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Calculus of Variations — *Regularity results for non-autonomous variational integrals with discontinuous coefficients*, by Antonia Passarelli di Napoli, communicated on 11 June 2015.[1]

Abstract. — We investigate the regularity properties of local minimizers of non autonomous convex integral functionals of the type

$$\mathscr{F}(u;\Omega) := \int_\Omega f(x, Du)\, dx,$$

with p-growth into the gradient variable and discontinuous dependence on the x variable. We prove a higher differentiability result for local minimizers of the functional $\mathscr{F}(u;\Omega)$ assuming that the function that measures the oscillation of the integrand with respect to the x variable belongs to a suitable Sobolev space.

Key words: Elliptic systems, discontinuous coefficients, higher differentiability

Mathematics Subject Classification: 49N15, 49N60, 49N99

1. Introduction

Classical multidimensional variational problems are related to the study of integral functionals of the type

$$\mathscr{F}(u;\Omega) := \int_\Omega f(x, Du)\, dx, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$, the integrand $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is such that $\xi \to f(\cdot, \xi)$ is a strictly convex function of class $C^1(\mathbb{R}^{n\times N})$ for almost every $x \in \Omega$ and satisfies the so called standard growth conditions, i.e.

$$\frac{1}{L}|\xi|^p \le f(x,\xi) \le L(1+|\xi|^p), \quad p > 1. \tag{F1}$$

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It is well known that $\mathscr{F}(u;\Omega)$ is lower semicontinuous and coercive on the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^{n\times N})$ and therefore admits a minimizer $u \in W^{1,p}(\Omega;\mathbb{R}^{n\times N})$.

The regularity properties of minimizers of integral functionals of the type (1.1) under standard growth conditions has been widely investigated in case the integrand $f(x,\xi)$ depends on the x-variable through a Hölder continuous function. Actually, the Hölder continuity of $f(x,\xi)$ with respect to x leads to the C^1 partial regularity of the minimizers with a quantitative modulus of continuity that can be determined in dependence on the modulus on continuity of the coefficients ([1, 2, 10, 16, 20, 26]). For an exhaustive treatment, we refer the interested reader to [19, 23] and the references therein.

It is worth pointing out that partial regularity results are a common feature when treating vectorial minimizers. Actually, in the vectorial setting everywhere regularity cannot be proven as it is shown by the counterexample due to De Giorgi and those due to Sverak and Yan ([12, 33, 34]).

In the last few years, the study of the regularity has been successfully carried out under weaker assumptions on the function that measures the continuity of the integrand $f(x,\xi)$ with respect to the x-variable. In particular, in [15] (see also [10, 14]), a $C^{0,\alpha}$ partial regularity result has been established relaxing the Hölder continuity with respect to x in a continuity assumption.

Further, the $C^{0,\alpha}$ partial regularity result of [15] has been extended in [5] and in [17] to operators that have discontinuous dependence on the x-variable, through a VMO coefficient and a Sobolev coefficient respectively (we refer to [25] for the regularity of the gradient of solutions of linear elliptic equations with VMO coefficients).

Our aim here is to investigate the regularity properties of the minimizers of integral functionals of the type (1.1), allowing a discontinuous dependence for the integrand $f(x,\xi)$ with respect to x-variable through a suitable Sobolev function.

More precisely, we shall assume that there exist constants $\ell, L, \nu > 0$ and an exponent $2 \le p \le n$ such that $f(x,\xi)$ satisfies the following assumptions:

$$\text{(F1)} \qquad \frac{1}{L}|\xi|^p \le f(x,\xi) \le L(1+|\xi|^p);$$

$$\text{(F2)} \qquad |D_\xi f(x,\xi) - D_\xi f(x,\eta)| \le \ell|\xi-\eta|(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}};$$

$$\text{(F3)} \qquad \nu(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \le \langle D_\xi f(x,\xi) - D_\xi f(x,\eta), \xi-\eta\rangle,$$

for every $\xi,\eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$. Concerning the dependence on the x-variable, we shall assume that there exists a function $k \in L^\sigma_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $1 < \sigma \le n$, such that

$$\text{(F4)} \qquad |D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le (|k(x)|+|k(y)|)|x-y|(1+|\xi|^{p-1}),$$

for every $\xi \in \mathbb{R}^{n\times N}$ and for almost every $x, y \in \Omega$.

By virtue of a characterization of the Sobolev functions due to Hajlasz ([24]), the function $k(x)$ plays the role of the derivative of the function $x \to D_\xi f(x,\xi)$.

