive Banach space, $\mathscr{M}_0 \subseteq X$ be a weakly complete "reflexive" pn-space, $X_0 \subseteq \mathscr{M}_0 \cap Y$ be a separable vector topological space. Let the following conditions be fulfilled:*

(i) $f : P_0 \to L_q(0, T; Y)$ *is a weakly compact* (*weakly continuous*) *mapping, where*

$$P_0 \equiv L_p(0, T; \mathscr{M}_0) \cap W_q^1(0, T; Y) \cap \{x(t) \mid x(0) = 0\},$$

$1 < \max\{q, q'\} \le p < \infty$, $q' = \frac{q}{q-1}$;

(ii) *there is a linear continuous operator* $A : W_m^s(0, T; X_0) \to W_m^s(0, T; Y^*)$, $s \ge 0$, $m \ge 1$ *such that A commutes with $\frac{\partial}{\partial t}$ and the conjugate operator A^* has* $\ker(A^*) = \{0\}$;

(iii) *operators f and A generate, in generalized sense, a coercive pair on space $L_p(0, T; X_0)$, i.e. there exist a number $r > 0$ and a function $\Psi : R_+^1 \to R_+^1$ such that $\Psi(\tau)/\tau \nearrow \infty$ as $\tau \nearrow \infty$ and for any $x \in L_p(0, T; X_0)$ under $[x]_{L_p(\mathscr{M}_0)} \ge r$ following inequality holds:*

$$\int_0^T \langle f(t, x(t)), Ax(t) \rangle \, dt \ge \Psi([x]_{L_p(\mathscr{M}_0)});$$

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[2] $c_2 \|u\|^2_{L_2(0,T;W_2^1(\Omega))} \le (\|Du\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(\Sigma_T)})$

[3] $\|u\|^2_{L_2(0,T;L_{p_0}(\Omega))} \le c_3 \|u\|^2_{L_2(0,T;W_2^1(\Omega))}$; $\|u\|_{L_2(\Omega)} \le c_6 \|u\|_{L_{p_0}(\Omega)}$

(iv) *there exist some constants* $C_0 > 0$, $C_1, C_2 \geq 0$, $\nu > 1$ *such that the inequalities*

$$\int_0^T \langle \xi(t), A\xi(t) \rangle \, dt \geq C_0 \|\xi\|^{\nu}_{L_q(0,T;Y)} - C_2,$$

$$\int_0^t \left\langle \frac{dx}{d\tau}, Ax(\tau) \right\rangle d\tau \geq C_1 \|x\|^{\nu}_{Y}(t) - C_2, \quad a.e.\ t \in [0,T]$$

hold for any $x \in W^1_p(0,T;X_0)$ *and* $\xi \in L_p(0,T;X_0)$.

Assume that conditions (i)–(iv) *are fulfilled. Then the Cauchy problem*

$$\frac{dx}{dt} + f(t, x(t)) = y(t), \quad y \in L_q(0,T;Y); \quad x(0) = 0$$

is solvable in P_0 *in the following sense*

$$\int_0^T \left\langle \frac{dx}{dt} + f(t,x(t)), y^*(t) \right\rangle dt = \int_0^T \langle y(t), y^*(t) \rangle \, dt, \quad \forall y^* \in L_{q'}(0,T;Y^*),$$

for any $y \in L_q(0,T;Y)$ *satisfying the inequality*

$$\sup\left\{ \frac{1}{[x]_{L_p(0,T;\mathscr{M}_0)}} \int_0^T \langle y(t), Ax(t) \rangle \, dt \mid x \in L_p(0,T;X_0) \right\} < \infty.$$

PROOF OF THEOREM 3.1. As we see in Theorem 3.2, for being able to apply it to our problem, firstly the initial condition should be zero. Therefore we can rewrite problem (1.1)–(1.3) as the following using the transformation $\tilde{u}(x,t) := u(x,t) - u_0(x)$:

$$\frac{\partial(\tilde{u}+u_0)}{\partial t} - \Delta(\tilde{u}+u_0) + g(x,t,\tilde{u}+u_0) + e(x,t)\|\tilde{u}+u_0\|_{L_2(\Omega)}(t) = h(x,t) \tag{3.1}$$

$$\tilde{u}(x,0) = 0 \tag{3.2}$$

$$\left(\frac{\partial(\tilde{u}+u_0)}{\partial \eta} + a(x',t)(\tilde{u}+u_0) \right)\Big|_{\Sigma_T} = \varphi(x',t) \tag{3.3}$$

Now let define corresponding mappings and acting spaces for problem (3.1)–(3.3):

$$f = \{f_1, f_2\} : P_0 \to L_2(0,T;(W^1_2(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$$

such that

$$f_1(\tilde{u}) := -\Delta(\tilde{u}+u_0) + g(x,t,\tilde{u}+u_0) + e(x,t)\|\tilde{u}+u_0\|_{L_2(\Omega)}(t), \tag{3.4}$$

$$f_2(\tilde{u}) := \frac{\partial(\tilde{u}+u_0)}{\partial \eta} + a(x',t)(\tilde{u}+u_0); \tag{3.5}$$

$$A : P_0 \to P_0$$
$$A \equiv Id \tag{3.6}$$

To see that the conditions of Theorem 3.2 are satisfied, we shall give the following lemmas: □

LEMMA 3.3. *f is weakly continuous from P_0 to $L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$.*

PROOF. Using condition $(1)'$ and Hölder inequality, we obtain that

$$\|g(x,t,\tilde{u}+u_0)\|_{L_2(0,T;L_{q_0}(\Omega))} \le \gamma_0(\|\tilde{u}+u_0\|_{L_2(0,T;L_{p_0}(\Omega))}),$$

where

$$\gamma_0(\|\tilde{u}+u_0\|_{L_2(0,T;L_{p_0}(\Omega))}) := \tilde{C}_0[\|c_1\|^2_{L_{s_1}(0,T;L_{r_1(\Omega)})}\|\tilde{u}+u_0\|^{2\alpha}_{L_2(0,T;L_{p_0}(\Omega))} + \|c_0\|^2_{L_2(0,T;L_{q_0}(\Omega))}]^{\frac{1}{2}},$$

$\tilde{C}_0$ is a positive constant. This means, g is a bounded mapping from P_0 to $L_2(0,T;L_{q_0}(\Omega))$, since $P_0 \subset L_2(0,T;W_2^1(\Omega)) \subset L_2(0,T;L_{p_0}(\Omega))$.

