*Rend. Lincei Mat. Appl.* 26 (2015), 465–473
DOI 10.4171/RLM/716

**Partial Differential Equations** — *Quasi-filling fractal layers*, by RAFFAELA CAPITANELLI and MARIA AGOSTINA VIVALDI, communicated on 11 June 2015.[1]

ABSTRACT. — We consider second order transmission problems across Koch-type curves formulated as boundary value problems for elliptic operators in a quasi-filling geometry for the fibers. We use a variational approach and the M-convergence methods. We prove that the solution of the transmission problem across a Koch-type curve is the limit of the solutions of suitable second order transmission problems across polygonal curves.

KEY WORDS: Fractal fibers, singular elliptic operators, variational convergence

MATHEMATICS SUBJECT CLASSIFICATION: 35J10, 35J75, 35P20, 28A80

## 1. INTRODUCTION

In this note, we investigate second order transmission problems across quasi-filling dynamical layers, from the variational point of view of the homogenization theory for elliptic operators in Euclidean domains.

More precisely, the fractal inclusion $\mathscr{K}_\alpha$ divides the domain $\Omega$ in two adjacent subdomains $\Omega_\alpha^j$, $j = 1, 2$, and the second order transmission problem across the curve $K_\alpha$ is formally stated as follows

$$\begin{cases} -\mathrm{div}(\nabla u) = f & \text{in } \Omega \backslash \mathscr{K}_\alpha \\ u^1 = u^2 & \text{on } \mathscr{K}_\alpha, \\ \left[\frac{\partial u}{\partial v}\right] = L_{\mathscr{K}_\alpha} u & \text{on } \mathscr{K}_\alpha \\ u = 0 & \text{on } \partial\Omega \\ u(P) = 0 & \text{on } \partial\mathscr{K}_\alpha. \end{cases} \tag{1.1}$$

We denote by $u^j$, $j = 1, 2$, the restriction of the function $u$ to the domains $\Omega_\alpha^j$ and by $\left[\frac{\partial u}{\partial v}\right]$ the jump of the normal derivative across $\mathscr{K}_\alpha$. Here $L_{\mathscr{K}_\alpha}$ is the self-adjoint operator defined by the intrinsic energy form on $\mathscr{K}_\alpha$ in the space $L^2(\mathscr{K}_\alpha, \mu_{\mathscr{K}_\alpha})$, where $\mu_{\mathscr{K}_\alpha}$ is the $d_H$-Hausdorff measure on $\mathscr{K}_\alpha$ (see (2.3)).

The interaction between the interior operator and the layer operator, both of second order, captures the main global dynamical features of the problem at hand. Moreover, the dimensional relationship between the domain and the layer is *unusual* when the layer's geometry is fractal. We have an open two-dimensional

[1] This paper is related to a talk given by M. A. Vivaldi at "XXV Convegno Nazionale di Calcolo delle Variazioni"—Levico, 2–6 February 2015.

Euclidean domain $\Omega$ in $\mathbb{R}^2$ with a layer, $\mathscr{K}_\alpha$, which has a Hausdorff dimension $d_H$ that is any number $(1 < d_H < 2)$ close to 2, as we wish, and which fills a two-dimensional open subset of $\Omega$ up to an arbitrarily small set.

In this note, we state that the variational solution of problem (1.1) is the limit of the solutions of suitable second order transmission problems across approximating polygonal curves. More precisely, we construct a sequence of energy forms that are sums of bulk energies in a bounded open domain $\Omega$ of the plane and layer energies on polygonal curves $K_\alpha^n$. The coefficient matrices $a^n Id$ of the bulk energies are adjusted to the geometry of a system of open thin fibers $\Sigma_{\varepsilon_n}^n$ and $\Sigma_{2\varepsilon_n}^n$, which is moved in $\Omega$ as $n \to +\infty$ by the iterated action of the contractive self-similarities of the fractal inclusion $\mathscr{K}_\alpha$ in $\Omega$. The external fibers have increasingly low conductivity. Under appropriate scaling assumptions, we prove that energy forms (sums of bulk energies in $\Omega$ and layer energies on the polygonal curves $K_\alpha^n$) converge to the energy form sum of the Dirichlet integral in $\Omega$ and layer energy on the Koch curve $\mathscr{K}_\alpha$, as $n \to +\infty$.