So the assumption (F4) describes the continuity of the operator $D_\xi f(x,\xi)$ with respect to the x-variable. Obviously, this is a weak form of continuity since the function k may blow up at some points.

The model case we have in mind is

$$\mathscr{G}(u,\Omega)=\int_\Omega a(x)g(Du)\,dx,$$

where $g:\mathbb{R}^{n\times N}\to\mathbb{R}$ is a C^1 function for which there exist constants L_1,L_2,L_3, $\tilde{\nu}>0$ and an exponent $2\le p\le n$ such that

$$\frac{1}{L_1}|\xi|^p\le g(\xi)\le L_1(1+|\xi|^p); \tag{G1}$$

$$|D_\xi g(\xi)-D_\xi g(\eta)|\le L_2|\xi-\eta|(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}; \tag{G2}$$

$$\tilde{\nu}(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2\le\langle D_\xi g(\xi)-D_\xi g(\eta),\xi-\eta\rangle, \tag{G3}$$

for every $\xi,\eta\in\mathbb{R}^{n\times N}$. The coefficient $a(x)$, appearing in the integrand of the functional $\mathscr{G}(u;\Omega)$, belongs to the space $W^{1,\sigma}_{\mathrm{loc}}\cap L^\infty(\Omega)$, $1<\sigma\le n$ and is such that

$$\frac{1}{L_3}\le a(x)\le L_3, \tag{1.2}$$

for a positive constant L_3.

Actually, $a(x)$ belongs to the Sobolev space $W^{1,\sigma}_{\mathrm{loc}}(\Omega)$, $1<\sigma<+\infty$, if and only if there exists a non negative function $K\in L^\sigma_{\mathrm{loc}}(\Omega)$ such that the following inequality

$$|a(x)-a(y)|\le c(n)(K(x)+K(y))|x-y|. \tag{1.3}$$

holds a.e. (see Theorem 1 in [24]). Therefore, one can easily check that assumptions (G1)–(G3) together with (1.2) and (1.3) imply (F1)–(F4).

In our previous papers ([30], [31], [18]) we investigated the regularity properties of solutions of elliptic systems as well as of local minimizers of integral functionals of the type (1.1) under the assumptions (F1)–(F4) in case $k\in L^n_{\mathrm{loc}}(\Omega)$. Actually we have shown that the $W^{1,n}$ assumption on the x-variable is sufficient to prove a higher differentiability result for the gradient. Namely, we established the following

THEOREM 1.1 ([30]). *Let f be an integrand such that $\xi\to f(\cdot,\xi)$ is of class $C^1(\mathbb{R}^{n\times N})$ for almost every $x\in\Omega$, satisfying the assumptions* (F1)–(F4)*, for an exponent $2\le p<n$ and for a function $k\in L^n_{\mathrm{loc}}(\Omega)$. If $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathscr{F}$, then*

$$(1+|Du|^2)^{\frac{p-2}{4}}Du\in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times N}).$$

Moreover there exists a radius $R_0 = R_0(n, N, \ell, \nu, L, p)$ *such that*

$$\int_{B_R} |D((1+|Du|^2)^{\frac{p-2}{4}} Du)|^2 \, dx \leq \frac{C}{R^2} \Big(\int_{B_{2R}} |k|^n \, dx \Big)^{\frac{1}{n}} \int_{B_{2R}} |Du|^p \, dx,$$

for every R *such that* $B_{2R} \subset B_{R_0}$.

The case $p = n > 2$ has been faced in [21], in case of degenerate elliptic systems, while the critical growth $p = n = 2$ needs a different study. Indeed, in this case (see Example 1 in [31]) we can not show that the second derivatives of u belongs to L^2 (which would be the analogous result of the case $2 \leq p < n$). However, we were able to prove that they belong to L^q, for every $q < 2$. In fact, we have proven the following

THEOREM 1.2 ([31]). *Let* f *be an integrand such that* $\xi \to f(\cdot, \xi)$ *is of class* $C^1(\mathbb{R}^{2\times N})$ *for almost every* $x \in \Omega$, *satisfying the assumptions* (F1)–(F4), *with* $p = n = 2$ *and for a function* $k \in L^2_{\mathrm{loc}}(\Omega)$. *If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *is a local minimizer of the functional* $\mathscr{F}$, *then*

$$Du \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^{2\times N}), \quad \forall q < 2.$$

Moreover there exists a radius $R_0 = R_0(N, \ell, \nu, L, p)$ *such that*

$$\int_{B_R} |D^2 u|^q \, dx \leq \frac{C}{R^2} \Big(\int_{B_{2R}} |k|^2 \, dx \Big)^{\frac{1}{2}} \int_{B_{2R}} |Du|^2 \, dx,$$

for every R *such that* $B_{2R} \subset B_{R_0}$.