Since linear parts of f are obviously bounded, they are already weakly continuous. It is enough to investigate the nonlinear part of f. Let $\{u_m\} \subset P_0$ and $u_m \rightharpoonup \bar{u}$ in P_0. Then $u_m \rightharpoonup \bar{u}$ in $L_2(0,T;L_{p_0}(\Omega))$. Since

$$L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \circlearrowleft L_2(Q_T) \tag{3.7}$$

then $\exists\{u_{m_l}\} \subset \{u_m\}$ such that $u_{m_l} \to \bar{u}$ almost everywhere in Q_T.

Using condition $(1)'$ we can say that

$$g(x,t,\bullet) : \mathbb{R}^1 \to \mathbb{R}^1$$

is a continuous function. Then according to a general result (1. Chapter, 1. Paragraph, Lemma 1.3 of [9]), $\exists\{u_{m_j}\} \subset \{u_m\}$ such that

$$g(x,t,u_{m_j}+u_0) \xrightharpoonup[L_2(0,T;L_{q_0}(\Omega))]{} g(x,t,\bar{u}+u_0).$$

Thus g is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)$. Now let $g_1(x,t,\tilde{u}+u_0) := e(x,t)\|\tilde{u}+u_0\|_{L_2(\Omega)}(t)$. Using the fact (3.7), we have

$$e(x,t)\|u_{m_k}+u_0\|_{L_2(\Omega)}(t) \xrightarrow[L_2(0,T;L_{q_0}(\Omega))]{} e(x,t)\|\bar{u}+u_0\|_{L_2(\Omega)}(t).$$

Therefore, g_1 is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)$. □

LEMMA 3.4. *Conditions* (ii), (iii), (iv) *of Theorem* 3.2 *are satisfied.*

PROOF. Since A is an identity mapping, it is obvious that condition (ii) is satisfied. Furthermore, for any $\tilde{u} \in W_2^1(0,T;W_2^1(\Omega))$ the following inequalities are satisfied:

$$\int_0^T \langle \tilde{u}, \tilde{u} \rangle_\Omega \, dt = \int_0^T \|\tilde{u}\|^2_{L_2(\Omega)} \, dt \geq c_6 \|\tilde{u}\|^2_{L_2(0,T;(W_2^1(\Omega))^*)}$$
$$\int_0^t \left\langle \frac{\partial \tilde{u}}{\partial \tau}, \tilde{u} \right\rangle_\Omega d\tau = \frac{1}{2}\|\tilde{u}\|^2_{L_2(\Omega)}(t) \geq \frac{1}{2} c_6 \|\tilde{u}\|^2_{(W_2^1(\Omega))^*}(t),$$

a.e. $t \in [0,T]$ ($c_6 > 0$ is constant[4] [1].) This means condition (iv) is also satisfied. It is enough to see that mapping f is coercive on $L_2(0,T;W_2^1(\Omega))$ for condition (iii), since A is an identity mapping:

Using conditions (1)$'$, (2), (3), (4) and (5) we obtain,

$$\langle f(\tilde{u}), \tilde{u} \rangle_{Q_T} \geq \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}),$$
$$\Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}) := Z_1(\theta_0 c_2 - c_6 c_3 \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} - \varepsilon)\|\tilde{u}\|^2_{L_2(0,T;W_2^1(\Omega))} - Z_2,$$

where $\theta_0 < \min\{\widetilde{b_0}, a_0\}$ with $\widetilde{b_0} < 1$, Z_1 is a positive constant,

$$Z_2 := Z_2(\|c_0\|_{L_{s_2}(0,T;L_{r_2}(\Omega))}, \|c_1\|_{L_{s_1}(0,T;L_{r_1}(\Omega))}, \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))}, \|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))},$$
$$\|u_0\|_{L_2(0,T;L_{p_0}(\Omega))}, \|u_0\|_{L_2(0,T;L_{\frac{2n-2}{n-2}}(\partial\Omega))}, \|Du_0\|_{L_2(Q_T)})$$

and ε is small enough. Hence, $\frac{\Psi(\|\tilde{u}\|)}{\|\tilde{u}\|} \nearrow \infty$ as $\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))} \nearrow \infty$. □

CONTINUATION OF THE PROOF OF THEOREM 3.1. We can apply Theorem 3.2 to problem (3.1)–(3.3) by virtue Lemma 3.3 and Lemma 3.4. Hence we obtain that problem (3.1)–(3.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ satisfying the following inequality

$$\sup\left\{ \frac{1}{\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}} \int_0^T \langle h, \tilde{u} \rangle_\Omega + \langle \varphi, \tilde{u} \rangle_{\partial\Omega} \, dt : \tilde{u} \in L_2(0,T;W_2^1(\Omega)) \right\} < \infty.$$

If we consider the norm definition of (h,φ) in $L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$, we see that problem (3.1)–(3.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega)$. This means (1.1)–(1.3) is also solvable. □

4. SOLVABILITY IN CASE OF $\alpha = 1$

Let $\alpha = 1$ for condition (1). In this case, $P_0 \equiv L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = u_0\}$. We consider the following conditions:

[4] $c_6 \|u\|^2_{(W_2^1(\Omega))^*} \leq \|u\|^2_{L_2(\Omega)}$

(1)″ Condition (1) is satisfied with parameters: $s_1 := \infty$, $r_1 := \frac{n}{2}$, $s_2 := 2$, $r_2 := q_0$.