The second order transmission problems across *the equilateral Koch curve*, that is $\mathscr{K}_\alpha$ for $\alpha = 3$, has been studied in [7], [8] and [13]. More precisely in [8] it is proved that the variational solution to problem (1.1) is the strong limit in $H_0^1(\Omega)$ of the solutions to the second order transmission problems across polygonal curves with the same bulk energy as the limit problem. On the other hand, in the framework of the singular homogenization, in [13], it is proved that the variational solution to problem (1.1) is the strong limit in $H_0^1(\Omega)$ of the solutions to minimum problems for functionals of *bulk energy alone*. An extension theorem, which is rooted in the geometrical properties *of the equilateral Koch curve* (see [3]), plays a crucial role in both these results. By contrast, in the present paper the quasi-filling geometry of our dynamical layers only allows us to prove the weak convergence in $H_{loc}^1(\Omega \backslash \mathscr{K}_\alpha)$. For the proofs we refer to the forthcoming paper [2].

## 2. The geometry

By $\Omega$ of $\mathbb{R}^2$ we denote the open rectangle with vertices $D = (0, -1)$, $E = (1, -1)$, $F = (1, 1)$ and $G = (0, 1)$. The domain $\Omega$ contains the segment of end points $A = (0, 0)$ and $B = (1, 0)$. The fractal inclusion $\mathscr{K}_\alpha$ is the invariant (self-similar) compact set of $\mathbb{R}^2$ associated with the family $\Psi = \{\psi_1, \ldots, \psi_4\}$ of 4 similarities in $\mathbb{R}^2$, which are contraction maps in $\mathbb{R}^2$ with a common contraction factor $\alpha^{-1}$, $\alpha \in (2, 4)$:

$$\begin{cases} \psi_1(z) = \frac{z}{\alpha}, \quad \psi_2(z) = \frac{z}{\alpha} e^{i\vartheta} + \frac{1}{\alpha}, \\ \psi_3(z) = \frac{z}{\alpha} e^{-i\vartheta} + \frac{1}{2} + \frac{i \sin\vartheta}{\alpha}, \quad \psi_4(z) = \frac{z + \alpha - 1}{\alpha}, \end{cases} \tag{2.1}$$

where

$$\vartheta = \arcsin\sqrt{\alpha - \frac{\alpha^2}{4}} \in \left(0, \frac{\pi}{2}\right) \tag{2.2}$$

and $z = x + iy \in \mathbb{C}$. The set of the essential fixed points of this family is $\Gamma = \{A, B\}$ and we set $\partial\mathscr{K}_\alpha = \Gamma$.

The invariant (self-similar) regular Borel measure in $\mathbb{R}^2$ supported on $\mathscr{K}_\alpha$, associated with $\Psi$, is given by

$$\mu = \mu_{\mathscr{K}_\alpha} := \frac{\mathbf{1}_{\mathscr{K}_\alpha}\mathscr{H}^d}{\mathscr{H}^d(\mathscr{K}_\alpha)}, \tag{2.3}$$

where

$$d = d_H = \ln 4/\ln\alpha \tag{2.4}$$

is the Hausdorff dimension of $\mathscr{K}_\alpha$ and $\mathscr{H}^d$ is the Hausdorff measure of dimension $d$ in $\mathbb{R}^2$. In particular,

$$\mu(\psi_{i|n}(\mathscr{K}_\alpha)) = \frac{1}{N^n}\mu(\mathscr{K}_\alpha) = \frac{1}{4^n} \tag{2.5}$$

where $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$ if $n > 0$. For these properties, and for the related theory of so-called nested fractals associated with similarity maps like the one considered here, we refer to Hutchinson [5], and Lindstrøm [9].

In the domain $\Omega$, we introduce a reference two-layer fiber. This fiber is made of two co-axial thin hexagons

$$\Sigma_\varepsilon^0 \subset \Sigma_{2\varepsilon}^0$$

whose largest transversal size is $\varepsilon > 0$ and $2\varepsilon$, respectively. The common axis of the fibers is the segment connecting points $A = (0,0)$ and $B = (1,0)$. We put $\varepsilon_0 = h_0/2$, $h_0 = \tan(\vartheta^*)$

$$\vartheta^* = \min\{\pi/2 - \vartheta, \vartheta/2\} \tag{2.6}$$

where $\vartheta$ is the rotation angle of the similarities that generate the Koch curve $\mathscr{K}_\alpha$ (see (2.2)).