In two forthcoming papers ([22], [9]), we will study the regularity properties of the local minimizers of the functional $\mathscr{F}(u; \Omega)$ under weaker assumptions on the summability of the function $k(x)$ appearing in assumption (F4). More precisely in [9], we prove a higher differentiability result analogous to Theorem 1.1, assuming that $k(x)$ belongs to a fractional order Sobolev space of the type $W^{\theta, n/\theta}$, with $0 < \theta < 1$.

Here we report a particular case of a result that will appear in the forthcoming paper ([22]) in which we are able to prove that the higher differentiability of Theorem 1.1 persists for locally bounded minimizers of the functional $\mathscr{F}(u; \Omega)$, under a weaker assumption on the summability of the function $k(x)$ in the scale of Lebesgue spaces.

More precisely, in this paper, we give the result only for scalar minimizers, i.e. for $N = 1$, assuming that $k \in L^{p+2}$, where p is the exponent appearing in the assumptions (F1)–(F4). Obviously, this is a weaker assumption on k with respect to the one in Theorem 1.1, only if $2 \leq p < n - 2$ that clearly excludes the critical growth case $p = n = 2$. More precisely, we establish the following

THEOREM 1.3. *Let* f *be an integrand such that* $\xi \to f(\cdot, \xi)$ *is of class* $C^1(\mathbb{R}^n)$ *for almost every* $x \in \Omega$, *satisfying the assumptions* (F1)–(F4), *for an exponent* $2 \leq p <$

$n-2$ and for a function $k \in L^{p+2}_{\mathrm{loc}}(\Omega)$. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional $\mathscr{F}$, then

$$(1+|Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^n).$$

Moreover

$$\begin{aligned}(1.4)\qquad &\int_{B_{R/4}} (1+|Du|^2)^{\frac{p-2}{2}} |D^2u|^2 \,\mathrm{d}x \\ &\leq \frac{c}{R^p}\int_{B_{R/2}} (1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x \\ &\quad + \frac{c}{R^p}\Big(\fint_{B_R} |u|^p\,dx\Big)\Big(\int_{B_{R/2}} (1+|k(x)|)^{p+2}\,dx\Big),\end{aligned}$$

for every ball $B_R \Subset \Omega$.

The proof of Theorem 1.3 is achieved combining a suitable a priori estimate for the second derivative of the local minimizers, obtained by the use of the difference quotient method, with a suitable approximation argument.

Our main idea in order to establish the a priori estimate is to treat the regularity of local solutions of systems with discontinuous coefficients with the tools needed to deal with functionals satisfying (p,q) growth conditions. Functionals with (p,q) growth conditions have been widely investigated both in the scalar and in the vectorial setting (see for example [3, 4, 6, 7, 8, 13, 27, 28, 29, 32]).

We take advantage from the assumption $N=1$, since, by virtue of a well known result due to De Giorgi, we have that the minimizers of the functionals $\mathscr{F}(u;\Omega)$ are locally bounded in Ω. The local boundedness allows us to use an interpolation inequality that gives L^{p+2} integrability of the gradient of the minimizers. Such higher integrability for $p < n-2$ is better than the one given by the Sobolev imbedding Theorem and is the key tool in order to weaken the assumption on k.

2. Preliminaries

We shall adopt the usual convention and denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^n$ will be the standard euclidean one and denoted by $|\cdot|$. In particular, for $\xi, \eta \in \mathbb{R}^n$ we write $\langle \xi, \eta \rangle$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. When $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ we write $a \otimes b \in \mathbb{R}^{n \times n}$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its r-th row and s-th column. Observe that $(a \otimes b)x = (b \cdot x)a$ for $x \in \mathbb{R}^n$, and $|a \otimes b| = |a|\,|b|$.

For a C^1 function $F : \mathbb{R}^n \to \mathbb{R}$, we write

$$D_\xi F(\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\xi + t\eta)$$

for $\xi, \eta \in \mathbb{R}^{N\times n}$.

We shall denote by $B_r(x_0)$ the ball centered at x_0 with radius r and by

$$(u)_{x_0,r} = \fint_{B_r(x_0)} u(x)\,dx,$$

the integral mean of u over the ball $B_r(x_0)$. We shall omit the dependence on the center when no confusion arises.