(6) One of the following conditions be satisfied:

I. There exists a number $a_0 > 0$ such that $a(x', t) \geq a_0$ for a.e. $(x', t) \in \Sigma_T$ and

$$c_6 \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} + \|c_1\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))} \leq \frac{\theta_1 c_2}{c_3}$$

where $\theta_1 < \min\{\widetilde{b_0}, a_0\}$ with $\widetilde{b_0} < 1$.

II. There exist some numbers $k_0 > 0$ and $k_1 \in \mathbb{R}^1$ such that

$$g(x, t, \xi)\xi \geq k_0|\xi|^2 - k_1$$

for a.e. $(x, t) \in Q_T$, for any $\xi \in \mathbb{R}^1$ and

$$c_5 \|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))} + c_3 c_6 \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} \leq \theta_2$$

where $\theta_2 < \min\{\widetilde{b_0}, k_0\}$ with $0 \ll \widetilde{b_0} < 1$.

(c_2, c_3, c_6 are like in Theorem 3.1 and c_5 is constant[5] [1].)

THEOREM 4.1 (Existence Theorem). *Let conditions* (1)″, (2), (3) *and* (6) *be fulfilled. Then problem* (1.1)–(1.3) *is solvable in* P_0 *for any* $(h, \varphi) \in L_2(0, T; (W_2^1(\Omega))^*) \times L_2(0, T; W_2^{-\frac{1}{2}}(\partial\Omega))$ *and* $u_0 \in W_2^1(\Omega)$.

PROOF. To prove this theorem we again make use of Theorem 3.2. We define corresponding mappings as (3.4), (3.5), (3.6) for problem (3.1)–(3.3). □

LEMMA 4.2. *f is weakly continuous from P_0 to $L_2(0, T; (W_2^1(\Omega))^*)$.*

PROOF. It is enough to show that $g : P_0 \subset L_2(0, T; L_{p_0}(\Omega)) \to L_2(0, T; L_{q_0}(\Omega))$ is a bounded mapping for $\alpha = 1$: Using condition (1)″ we obtain,

$$\|g\|_{L_2(0,T;L_{q_0}(\Omega))} \leq \gamma_1(\|u\|_{L_2(0,T;L_{p_0}(\Omega))}),$$

$$\gamma_1(\|\tilde{u} + u_0\|_{L_2(0,T;L_{p_0}(\Omega))}) := \tilde{C}_1[\|c_1\|^2_{L_{s_1}(0,T;L_{r_1}(\Omega))} \|\tilde{u} + u_0\|^2_{L_2(0,T;L_{p_0}(\Omega))} + \|c_0\|^2_{L_2(0,T;L_{q_0}(\Omega))}]^{\frac{1}{2}},$$

$\tilde{C}_1$ is a positive constant. The rest of this proof is similar with the proof of Lemma 3.3. □

LEMMA 4.3. *Conditions* (ii), (iii), (iv) *of Theorem* 3.2 *are satisfied.*

PROOF. This proof is similar with the proof of Lemma 3.4. As a different part, we show that f is coercive on $L_2(0, T; W_2^1(\Omega))$:

[5] $\|u\|^2_{L_2(0,T;L_{\frac{2n-2}{n-2}}(\partial\Omega))} \leq c_5 \|u\|^2_{L_2(0,T;W_2^1(\Omega))}$

If we consider conditions $(1)''$, (2), (3) and (6)-I, we obtain,

$$\begin{aligned}\langle f(\tilde{u}),\tilde{u}\rangle_{Q_T} \geq \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}),\ & \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}) \\ := Z_3(\theta_1 c_2 - c_3\|c_1\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))} & - c_6 c_3\|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} - \varepsilon) \\ \times \|\tilde{u}\|^2_{L_2(0,T;W_2^1(\Omega))} & - Z_4,\end{aligned}$$

where $\theta_1 < \min\{\widetilde{b_0}, a_0\}$ with $\widetilde{b_0} < 1$, Z_3 is a positive constant,

$$\begin{aligned}Z_4 := Z_4(&\|c_0\|_{L_{s_2}(0,T;L_{r_2}(\Omega))}, \|c_1\|_{L_{s_1}(0,T;L_{r_1}(\Omega))}, \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))}, \|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}, \\ &\|u_0\|_{L_2(0,T;L_{p_0}(\Omega))}, \|u_0\|_{L_2(0,T;L_{\frac{2n-2}{n-2}}(\partial\Omega))}, \|Du_0\|_{L_2(Q_T)})\end{aligned}$$

and ε is small enough.

If we consider condition (6)-II, we obtain,

$$\begin{aligned}\langle f(\tilde{u}),\tilde{u}\rangle_{Q_T} \geq \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}),\ & \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))}) \\ := Z_5(\theta_2 - c_5\|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))} & - c_6 c_3\|e\|_{L_\infty(0,T;L_{q_0}(\Omega))} - \varepsilon) \\ \times \|\tilde{u}\|^2_{L_2(0,T;W_2^1(\Omega))} & - Z_6,\end{aligned}$$

where $\theta_2 < \min\{\widetilde{b_0}, k_0\}$ with $\widetilde{b_0} < 1$, Z_5 is a positive constant,

$$\begin{aligned}Z_6 := Z_6(&\|c_0\|_{L_{s_2}(0,T;L_{r_2}(\Omega))}, \|c_1\|_{L_{s_1}(0,T;L_{r_1}(\Omega))}, \|e\|_{L_\infty(0,T;L_{q_0}(\Omega))}, \|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))}, \\ &\|u_0\|_{L_2(0,T;L_{p_0}(\Omega))}, \|u_0\|_{L_2(0,T;L_{\frac{2n-2}{n-2}}(\partial\Omega))}, \|Du_0\|_{L_2(Q_T)}, k_1, T, \operatorname{mes}\Omega)\end{aligned}$$

and ε is small enough.