For every $0 < \varepsilon \le \varepsilon_0$, we define the thin fiber $\Sigma_{2\varepsilon}^0$ to be the hexagon with vertices, listed clockwise, $A$, $Q_1(\varepsilon)$, $Q_2(\varepsilon)$, $B$, $Q_3(\varepsilon)$, $Q_4(\varepsilon)$, where $Q_1(\varepsilon) = (\varepsilon/h_0, \varepsilon)$, $Q_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon)$, $Q_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon)$, $Q_4(\varepsilon) = (\varepsilon/h_0, -\varepsilon)$. The perimeter of hexagon $\Sigma_{2\varepsilon}^0$ gives the external profile of our two-layer fiber. Inside hexagon $\Sigma_{2\varepsilon}^0$, we now insert a smaller hexagon $\Sigma_\varepsilon^0$. We define the thin fiber $\Sigma_\varepsilon^0$ to be the hexagon with vertices, listed clockwise $A$, $P_1(\varepsilon)$, $P_2(\varepsilon)$, $B$, $P_3(\varepsilon)$, $P_4(\varepsilon)$, where $P_1(\varepsilon) = (\varepsilon/h_0, \varepsilon/2)$, $P_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon/2)$, $P_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon/2)$, $P_4(\varepsilon) = (\varepsilon/h_0, -\varepsilon/2)$.

We now iteratively transform the arrays $\Sigma_\varepsilon^0$ into increasingly fine arrays, by the action, for each integer $n > 0$ of the maps $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$ associated with arbitrary $n$-tuples of indices $i\,|\,n = (i_1, i_2, \ldots, i_n) \in \{1, \ldots, 4\}^n$. If $n = 0$, we define $\psi_{i|n}$ to be the identity map in $\mathbb{R}^2$. For every set $\mathscr{O} \subseteq \mathbb{R}^2$, we define $\mathscr{O}^{i|n} = \psi_{i|n}(\mathscr{O})$, and, occasionally, we call $i\,|\,n$ the $n$-address of the set $\mathscr{O}^{i|n}$. With this notation, for every $\varepsilon$ and every $n \geq 0$, we then define the arrays of co-axial fibers $\Sigma^n_{2\varepsilon}$ and $\Sigma^n_\varepsilon$ by setting

$$\Sigma^n_{2\varepsilon} = \bigcup_{i|n} \Sigma^{i|n}_{2\varepsilon}, \quad \Sigma^{i|n}_{2\varepsilon} = \psi_{i|n}(\Sigma^0_{2\varepsilon}), \tag{2.7}$$

$$\Sigma^n_{\varepsilon} = \bigcup_{i|n} \Sigma^{i|n}_{\varepsilon}, \quad \Sigma^{i|n}_{\varepsilon} = \psi_{i|n}(\Sigma^0_{\varepsilon}). \tag{2.8}$$

Let $K^0$ be the line segment of unit length whose endpoints are $A = (0,0)$ and $B = (1,0)$. We set, for each $n$ in $\mathbb{N}$,

$$K^1_\alpha = \bigcup_{i=1}^{4} \psi_i(K^0), \quad K^n_\alpha = \bigcup_{i|n} K^{i|n}_\alpha, \quad K^{i|n}_\alpha = \psi_{i|n}(K^0); \tag{2.9}$$

$K^n_\alpha$ is the so-called $n$-th pre-fractal curve. We recall that the curves $K^n_\alpha$ converge to the curve $\mathscr{K}_\alpha$ in the Hausdorff metric [5].

## 3. Main result

Before stating our results, we recall some definitions and properties that refer to the fractal $\mathscr{K}_\alpha$. The fractal $\mathscr{K}_\alpha$ is the closure in $\mathbb{R}^2$ of the set $V^\infty = \bigcup_{n=0}^{+\infty} V^n$ where for every $n \geq 0$

$$V^n = \bigcup_{i|n} \psi_{i|n}(\Gamma). \tag{3.1}$$

The fractal energy $\mathscr{E}[u] = \mathscr{E}_{\mathscr{K}_\alpha}[u]$ is the limit of the increasing sequence

$$\mathscr{E}[u] = \lim_{n\to+\infty} \mathscr{E}^n[u], \tag{3.2}$$

with

$$\mathscr{E}^n[u] = 4^n \sum_{i|n} (u(\psi_{i|n}(A)) - u(\psi_{i|n}(B)))^2, \tag{3.3}$$

on the domain

$$D[\mathscr{E}] = D[\mathscr{E}_{\mathscr{K}_\alpha}] = \left\{ u \in C(\mathscr{K}_\alpha) \mid \sup_{n\geq 0} \mathscr{E}^n[u|_{V^n}] < +\infty \right\}. \tag{3.4}$$