Let us recall the definition of local minimizer.

DEFINITION 2.1. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathscr{F}$ if

$$\int_{\operatorname{supp}\varphi} f(x, Du)\,dx \le \int_{\operatorname{supp}\varphi} f(x, Du + D\varphi)\,dx,$$

for any $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ with $\operatorname{supp}\varphi \Subset \Omega$.

To shorten the notations, we shall use the following auxiliary function defined for $\xi \in \mathbb{R}^n$

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}}\xi.$$

We recall some useful properties of the function V that can be easily checked. More precisely, we shall use that

$$|V(\xi)| \text{ is a non-decreasing function of } |\xi|; \tag{2.1}$$

$$|V(\xi + \eta)| \le c(p)(|V(\xi)| + |V(\eta)|); \tag{2.2}$$

$$c(p)(|\xi|^2 + |\xi|^p) \le |V(\xi)|^2 \le C(p)(|\xi|^2 + |\xi|^p) \quad \text{if } p \ge 2; \tag{2.3}$$

Next Lemma has been proven in [20].

LEMMA 2.2. *Let $2 \le p < \infty$. There exists a constant $c = c(n, p) > 0$ such that*

$$c^{-1}(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \le \frac{|V(\xi) - V(\eta)|^2}{|\xi - \eta|^2} \le c(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for every $\xi, \eta \in \mathbb{R}^n$.

For a C^2 function g, it is a routine matter to check that there exists a positive constant $C(p)$ such that

$$C^{-1}|D^2 g|^2(1 + |Dg|^2)^{\frac{p-2}{2}} \le |D(V(Dg))|^2 \le C|D^2 g|^2(1 + |Dg|^2)^{\frac{p-2}{2}}. \tag{2.4}$$

Next Lemma finds an important application in the so called hole-filling method. Its proof can be found for example in [23, Lemma 6.1] .

LEMMA 2.3. *Let $h : [\rho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(r) \leq \vartheta h(d) + \frac{A}{(d-r)^{\beta}} + B,$$

for all $\rho \leq r < d \leq R_0$. Then

$$h(\rho) \leq \frac{cA}{(R_0 - \rho)^{\beta}} + B,$$

where $c = c(\vartheta, \beta) > 0$.

2.1. Difference quotient

In order to get a suitable Caccioppoli type inequality for local minimizers of the functional $\mathscr{F}(u, \Omega)$, we shall use the difference quotient method. To this aim, let us briefly recall the definition and the basic properties of the finite difference operator.

DEFINITION 2.4. For every vector valued function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator is defined by

$$\tau_{s,h}F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}$, e_s is the unit vector in the x_s direction and $s \in \{1, \ldots, n\}$.

The following proposition describes some elementary properties of the finite difference operator and can be found, for example, in [23].

PROPOSITION 2.5. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega; \mathbb{R}^N)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

(*d*1) *$\tau_{s,h}F \in W^{1,p}(\Omega)$ and*

$$D_i(\tau_{s,h}F) = \tau_{s,h}(D_iF).$$

(*d*2) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then*

$$\int_{\Omega} F\tau_{s,h}G\,dx = -\int_{\Omega} G\tau_{s,-h}F\,dx.$$

(*d*3) *We have*

$$\tau_{s,h}(FG)(x) = F(x + he_s)\tau_{s,h}G(x) + G(x)\tau_{s,h}F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

LEMMA 2.6. *If* $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, $s \in \{1, \dots, n\}$ *and* $F, D_sF \in L^p(B_R)$ *then*

$$\int_{B_\rho} |\tau_{s,h}F(x)|^p \, dx \le |h|^p \int_{B_R} |D_sF(x)|^p \, dx.$$

Moreover

$$\int_{B_\rho} |F(x + he_s)|^p \, dx \le c(n, p) \int_{B_R} |F(x)|^p \, dx.$$

Now, we recall the fundamental Sobolev embedding property.

LEMMA 2.7. *Let* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ *with* $1 < p < +\infty$. *Suppose that there exist* $\rho \in (0, R)$ *and* $M > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h}F(x)|^p \, dx \le M^p |h|^p,$$

for every h *with* $|h| < \frac{R-\rho}{2}$. *Then* $F \in W^{1,p}(B_\rho; \mathbb{R}^N) \cap L^{\frac{np}{n-p}}(B_\rho; \mathbb{R}^N)$. *Moreover*

$$\|DF\|_{L^p(B_\rho)} \le M$$

and

$$\|F\|_{L^{\frac{np}{n-p}}(B_\rho)} \le c(M + \|F\|_{L^p(B_R)}),$$

with $c \equiv c(n, N, p)$.