Hence, $\frac{\Psi(\|\tilde{u}\|)}{\|\tilde{u}\|} \nearrow \infty$ as $\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega))} \nearrow \infty$. □

Continuation of the Proof of Theorem 4.1. We can apply Theorem 3.2 to problem (3.1)–(3.3) by virtue Lemma 4.2 and Lemma 4.3. Hence we obtain that problem (3.1)–(3.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega)$ and $u_0 \in W_2^1(\Omega)$. Therefore (1.1)–(1.3) is also solvable. □

5. Solvability in case of $\alpha > 1$

Let $\alpha > 1$. This case is super linear case for mapping g and

$$\begin{aligned}P_0 = L_2(0,T;W_2^1(\Omega)) &\cap L_{\alpha+1}(Q_T) \\ &\cap [W_2^1(0,T;(W_2^1(\Omega))^*) + W^1_{\frac{\alpha+1}{\alpha}}(0,T;L_{\frac{\alpha+1}{\alpha}}(\Omega))] \\ &\cap \{u : u(x,0) = u_0\}.\end{aligned}$$

We consider the following conditions:

(1)‴ Condition (1) is satisfied with parameters: $s_1 := \infty$, $r_1 := \infty$, $s_2 := \frac{\alpha+1}{\alpha}$, $r_2 := \frac{\alpha+1}{\alpha}$.

(7) There exist some numbers $k_0 > 0$ and $k_1 \in \mathbb{R}^1$ such that

$$g(x,t,\xi)\xi \geq k_0|\xi|^{\alpha+1} - k_1$$

for a.e. $(x,t) \in Q_T$, for any $\xi \in \mathbb{R}^1$.

(8) There exists a number $a_0 > 0$ such that $a(x',t) \geq -a_0$ for a.e. $(x',t) \in \Sigma_T$ and $a_0 < \frac{\theta_3}{c_4}$ where $\theta_3 < \min\{\widetilde{b_0}, \widetilde{k_0}\}$ with $0 \ll \widetilde{b_0} < 1$ and $0 \ll \widetilde{k_0} < k_0$. ($c_4$ is constant[6] [1].)

Theorem 5.1 (Existence Theorem). *Let conditions* (1)‴, (2), (3), (7) *and* (8) *be fulfilled. Then problem* (1.1)–(1.3) *is solvable in* P_0 *for any* $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ *and* $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(Q_T)$.

Proof. To prove this theorem let recall (3.4), (3.5), (3.6). □

Lemma 5.2. *f is weakly continuous from P_0 to $L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)$.*

Proof. Using condition (1)‴ and Hölder inequality, we obtain that

$$\|g(x,t,\tilde{u}+u_0)\|_{L_{\frac{\alpha+1}{\alpha}}(Q_T)} \leq \gamma_2(\|\tilde{u}+u_0\|_{L_{\alpha+1}(Q_T)})$$

where

$$\gamma_2(\|\tilde{u}+u_0\|_{L_{\alpha+1}(Q_T)}) := 2[\|c_1\|^{\frac{\alpha+1}{\alpha}}_{L_\infty(Q_T)}\|\tilde{u}+u_0\|^{\alpha+1}_{L_{\alpha+1}(Q_T)} + \|c_0\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}(Q_T)}]^{\frac{\alpha}{\alpha+1}}.$$

This means, g is a bounded mapping from $P_0 \subset L_{\alpha+1}(Q_T)$ to $L_{\frac{\alpha+1}{\alpha}}(Q_T)$.

Since linear parts of f are obviously bounded, they are already weakly continuous. It is enough to investigate the nonlinear part of f. Let $\{u_m\} \subset P_0$ and $u_m \rightharpoonup \bar{u}$ in P_0. Then $u_m \rightharpoonup \bar{u}$ in $L_{\alpha+1}(Q_T)$. From (3.7), $\exists\{u_{m_l}\} \subset \{u_m\}$ such that $u_{m_l} \to \bar{u}$ almost everywhere in Q_T.

Recalling condition (1)‴ we can say that

$$g(x,t,\bullet): \mathbb{R}^1 \to \mathbb{R}^1$$

is a continuous function. Then according to a general result in [8], $\exists\{u_{m_j}\} \subset \{u_m\}$ such that

$$g(x,t,u_{m_j}+u_0) \xrightharpoonup[L_2(0,T;L_{q_0}(\Omega))]{} g(x,t,\bar{u}+u_0).$$

Thus g is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)$. Now let $g_1(x,t,\tilde{u}+u_0) := e(x,t)\|\tilde{u}+u_0\|_{L_2(\Omega)}(t)$. Using the fact (3.7), we have

$$e(x,t)\|u_{m_k}+u_0\|_{L_2(\Omega)}(t) \xrightarrow[L_{\frac{\alpha+1}{\alpha}}(Q_T)]{} e(x,t)\|\bar{u}+u_0\|_{L_2(\Omega)}(t).$$

[6] $\|u\|^2_{L_2(\Sigma_T)} \leq c_4\|u\|^2_{L_2(0,T;W_2^1(\Omega))}$

Therefore, g_1 is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)$. □

LEMMA 5.3. *Conditions* (ii), (iii), (iv) *of Theorem* 3.2 *are satisfied.*

PROOF. Since A is an identity mapping, it is obvious that condition (ii) is satisfied. Furthermore, for any $\tilde{u} \in W_2^1(0,T;W_2^1(\Omega))$ the following inequalities are satisfied:

$$\int_0^T \langle \tilde{u}, \tilde{u} \rangle_\Omega \, dt = \int_0^T \|\tilde{u}\|_{L_2(\Omega)}^2 \, dt \geq c_6 \|\tilde{u}\|^2_{L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)}$$

$$\int_0^t \left\langle \frac{\partial \tilde{u}}{\partial \tau}, \tilde{u} \right\rangle_\Omega d\tau = \frac{1}{2}\|\tilde{u}\|^2_{L_2(\Omega)}(t) \geq \frac{1}{2} c_6 \|\tilde{u}\|^2_{(W_2^1(\Omega))^*}(t),$$

a.e. $t \in [0,T]$ ($c_6 > 0$ constant[7] [1])