$D[\mathscr{E}] \subset C^\beta(\mathscr{K}_\alpha)$ and the estimate

$$|u(P) - u(Q)| \leq C_H \sqrt{\mathscr{E}[u]} |P - Q|^\beta \tag{3.5}$$

with $\beta = \frac{\ln 4}{2 \ln \alpha}$ holds for every $P, Q \in \mathscr{K}_\alpha$. For these Hölder estimates, we refer to Kozlov [6] (see also [12], where Kozlov's result is interpreted as an intrinsic Morrey's imbedding).

The *Dirichlet* operator $L_{\mathscr{K}_\alpha}$ on $\mathscr{K}_\alpha$ is the self-adjoint operator $L_{\mathscr{K}_\alpha}$ defined by the identity

$$\mathscr{E}_{\mathscr{K}_\alpha}(u, v) = \int_{\mathscr{K}_\alpha} L_{\mathscr{K}_\alpha} uv \, d\mu \quad \forall u \in D[L_{\mathscr{K}_\alpha}], \quad v \in D_0[\mathscr{E}] \tag{3.6}$$

where the domain $D_0[\mathscr{E}] = \{u \in D[\mathscr{E}] : u(P) = 0 \; \forall P \in \Gamma\}$ is dense in $L^2(\mathscr{K}_\alpha, \mu)$.

We define the functional $\mathscr{F} : L^2(\Omega) \mapsto (-\infty, +\infty]$, by

$$\mathscr{F}[u] = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 \, dx \, dy + \mathscr{E}_{\mathscr{K}_\alpha}(u|_{\mathscr{K}_\alpha}, u|_{\mathscr{K}_\alpha}) & \text{if } u \in D_0[\mathscr{F}] \\ +\infty & \text{if } u \in L^2(\Omega) \backslash D_0[\mathscr{F}] \end{cases} \tag{3.7}$$

where $D_0[\mathscr{F}] = \{u \in H_0^1(\Omega), u|_{\mathscr{K}_\alpha} \in D_0[\mathscr{E}]\}$. We note that the domain $D_0[\mathscr{F}]$ is a Hilbert space with respect to the norm

$$\|u\|_{D_0[\mathscr{F}]} = \left\{ \int_\Omega |\nabla u|^2 \, dx \, dy + \mathscr{E}[u] \right\}^{\frac{1}{2}}. \tag{3.8}$$

For any given $f \in L^2(\Omega)$, the function $u$ that minimizes on $D_0[\mathscr{F}]$ the energy functional

$$\mathscr{F}[u] - 2 \int_\Omega fu \, dx \, dy$$

formally solves the following boundary value problem

$$\begin{cases} -\mathrm{div}(\nabla u) = f & \text{in } \Omega \backslash \mathscr{K}_\alpha \\ u^1 = u^2 & \text{on } \mathscr{K}_\alpha, \\ \left[\frac{\partial u}{\partial \nu}\right] = L_{\mathscr{K}_\alpha} & \text{on } \mathscr{K}_\alpha \\ u = 0 & \text{on } \partial\Omega \\ u(P) = 0 & \text{on } \Gamma. \end{cases} \tag{3.9}$$

The fractal inclusion $\mathscr{K}_\alpha$ divides the domain $\Omega$ into two adjacent subdomains $\Omega_\alpha^j$, $j = 1, 2$ (where $\Omega_\alpha^1$ is the domain above the curve $\mathscr{K}_\alpha$), and we denote by $u^j$ the restriction of $u$ to the subdomain $u_{|_{\Omega_\alpha^j}}$, $j = 1, 2$. By $\left[\frac{\partial u}{\partial \nu}\right] = \frac{\partial u^1}{\partial \nu^1} + \frac{\partial u^2}{\partial \nu^2}$, we denote the jump of the normal derivative (inward to $\Omega_\alpha^1$) across $\mathscr{K}_\alpha$, and $L_{\mathscr{K}_\alpha}$ is the self-adjoint operator defined in (3.6). As this strong formulation of the boundary value problem only serves an illustrative purpose in this paper, we do not go into the regularity and geometrical details that would be needed to rigorously formulate all the preceding equations and boundary conditions.