For the proof see, for example, [23, Lemma 8.2].

2.2. A higher integrability result

In this section, we combine a fundamental result of De Giorgi [11], that gives the local boundedness of minimizers with the existence of the second derivatives to deduce a higher integrability result for the gradient of the minimizers. More precisely, we recall the following

THEOREM 2.8. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of the functional* (1.1), *under the assumption* (F1). *Then* u *is locally bounded in* Ω. *Moreover the following*

estimate holds

$$\sup_{B_\rho} |u| \le \frac{c(p)}{(R-\rho)^{\frac{n}{p}}} \Big(\int_{B_R} |u|^p \Big)^{\frac{1}{p}}, \tag{2.5}$$

for every $B_\rho \subset B_R \Subset \Omega$.

For the proof we refer to [23], Theorem 7.5.

The following interpolation type inequality has been proven in Lemma 10 in [6], in a slightly different form (see also [18, 30]). We report it here for the sake of completeness.

LEMMA 2.9. *Let* $\eta \in C^1_c(\Omega)$ *be such that* $\eta \ge 0$ *and let* $u \in C^2(\Omega)$. *For every* $p \ge 2$ *there exists a positive constant* $c = c(p)$ *such that*

$$\begin{aligned} \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p}{2}} |Du|^2 \, dx \\ \le c(p) \|u\|^2_{L^\infty(\operatorname{supp}\eta)} \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \, dx \\ + c \|u\|^2_{L^\infty(\operatorname{supp}\eta)} \int_\Omega (|\eta|^2 + |\nabla \eta|^2)(1+|Du|^2)^{\frac{p}{2}} \, dx. \end{aligned} \tag{2.6}$$

PROOF. Integration by parts yields

$$\begin{aligned} \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p}{2}} |Du|^2 \, dx \\ = \int_\Omega \langle \eta^2 (1+|Du|^2)^{\frac{p}{2}} Du, Du \rangle \, dx \\ = - \int_\Omega D[\eta^2 Du (1+|Du|^2)^{\frac{p}{2}}] \cdot u \, dx \\ \le c(p) \int_\Omega \eta^2 |u| (1+|Du|^2)^{\frac{p}{2}} |D^2 u| \, dx \\ + 2 \int_\Omega \eta |u| \, |\nabla \eta| (1+|Du|^2)^{\frac{p}{2}} |Du| \, dx \\ =: I_1 + I_2. \end{aligned} \tag{2.7}$$

We estimate I_1 by using the Young's inequality as follows

$$\begin{aligned} I_1 \le \frac{1}{4} \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p}{2}} |Du|^2 \, dx + c \int_\Omega \eta^2 (1+|Du|^2)^{\frac{p}{2}} \\ + c(p) \int_\Omega \eta^2 |u|^2 (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \, dx. \end{aligned} \tag{2.8}$$

Similarly, we have

$$I_2 \leq \frac{1}{4}\int_\Omega \eta^2(1+|Du|^2)^{\frac{p}{2}}|Du|^2\,\mathrm{d}x + c\int_\Omega |u|^2|\nabla\eta|^2(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x. \tag{2.9}$$

Hence, inserting (2.8) and (2.9) in (2.7), we get

$$\begin{aligned}\int_\Omega \eta^2(1+|Du|^2)^{\frac{p}{2}}|Du|^2\,\mathrm{d}x &\\ \leq \frac{1}{2}\int_\Omega \eta^2(1+|Du|^2)^{\frac{p}{2}}|Du|^2\,\mathrm{d}x + c(p)\int_\Omega \eta^2|u|^2(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x &\\ + c\int_\Omega |u|^2(\eta^2+|\nabla\eta|^2)(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x.&\end{aligned}$$

Reabsorbing the first integral in the right hand side by the left hand side in previous estimate and using the local boundedness of the function u, we have that

$$\begin{aligned}\int_\Omega \eta^2(1+|Du|^2)^{\frac{p}{2}}|Du|^2\,\mathrm{d}x &\leq c(p)\int_\Omega \eta^2|u|^2(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x \\ &\quad + c\int_\Omega |u|^2(\eta^2+|\nabla\eta|^2)(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x \\ &\leq c(p)\|u\|^2_{L^\infty(\operatorname{supp}\eta)}\int_\Omega \eta^2(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x \\ &\quad + c\|u\|^2_{L^\infty(\operatorname{supp}\eta)}\int_\Omega (\eta^2+|\nabla\eta|^2)(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x,\end{aligned}$$

i.e. the conclusion. □

Combining Theorem 2.8 with Lemma 2.9 we have the following higher integrability result.