This means condition (iv) is also satisfied. Now let see that mapping f is coercive on $L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)$ for condition (iii), since A is an identity mapping: If we consider conditions $(1)'''$, (2), (3), (7) and (8) we obtain,

$$\langle f(\tilde{u}), \tilde{u} \rangle_{Q_T} \geq \Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)}),$$
$$\Psi(\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)}) := Z_7(\theta_3 - a_0 c_4 - \varepsilon)\|\tilde{u}\|^2_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)} - Z_8,$$

where $\theta_3 < \min\{\widetilde{b_0}, \widetilde{k_0}\}$ with $\widetilde{b_0} < 1$ and $\widetilde{k_0} < k_0$; Z_7 is a positive constant,

$$Z_8 := Z_8(\|c_0\|_{L_{s_2}(0,T;L_{r_2}(\Omega))}, \|c_1\|_{L_{s_1}(0,T;L_{r_1}(\Omega))}, \|e\|_{L_\infty(0,T;L_{\tilde{q}}(\Omega))}, \|a\|_{L_\infty(0,T;L_{n-1}(\partial\Omega))},$$
$$\|u_0\|_{L_{\alpha+1}(Q_T)}, \|u_0\|_{L_2(0,T;L_{\frac{2n-2}{n-2}}(\partial\Omega))}, \|Du_0\|_{L_2(Q_T)}, k_1, T, \operatorname{mes}\Omega)$$

and ε is small enough. □

Hence, $\frac{\Psi(\|\tilde{u}\|)}{\|\tilde{u}\|} \nearrow \infty$ as $\|\tilde{u}\|_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)} \nearrow \infty$.

CONTINUATION OF THE PROOF OF THEOREM 5.1. We can apply Theorem 3.2 to problem (1.1)–(1.3) from Lemma 5.2 and Lemma 5.3. Hence we obtain that problem (3.1)–(3.3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ satisfying the following inequality

$$\sup\left\{ \frac{1}{\|u\|_{L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)}} \int_0^T \langle h, u \rangle_\Omega + \langle \varphi, u \rangle_{\partial\Omega} \, dt : \right.$$
$$\left. u \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \right\} < \infty.$$

[7] $c_6\|u\|^2_{(W_2^1(\Omega))^*} \leq \|u\|^2_{L_2(\Omega)}$

If we consider the norm definition of (h, φ) in $[L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$, we see that problem (3.1)–(3.3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ and $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$. Therefore (1.1)–(1.3) is also solvable. □

6. Uniqueness theorem for problem (1.1)–(1.3)

Theorem 6.1. *Let existence theorems in sections* 3, 4 *and* 5 *be fulfilled and moreover assume the following conditions:*

(3)′ $e \in L_\infty(0,T;L_2(\Omega))$

(9) *Let* $g(x,t,\xi)$ *is differentiable with respect to* ξ *and* $g_\xi \in L_\infty(0,T;L_{\frac{n}{2}}(\Omega))$, *moreover there exists a positive number* g_0 *such that* $g_\xi(x,t,\xi) \geq -g_0$ *for a.e.* $(x,t) \in Q_T$ *and for all* $\xi \in \mathbb{R}$.

Then the solution of (1.1)–(1.3) *in* P_0 *is unique.*

Proof. Let define $w := u_1 - u_2$ assuming that u_1 and u_2 are two different solutions of (1.1)–(1.3). Then we can obtain the following problem:

$$\frac{\partial w}{\partial t} - \Delta w + [g(x,t,u_1) - g(x,t,u_2)] + e(x,t)(\|u_1\|_{L_2(\Omega)} - \|u_2\|_{L_2(\Omega)})(t) = 0 \tag{6.1}$$

$$w(x,0) = 0 \tag{6.2}$$

$$\left(\frac{\partial w}{\partial \eta} + a(x',t)w\right)\Big|_{\Sigma_T} = 0 \tag{6.3}$$

After multiplying (6.1) by w under the integral of Ω, if we use the conditions of Theorem 6.1 and make some calculations, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L_2(\Omega)} \leq (g_0 + \|e\|_{L_\infty(0,T;L_2(\Omega))} + a_0 c_4)\|w\|^2_{L_2(\Omega)}.$$

By solving the last inequality,

$$\|w\|^2_{L_2(\Omega)}(t) \leq \|w(0)\|^2_{L_2(\Omega)} \exp\{2[g_0 + \|e\|_{L_\infty(0,T;L_2(\Omega))} + a_0 c_4]t\} \tag{6.4}$$

is obtained. Since $w(0) = 0$, the solution is unique. □

7. Behavior of solution for problem (1.1)–(1.3)

We investigate the behavior of solution for problem (1.1)–(1.3) in two subsections as homogenous case and nonhomogenous, autonomous case.

7.1. Homogenous Case ($h(x,t)=0, \varphi(x',t)=0$). Let $h(x,t)=0, \varphi(x',t)=0$ for problem (1.1)–(1.3) and assume the following conditions:

(d_1) $a \in L_\infty(0,T;L_{n-1}(\partial\Omega))$, $e \in L_\infty(0,T;L_2(\Omega))$.
(d_2) Condition (1) is satisfied with $s_1 := \infty$, $r_1 := \infty$, $s_2 := \frac{\alpha+1}{\alpha}$, $r_2 := \frac{\alpha+1}{\alpha}$ and $\alpha > 1$.
(d_3) Condition (7) is satisfied with $k_1 = 0$.
(d_4) There exists a number $a_0 > 0$ such that $a(x',t) \geq a_0$ for a.e. $(x',t) \in \Sigma_T$.