We prove that the variational solution to problem (3.9) is the limit of the solutions to suitable second order transmission problems across approximating polygonal curves. Our approach to asymptotic convergence is of variational nature and is based on the tools developed in [10] and [11]. More precisely, in $\Omega$ we now introduce a sequence of energy functionals.

For each given $n$, we introduce in $\Omega$ the fibers $\Sigma^n_\varepsilon$ and $\Sigma^n_{2\varepsilon}$ described in Section 2, by now choosing $\varepsilon = \varepsilon_n$. For every $n$, we define the functional $\mathscr{F}^n : L^2(\Omega) \mapsto (-\infty, +\infty]$ by

$$\mathscr{F}^n[u] = \begin{cases} \displaystyle\int_\Omega a^n(x,y)|\nabla u|^2\,dx\,dy + \sigma_n \int_{K^n_\alpha} |\nabla_\tau u|^2\,ds & \text{if } u \in D_0[\mathscr{F}^n] \\ +\infty & \text{if } u \in L^2(\Omega)\backslash D_0[\mathscr{F}^n] \end{cases} \tag{3.10}$$

where the coefficient matrix $a^n Id$ is defined at every $(x,y) \in \Omega$ by

$$a^n(x,y)Id = \mathbf{1}_{\Omega\backslash\Sigma^n_{2\varepsilon_n}}(x,y)Id + \chi_n \mathbf{1}_{\Sigma^n_{2\varepsilon_n}\backslash\Sigma^n_{\varepsilon_n}}(x,y)Id + \mathbf{1}_{\Sigma^n_{\varepsilon_n}}(x,y)Id, \tag{3.11}$$

with $\sigma_n > 0$, $\nabla_\tau u$ is the tangential derivative of $u$ along the polygonal curve $K^n_\alpha$. The domain $D_0[\mathscr{F}^n] = \{u \in H^1_0(\Omega) : u|_{K^n_\alpha} \in H^1_0(K^n_\alpha)\}$ is a Hilbert space with respect to the norm

$$\|u\|_{D_0[\mathscr{F}^n]} = \left\{ \int_\Omega a^n |\nabla u|^2\,dx\,dy + \sigma_n \int_{K^n_\alpha} |\nabla_\tau u|^2\,ds \right\}^{\frac{1}{2}}. \tag{3.12}$$

Here $H^1_0(\Omega)$ and $H^1_0(K^n_\alpha)$ denote the Sobolev spaces on the domain $\Omega$ and on the polygonal curve $K^n_\alpha$, and we refer to [1] for definitions and properties. In this description, the important parameters are the positive real numbers $\varepsilon_n > 0$ and $\chi_n > 0$, which will be specified later on, both converging to zero as $n \to +\infty$. The coefficients $a^n(x,y)$ present discontinuities across the fibers $\Sigma^n_{\varepsilon_n}$ and $\Sigma^n_{2\varepsilon_n}$. The external fiber $\Sigma^n_{2\varepsilon_n}\backslash\Sigma^n_{\varepsilon_n}$ is a region of increasingly low conductivity as $n \to +\infty$.

For any given $f \in L^2(\Omega)$, the function $u$ that minimizes on $D_0[\mathscr{F}^n]$ the energy functional

$$\mathscr{F}^n[u] - 2\int_\Omega fu\,dx\,dy$$

formally solves the following boundary value problem

$$\begin{cases} -\mathrm{div}(\nabla u) = f & \text{in } \Omega\backslash\Sigma^n_{2\varepsilon_n} \\ -\chi_n\,\mathrm{div}(\nabla u) = f & \text{in } \Sigma^n_{2\varepsilon_n}\backslash\Sigma^n_{\varepsilon_n} \\ -\mathrm{div}(\nabla u) = f & \text{in } \Sigma^n_{\varepsilon_n}\backslash K^n_\alpha \\ u^1 = u^2 & \text{on } K^n_\alpha, \\ \left[\frac{\partial u}{\partial \nu}\right] = \sigma_n \cdot \triangle_\tau u & \text{on } K^n_\alpha \\ u = 0 & \text{on } \partial\Omega \\ u(P) = 0 & \text{on } \Gamma \\ \text{natural transmission conditions on } \partial(\Sigma^n_{2\varepsilon_n}\backslash\Sigma^n_{\varepsilon_n}). \end{cases} \tag{3.13}$$