THEOREM 2.10. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathscr{F}(u)$ under the asumption* (F1), *such that $(1+|Du|^2)^{\frac{p-2}{4}}D^2u \in L^2_{\mathrm{loc}}(\Omega)$. Then $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$ and the following estimate*

$$\begin{aligned}\int_{B_\rho}(1+|Du|^2)^{\frac{p}{2}}|Du|^2\,\mathrm{d}x &\leq c\Big(\fint_{B_{2R}}|u|^p\Big)^{\frac{2}{p}}\int_{B_R}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x \\ &\quad + \frac{c(p)}{(R-\rho)^2}\Big(\fint_{B_{2R}}|u|^p\Big)^{\frac{2}{p}}\int_{B_R}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x\end{aligned}$$

holds for every balls $B_\rho \subset B_R \subset B_{2R} \Subset \Omega$.

PROOF. Fix balls $B_\rho \subset B_R \Subset \Omega$ and let $\eta \in C_0^\infty(B_R)$ be a cut off function between B_ρ and B_R, i.e. $0 \le \eta \le 1$, $\eta = 1$ on B_ρ and $|\nabla \eta| \le \frac{c}{R-\rho}$. With such a choice of η, the interpolation inequality at (2.6) becomes

$$\int_{B_\rho} (1+|Du|^2)^{\frac{p}{2}} |Du|^2 \,\mathrm{d}x \le c\|u\|^2_{L^\infty(B_R)} \int_{B_R} (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x + \frac{c}{(R-\rho)^2} \|u\|^2_{L^\infty(B_R)} \int_{B_R} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x. \tag{2.10}$$

Theorem 2.8, applied for concentric balls $B_R \subset B_{2R}$, yields

$$\sup_{B_R} |u| \le c \Big(\fint_{B_{2R}} |u|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \tag{2.11}$$

Therefore, inserting (2.11) in (2.10), we get the conclusion. □

3. PROOF OF THEOREM 1.3

This section is devoted to the proof of our main result. It will be divided in two steps: in the first one, we will establish the a priori estimate, while in the second one we will conclude through an approximation argument.

PROOF. **Step 1. The a priori estimate**
Suppose that the local minimizer u is such that $(1+|Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. Recall that local minimizers of the functional (1.1) are solutions of the corresponding Euler Lagrange equation

$$\int_\Omega \langle D_\xi f(x, Du) D\varphi \rangle \,\mathrm{d}x = 0. \tag{3.1}$$

Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} < r < s < t < \lambda r < R$, with $1 < \lambda < 2$ and consider a cut off function $\rho \in C_0^\infty(B_t)$ such that $\rho = 1$ on B_s, $|\nabla \rho| \le \frac{c}{t-s}$. Using $\varphi = \tau_{s,-h}(\rho^p \tau_{s,h} u)$ as a test function in the equation (3.1), we get

$$\int_{B_t} \langle D_\xi f(x, Du), D\tau_{s,-h}(\rho^p \tau_{s,h} u) \rangle \,\mathrm{d}x = 0,$$

which, by virtue of $(d2)$ of Proposition 2.5, is equivalent to the following

$$\int_{B_t} \langle \tau_{s,h} D_\xi f(x, Du), D(\rho^p \tau_{s,h} u) \rangle \,\mathrm{d}x = 0. \tag{3.2}$$