THEOREM 7.1. *Let* (d_1)–(d_4) *be fulfilled. Then the solution in P_0 satisfies the following inequalities for all $t \geq 0$:*

(a) $$\|u\|^2_{L_2(\Omega)}(t) \leq \frac{2\|u_0\|^2_{L_2(\Omega)}}{\left[2^{\frac{\alpha-1}{2}}\exp\left\{\frac{1-\alpha}{2}K_1 t\right\}+\frac{K_2}{K_1}\|u_0\|^{\alpha-1}_{L_2(\Omega)}\left(\exp\left\{\frac{1-\alpha}{2}K_1 t\right\}-1\right)\right]^{\frac{2}{\alpha-1}}},\ K_1 \neq 0$$

(b) $$\|u\|_{L_2(\Omega)}(t) \leq \frac{\|u_0\|_{L_2(\Omega)}}{\left[2^{-\frac{1+\alpha}{2}}(1-\alpha)K_2 t\|u_0\|^{\alpha-1}_{L_2(\Omega)}+1\right]^{\frac{1}{\alpha-1}}},\ K_1 = 0$$

Here $K_1 = K_1(\|e(x,t)\|_{L_\infty(0,T;L_2(\Omega))}, c_2, a_0)$ is a constant and $K_2 = K_2(k_0, c_7)$ is a negative constant (c_2 and c_7 are constants coming from the inequalities[8] [19], [1]).

PROOF. Conditions of Theorem 7.1 provide that (1.1)–(1.3) has a solution in P_0. Let make use of Lyapunov functional $E(u(t)) := \frac{1}{2}\int_\Omega u^2\,dx$. If we write the equality $E'(t) = \langle u, u_t\rangle_\Omega$ for the solution, then we obtain the following Cauchy problem by using conditions of the theorem and making some calculations:

$$y' - K_1 y \leq K_2 y^{\frac{\alpha+1}{2}} \tag{7.1}$$

$$y(0) = \frac{1}{2}\|u_0\|^2_{L_2(\Omega)} \tag{7.2}$$

where $y := E(t)$, $K_1 := 2(\|e\|_{L_\infty(0,T;L_2(\Omega))} - \theta_0 c_2)$ and $K_2 := -2^{\frac{\alpha+1}{2}} k_0 c_7^{-1}$.

From here follows as the solution of (7.1)–(7.2) inequality (a) under assuming $K_1 \neq 0$ and inequality (b) under assuming $K_1 = 0$. □

COROLLARY 7.2. *If $u_0 = 0$, then the solution is zero regardless of the sign of K_1. Moreover if $\|u_0\|_{L_2(\Omega)}$ is bounded then $\|u(t)\|_{L_2(\Omega)}$ bounded for all $t > 0$.*

COROLLARY 7.3. *If $K_1 > 0$ then $\|u\|^2_{L_2(\Omega)}(t) \leq M_0^2$ as $t \to \infty$, where $M_0^2 = 2\left(-\frac{K_1}{K_2}\right)^{\frac{2}{\alpha-1}}$. Moreover, $\|u\|^2_{L_2(\Omega)}(t) \leq M_0^2$ when $\|u_0\|^2_{L_2(\Omega)} \leq M_0^2$ for all $t > 0$.*

COROLLARY 7.4. *If $K_1 < 0$, $\|u\|^2_{L_2(\Omega)}(t) \leq \|u_0\|^2_{L_2(\Omega)}$ is satisfied for all $t > 0$.*

COROLLARY 7.5. *If $K_1 \leq 0$, the solution goes to zero as $t \to \infty$ regardless of initial function u_0.*

[8] $c_2\|u\|^2_{W_2^1(\Omega)} \leq (\|Du\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\partial\Omega)})$; $\|u\|_{L_2(\Omega)} \leq c_7\|u\|_{L_{\alpha+1}(\Omega)}$

7.2. Nonhomogenous and Autonomous Case. Let consider problem (1.1)–(1.3) in autonomous case with h and φ are different from zero as the following:

$$\frac{\partial u}{\partial t} - \Delta u + g(x,u) + e(x)\|u\|_{L_2(\Omega)}(t) = h(x) \tag{7.3}$$

$$u(x,0) = u_0 \tag{7.4}$$

$$\left(\frac{\partial u}{\partial \eta} + a(x')u\right)\Big|_{\Sigma_T} = \varphi(x'), \tag{7.5}$$

We can rewrite the conditions to provide that (7.3)–(7.5) has a unique solition:

(1_0) $g(x,u)$ satisfies $(1)'''$ with $c_1 \in L_{r_1}(\Omega)$ and $c_0 \in L_{r_2}(\Omega)$.
(2_0) $a \in L_{n-1}(\partial\Omega)$
(3_0) $e \in L_2(\Omega)$
(7_0) There exist some numbers $k_0 > 0$, $k_1 \in \mathbb{R}^1$ such that

$$g(x,\xi)\xi \geq k_0|\xi|^{\alpha+1} - k_1$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}$.
(8_0) There exists a number $a_0 > 0$ such that $a(x',t) \geq -a_0$ for a.e. $x' \in \partial\Omega$ and $0 < a_0 \leq \frac{\theta_4}{c_4}$ where $\theta_4 < \min\{\widetilde{b_0}, \widetilde{k_0}\}$ with $0 \ll \widetilde{b_0} < 1$ and $0 \ll \widetilde{k_0} < k_0$. ($c_4$ is constant[9] [1].)
(9_0) Condition (9) is satisfies with $g_\xi \in L_{\frac{n}{2}}(\Omega)$.