The polygonal curve $K_\alpha^n$ divides the domain $\Omega$ into two adjacent subdomains $\Omega_\alpha^{n,j}$, $j = 1, 2$ (where $\Omega_\alpha^{n,1}$ is the domain above the curve $K_\alpha^n$). Here by $u^j$ we denote the restriction of $u$ to the subdomain $\Omega_\alpha^{n,j}$, $j = 1, 2$. By $\left[\frac{\partial u}{\partial \nu}\right] = \frac{\partial u^1}{\partial \nu^1} + \frac{\partial u^2}{\partial \nu^2}$ we denote the jump of the normal derivative across $K_\alpha^n$, by $\nu^j$ the inward normal vector to the boundary of $\Omega_\alpha^{n,j}$ and by $\triangle_\tau u$ the tangential Laplacian of $u$ along the polygonal curve $K_\alpha^n$. As this strong formulation of the boundary value problem only serves an illustrative purpose in this paper, here we do not go into the regularity and geometrical details that would be needed to rigorously formulate all the preceding equations and boundary conditions.

Assume

$$\sigma_n = \left(\frac{4}{\alpha}\right)^n, \tag{3.14}$$

$$\frac{\chi_n}{\varepsilon_n} \to 0 \tag{3.15}$$

and

$$\frac{\chi_n}{(4^n \varepsilon_n \cdot \alpha^{-2n})^q} \geq C_0 \tag{3.16}$$

as $n \to +\infty$, with $C_0 > 0$ and $q \in \left(0, \frac{1}{2}\right)$. We note that the total area of the fibers $\Sigma_{2\varepsilon_n}^n$ behaves like $4^n \varepsilon_n \cdot \alpha^{-2n}$ as $n$ tends to $+\infty$.

REMARK 3.1. A possible choice of coefficients satisfying assumptions (3.15), (3.16) is:

$$\varepsilon = \varepsilon_n = 4^{-s_1 n}, \quad \chi_n = \frac{4^{-s_1 n}}{\ln n}, \tag{3.17}$$

where $0 < s_1 < 2 \ln \alpha / \ln 4 - 1$.

Our main result is the following

THEOREM 3.1. *We assume conditions* (3.14), (3.15) *and* (3.16). *Let $u$ be the variational solution of the fractal transmission problem* (3.9). *For every integer $n \geq 1$, let $u_n$ be the variational solution to the transmission problem* (3.13). *Then*

$$u_n \rightharpoonup u \quad \textit{weakly in } H^1_{loc}(\Omega \backslash \mathscr{K}_\alpha), \tag{3.18}$$

$$u_n \rightharpoonup u \quad \textit{weakly in } W^{1,p}(\Omega), \quad p = \frac{2}{q+1}, \tag{3.19}$$

$$(u_n)|_{\mathscr{K}_\alpha} \to u|_{\mathscr{K}_\alpha} \quad \textit{strongly in } B_s^{2,2}(\mathscr{K}_\alpha), \quad s < d/2 - q \tag{3.20}$$

$$\langle \sigma_n \triangle_\tau u_n, \phi_{I_n} \rangle_{(H^{-1}(K_\alpha^n), H_0^1(K_\alpha^n))} \to \langle L_{\mathscr{K}_\alpha} u, \phi \rangle_{((D_0[\mathscr{E}])', D_0[\mathscr{E}])} \quad \forall \phi \in D_0[\mathscr{F}], \tag{3.21}$$

*as $n \to +\infty$.*

By $\phi_{I_n}$ we denote the function continuous on $K_\alpha^n$ and affine on each side of $K_\alpha^n$ obtained by interpolating the values of $\phi$ at the vertices of $K_\alpha^n$. Here $d = \ln 4/\ln \alpha$ is the Hausdorff dimension of $\mathscr{K}_\alpha$. For the definition of the Besov spaces $B_s^{2,2}(\mathscr{K}_\alpha)$ we refer to [4].

Acknowledgements. The authors are members of GNAMPA (INdAM). The authors are partially supported by Grant Ateneo "Sapienza" 2014.

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Received 23 April 2015,
and in revised form 20 May 2015.

Raffaela Capitanelli
Dipartimento di Scienze di Base e Applicate per l'Ingegneria
SapienzaUniversità di Roma
Via A. Scarpa 16
00161 Roma, Italy
raffaela.capitanelli@uniroma1.it

Maria Agostina Vivaldi
Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Sapienza Università di Roma
Via A. Scarpa 16
00161 Roma, Italy
maria.vivaldi@sbai.uniroma1.it