We write the left hand side of (3.2) as follows

$$
\begin{aligned}
(3.3)\quad & \int_{B_t} \langle \tau_{s,h} D_\xi f(x, Du), D(\rho^p \tau_{s,h} u) \rangle \, \mathrm{d}x \\
&= \int_{B_t} \langle D_\xi f(x+sh, Du(x+sh)) - D_\xi f(x, Du(x)), D(\rho^p \tau_{s,h} u) \rangle \, \mathrm{d}x \\
&= \int_{B_t} \langle D_\xi f(x+sh, Du(x+sh)) - D_\xi f(x+sh, Du(x)), D(\rho^p \tau_{s,h} u) \rangle \, \mathrm{d}x \\
&\quad + \int_{B_t} \langle D_\xi f(x+sh, Du(x)) - D_\xi f(x, Du(x)), D(\rho^p \tau_{s,h} u) \rangle \, \mathrm{d}x \\
&= \int_{B_t} \langle D_\xi f(x+sh, Du(x+sh)) - D_\xi f(x+sh, Du(x)), \rho^p D(\tau_{s,h} u) \rangle \, \mathrm{d}x \\
&\quad + p \int_{B_t} \langle D_\xi f(x+sh, Du(x+sh)) \\
&\quad - D_\xi f(x+sh, Du(x)), \rho^{p-1} \nabla \rho \tau_{s,h} u \rangle \, \mathrm{d}x \\
&\quad + \int_{B_t} \langle D_\xi f(x+sh, Du(x)) - D_\xi f(x, Du(x)), \rho^p D(\tau_{s,h} u) \rangle \, \mathrm{d}x \\
&\quad + p \int_{B_t} \langle D_\xi f(x+sh, Du(x)) - D_\xi f(x, Du(x)), \rho^{p-1} \nabla \rho \tau_{s,h} u \rangle \, \mathrm{d}x.
\end{aligned}
$$

Combining (3.3) with (3.2), we have

$$
\begin{aligned}
(3.4)\quad & \int_{B_t} \rho^p \langle D_\xi f(x+sh, Du(x+sh)) - D_\xi f(x+sh, Du(x)), D(\tau_{s,h} u) \rangle \, \mathrm{d}x \\
&= -p \int_{B_t} \langle D_\xi f(x+sh, Du(x+sh)) \\
&\quad - D_\xi f(x+sh, Du(x)), \rho^{p-1} \nabla \rho \tau_{s,h} u \rangle \, \mathrm{d}x \\
&\quad - \int_{B_t} \langle D_\xi f(x+sh, Du(x)) - D_\xi f(x, Du(x)), \rho^p D(\tau_{s,h} u) \rangle \, \mathrm{d}x \\
&\quad - p \int_{B_t} \langle D_\xi f(x+sh, Du(x)) - D_\xi f(x, Du(x)), \rho^{p-1} \nabla \rho \tau_{s,h} u \rangle \, \mathrm{d}x.
\end{aligned}
$$

The left hand side of (3.4) can be estimated by the monotonicity assumption (F3) as follows

$$
\begin{aligned}
(3.5)\quad & \int_{B_t} \rho^p \langle D_\xi f(x+sh, Du(x+sh)) - D_\xi f(x+sh, Du(x)), D(\tau_{s,h} u) \rangle \, \mathrm{d}x \\
&\geq \nu \int_{B_t} \rho^p (1 + |Du(x+sh)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h} u)|^2 \, \mathrm{d}x.
\end{aligned}
$$

Inserting (3.5) in (3.4) and using the properties of ρ, we get

$$
\begin{aligned}
(3.6)\qquad & v\int_{B_t}\rho^p(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&\le \frac{c}{t-s}\int_{B_t\setminus B_s}|D_\xi f(x+sh,Du(x+sh))\\
&\qquad\qquad - D_\xi f(x+sh,Du(x))|\,|\tau_{s,h}u|\,\mathrm{d}x\\
&\quad+\int_{B_t}\rho^p|D_\xi f(x+sh,Du(x))-D_\xi f(x,Du(x))|\,|D(\tau_{s,h}u)|\,\mathrm{d}x\\
&\quad+\frac{c}{t-s}\int_{B_t\setminus B_s}\rho^{p-1}|D_\xi f(x+sh,Du(x))-D_\xi f(x,Du(x))|\,|\tau_{s,h}u|\,\mathrm{d}x\\
&=: I+II+III.
\end{aligned}
$$

In order to estimate I, we use the assumption (F2), Young's and Hölder's inequalities as follows

$$
\begin{aligned}
(3.7)\quad I &\le \frac{c(\ell)}{t-s}\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|\,|\tau_{s,h}u|\,\mathrm{d}x\\
&\le c(\ell)\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&\quad+\frac{c(\ell)}{(t-s)^2}\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|\tau_{s,h}u|^2\,\mathrm{d}x\\
&\le c(\ell)\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&\quad+\frac{c(\ell)}{(t-s)^2}\Big(\int_{B_t}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\Big(\int_{B_t}|\tau_{s,h}u|^p\,\mathrm{d}x\Big)^{\frac{2}{p}}\\
&\le c(\ell)\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&\quad+\frac{c|h|^2}{(t-s)^2}\Big(\int_{B_{\lambda r}}(1+|Du(x)|^2)^{\frac{p}{2}}\,\mathrm{d}x\Big)^{\frac{p-2}{p}}\Big(\int_{B_t}|Du|^p\,\mathrm{d}x\Big)^{\frac{2}{p}}\\
&\le c(\ell)\int_{B_t\setminus B_s}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&\quad+\frac{c|h|^2}{(t-s)^2}\Big(\int_{B_{\lambda r}}(1+|Du(x)|^2)^{\frac{p}{2}}\,\mathrm{d}x\Big),
\end{aligned}
$$