THEOREM 7.6. *Let conditions* (1_0), (2_0), (3_0), (7_0), (8_0) *and* (9_0) *be fulfilled and let* $h \in (W_2^1(\Omega))^* + L_{\frac{\alpha+1}{\alpha}}(\Omega)$, $\varphi \in W_2^{-\frac{1}{2}}(\partial\Omega)$. *Then for all bounded set* $B \subset W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ *and for any fixed number* $\rho > 0$, *there exists a* $t_0 = t_0(B,\rho) > 0$ *such that* $S(t)B \subset B_0^\rho$ *when* $t \geq t_0$. *Here*

$$B_0^\rho := \{u \in L_2(\Omega) : \|u\|_{L_2(\Omega)}(t) \leq (M_1 + \rho)^{\frac{1}{2}},$$
$$M_1 = M_1(\|h\|, \|\varphi\|, \|e\|, k_1, k_0, a_0, c_4, c_7, c, \operatorname{mes}\Omega)\}.$$

That means, there exists an absorbing set in $L_2(\Omega)$ *for* $\{S(t)\}_{t\geq 0}$. (c *is constant*[10], c_4 *and* c_7 *are constants*[11] [1].)

PROOF. This proof is similarly with the proof of Theorem 7.1 as making use of Lypunov functional $E(u(t)) := \frac{1}{2}\int_\Omega u^2\,dx$ and equality $E'(t) = \langle u, u_t\rangle_\Omega$. By using conditions of the theorem and making some calculations, we obtain the following Cauchy problem:

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[9] $\|u\|^2_{L_2(\partial\Omega)} \leq c_4\|u\|^2_{W_2^1(\Omega)}$

[10] $\|u\|^2_{W_2^1(\Omega)} \leq \|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)} + \|Du\|^2_{L_2(\Omega)} + c$

[11] $\|u\|^2_{L_2(\partial\Omega)} \leq c_4\|u\|^2_{W_2^1(\Omega)}$; $\|u\|_{L_2(\Omega)} \leq c_7\|u\|_{L_{\alpha+1}(\Omega)}$

$$\frac{dy}{dt} + M_2 y \leq \tilde{M}_3$$
$$y(0) = \|u_0\|^2_{L_2(\Omega)}$$

here $y := E(t)$, with $0 < M_2 \leq 2(\theta - a_0 c_4 - \varepsilon)$ for $\theta := \min\{1, \widetilde{k_0}\}$ and sufficiently small ε and $0 < \tilde{M}_3(\|h\|, \|\varphi\|, \|e\|, k_1, \theta_1, c, c_7, \operatorname{mes}\Omega)$. Solving this problem,

$$\|u\|^2_{L_2(\Omega)}(t) \leq \|u_0\|^2_{L_2(\Omega)} \exp\{-M_2 t\} + M_1,$$

is obtained where $M_1 := M_1(\|h\|, \|\varphi\|, \|e\|, k_1, \theta, c, c_7, \operatorname{mes}\Omega, M_2)$.

Then there exists set

$$B_0 := \{u \in L_2(\Omega) : \|u\|_{L_2(\Omega)}(t) \leq M_1^{\frac{1}{2}},$$
$$M_1 = M_1(\|h\|, \|\varphi\|, \|e\|, k_1, k_0, a_0, c_4, c_7, c, \operatorname{mes}\Omega)\}$$

when $t \to \infty$.

If we search the values of t which satisfies

$$\|u_0\|^2_{L_2(\Omega)} \exp\{-M_2 t\} \leq \rho, \tag{7.6}$$

then we find that (7.6) is satisfied $\forall t \geq t_0$, $t_0 := \frac{1}{M_2} \ln\left(\frac{\|u_0\|^2_{L_2(\Omega)}}{\rho}\right)$. That means

$$\|S(t)u_0\|_{L_2(\Omega)} \leq (\rho + M_1)^{\frac{1}{2}}, \quad \forall t \geq t_0.$$

Hence, $B_0^\rho := \{u \in L_2(\Omega) : \|u\|_{L_2(\Omega)} \leq (M_1 + \rho)^{\frac{1}{2}}\}$ is an absorbing set in $L_2(\Omega)$ for $\{S(t)\}_{t \geq 0}$. □

From previous results we obtain that for all $u_0 \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ and $T > 0$ there exists a continuous mapping $u_0 \to u(t)$ that determine the solution of the problem in P_0. Denoting this mapping by $S(t)$ we get, that problem (7.3)–(7.5) defines a semiflow $\{S(t)\}_{t \geq 0}$ with $S(t)u_0 = u(t)$. Now we will show that under some complementary conditions on the dates of the posed problem if its solutions from P_0 possess some smoothness then this semiflow will act on $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ and possess an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$.

THEOREM 7.7. *Let* (1_0), (2_0), (3_0), (7_0), (9_0) *and the following conditions be satisfied for problem* (7.3)–(7.5):

(A_1) *There exists a number* $a_0 > 0$ *such that* $a(x') \geq a_0 > 0$ *for a.e.* $x' \in \partial\Omega$.

(A_2)

$$0 < (\alpha + 1)G(x, \tau) \leq g(x, \tau)\tau, \quad \tau \in \mathbb{R} - \{0\}$$

is satisfied for $G(x, u) := \int_0^u g(x, \vartheta)\, d\vartheta$ *with* $g(x, 0) = 0$.

(A_3) $(h, \varphi) \in L_2(\Omega) \times L_2(\partial\Omega)$

Then, if the solution such that $u_t \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q)$, *then there exists an absorbing set in* $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ *for* $\{S(t)\}_{t\geq 0}$*: for all bounded set* $B \subset W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ *and for any fixed number* $\rho > 0$, *there exists a* $t_0 = t_0(B,\rho) > 0$ *such that* $S(t)B \subset B_1^\rho$ *when* $t \geq t_0$.

Here $B_1^\rho := \{u \in W_2^1(\Omega) \cap L_{\alpha+1}(\Omega) : \|u\|_{W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)}(t) \leq \tilde{M}_1^{\frac{1}{2}}\}$

($\tilde{M}_1 = \tilde{M}_1(M_1,\rho,k_3,k_2,c_2)$ *with constants* $k_2 > 0$, $k_3 \in \mathbb{R}^1$*; moreover* c_2 *is coming from the inequality*[12] [1])

PROOF. As considering that u is a smooth function enough, let multiply 7.3 with u_t under the integral Ω.

$$\|u_t\|_{L_2(\Omega)}^2 - \int_\Omega u_t \Delta u\,dx + \int_\Omega g(x,u)u_t\,dx + \|u\|_{L_2(\Omega)}(t)\int_\Omega e(x)u_t\,dx = \int_\Omega h(x)u_t\,dx.$$

Using integration by parts with (7.5), Hölder inequality and (A_2), we obtain the following inequality:

$$\begin{aligned}\|u_t\|_{L_2(\Omega)}^2 &+ \frac{1}{2}\frac{d}{dt}\|Du\|_{L_2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} a(x')u^2\,dx' + \frac{d}{dt}\int_\Omega G(x,u)\,dx \\ &- \frac{d}{dt}\int_\Omega h(x)u\,dx - \frac{d}{dt}\int_{\partial\Omega}\varphi(x')u\,dx' \leq \|e\|_{L_2(\Omega)}\|u\|_{L_2(\Omega)}\|u_t\|_{L_2(\Omega)}.\end{aligned}$$

Here let denote

$$\begin{aligned}\frac{d}{dt}F(t) := \frac{1}{2}\frac{d}{dt}\|Du\|_{L_2(\Omega)}^2 &+ \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} a(x')u^2\,dx' + \frac{d}{dt}\int_\Omega G(x,u) \\ &- \frac{d}{dt}\int_\Omega h(x)u\,dx - \frac{d}{dt}\int_{\partial\Omega}\varphi(x')u\,dx'.\end{aligned}$$

Now if we consider Theorem 7.6 after applying Young inequality to the right hand side of the last inequality, we get that there exists a $t_0(\rho) > 0$ such that for all $t \geq t_0$

$$(1-\varepsilon_1)\|u_t\|_{L_2(\Omega)}^2 + \frac{d}{dt}(F(t)) \leq c(\varepsilon_1)\|e\|_{L_2(\Omega)}^2(M_1+\rho).$$

Choosing ε_1 positive number is less than 1, we obtain that $\frac{d}{dt}(F(t))$ is bounded for all $t \geq t_0$.

On the other hand, as multiplying (7.3) with u under the integral Ω and using integration by parts, Hölder inequality and (A_2), we obtain

[12] $c_2\|u\|_{W_2^1(\Omega)}^2 \leq (\|Du\|_{L_2(\Omega)}^2 + \|u\|_{L_2(\partial\Omega)}^2)$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)} + \|Du\|^2_{L_2(\Omega)} + \int_{\partial\Omega} a(x')u^2\,dx' + (\alpha+1)\int_\Omega G(x,u)\,dx$$
$$- \int_\Omega h(x)u\,dx - \int_{\partial\Omega} \varphi(x')u\,dx' \le -\|u\|_{L_2(\Omega)}(t)\int_\Omega e(x)u\,dx.$$

Now considering the proof of Theorem 7.6 we have:

$$F(t) \le \|u\|_{L_2(\Omega)}(t)\int_\Omega |e(x)|\,|u|\,dx + \left|\frac{1}{2}\frac{d}{dt}\|u\|^2_{L_2(\Omega)}\right|$$
$$\le (M_1+\rho)[\|e\|_{L_2(\Omega)} + M_2] + \tilde{M}_3.$$

Then we obtain that

$$F(t) \le L \tag{7.7}$$

where $L := (M_1+\rho)[\|e\|_{L_2(\Omega)} + M_2] + \tilde{M}_3$ for all $t \ge t_0$.

So we obtain that $\frac{d}{dt}F(t)$ and $F(t)$ are bounded by some numbers that are independent at t if $t \ge t_0$ defined in previous result. Consequently $F(t)$ is a bounded function for $t \ge t_0$. Now if we use condition (A_1) and define $\theta := \min\{1, a_0 - \varepsilon_2\}$ with $0 < \varepsilon_2 < a_0$, then the following inequality is obtained from (7.7) for all $t \ge t_0$:

$$\theta c_2\|u\|^2_{W_2^1(\Omega)}(t) + 2\int_\Omega G(x,u)\,dx$$
$$\le 2L + c(\varepsilon_2)\|\varphi\|^2_{L_2(\partial\Omega)} + \varepsilon_3(M_1+\rho) + c(\varepsilon_3)\|h\|^2_{L_2(\Omega)}.$$

Since we have the fact from (A_2) that $G(x,u) \ge k_2|u|^{\alpha+1} - k_3$ for some numbers $k_2 > 0$, $k_3 \in \mathbb{R}^1$,

$$\theta c_2\|u\|^2_{W_2^1(\Omega)}(t) + 2k_2\|u\|^{\alpha+1}_{L_{\alpha+1}(\Omega)}(t) - 2k_3 \operatorname{mes}\Omega$$
$$\le 2L + c(\varepsilon_2)\|\varphi\|^2_{L_2(\partial\Omega)} + \varepsilon_3(M_1+\rho) + c(\varepsilon_3)\|h\|^2_{L_2(\Omega)}, \quad t \ge t_0$$

is satisfied from the last inequality. Then we get,

$$\|u\|_{W_2^1(\Omega)\cap L_{\alpha+1}(\Omega)}(t) \le \tilde{M}_1^{\frac{1}{2}} \quad t \ge t_0 \tag{7.8}$$

where $\tilde{M}_1(M_1, \rho, k_3, k_2, c_2)$.

(7.8) says that there exists an absorbing set in $W_2^1(\Omega) \cap L_{\alpha+1}(\Omega)$ for $\{S(t)\}_{t\ge 0}$. □

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