where we also used Lemma 2.6. In order to estimate II, we use the assumption (F4), the fact that $k(x) \in L^{p+2}(\Omega)$, Young's and Hölder's inequalities thus obtaining

$$\begin{aligned} II &\le |h| \int_{B_t} \rho^p (|k(x+sh)| + |k(x)|)(1 + |Du(x)|)^{p-1} |D(\tau_{s,h}u)| \, \mathrm{d}x \\ &\le \frac{\nu}{4} \int_{B_t} \rho^p (1 + |Du(x)|^2 + |Du(x+sh)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \, \mathrm{d}x \\ &\quad + c|h|^2 \int_{B_t} \rho^p (|k(x+sh)| + |k(x)|)^2 (1 + |Du(x)|^2)^{\frac{p}{2}} \, \mathrm{d}x \\ &\le \frac{\nu}{4} \int_{B_t} \rho^p (1 + |Du(x)|^2 + |Du(x+sh)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \, \mathrm{d}x \\ &\quad + c|h|^2 \Big(\int_{B_R} \rho^p |k(x)|^{p+2} \, dx \Big)^{\frac{2}{p+2}} \Big(\int_{B_t} \rho^p (1 + |Du(x)|^2)^{\frac{p+2}{2}} \, \mathrm{d}x \Big)^{\frac{p}{p+2}}. \end{aligned}$$

The interpolation inequality of Lemma 2.9, used with $\rho^{\frac{p}{2}}$ in place of η, yields that

$$\begin{aligned} II &\le \frac{\nu}{4} \int_{B_t} \rho^p (1 + |Du(x)|^2 + |Du(x+sh)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \, \mathrm{d}x \\ &\quad + c|h|^2 \Big(\int_{B_R} |k(x)|^{p+2} \, dx \Big)^{\frac{2}{p+2}} \Big(\|u\|^2_{L^\infty(B_t)} \int_{B_t} (1 + |Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \, \mathrm{d}x \Big)^{\frac{p}{p+2}} \\ &\quad + c|h|^2 \Big(\int_{B_R} |k(x)|^{p+2} \, dx \Big)^{\frac{2}{p+2}} \Big(\frac{\|u\|^2_{L^\infty(B_t)}}{(t-s)^2} \int_{B_t} (1 + |Du|^2)^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{p}{p+2}} \\ &\le \frac{\nu}{4} \int_{B_t} \rho^p (1 + |Du(x)|^2 + |Du(x+sh)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \, \mathrm{d}x \\ &\quad + c|h|^2 \|u\|^{\frac{2p}{p+2}}_{L^\infty(B_t)} \Big(\int_{B_R} |k(x)|^{p+2} \, dx \Big)^{\frac{2}{p+2}} \Big(\int_{B_t} (1 + |Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \, \mathrm{d}x \Big)^{\frac{p}{p+2}} \\ &\quad + c|h|^2 \frac{\|u\|^{\frac{2p}{p+2}}_{L^\infty(B_t)}}{(t-s)^{\frac{2p}{p+2}}} \Big(\int_{B_R} |k(x)|^{p+2} \, dx \Big)^{\frac{2}{p+2}} \Big(\int_{B_t} (1 + |Du|^2)^{\frac{p}{2}} \, \mathrm{d}x \Big)^{\frac{p}{p+2}}. \end{aligned}$$

Therefore, by the use of Young's inequality with exponents $\frac{p+2}{p}$ and $\frac{p+2}{2}$ in the last two integrals of the right hand side of previous estimate, we conclude that

$$
\begin{aligned}
(3.8)\qquad II \le{}& \frac{\nu}{4}\int_{B_t}\rho^p\big(1+|Du(x)|^2+|Du(x+sh)|^2\big)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x\\
&+c|h|^2\frac{\|u\|^p_{L^\infty(B_t)}}{(t-s)^p}\int_{B_R}|k(x)|^{p+2}\,dx+c|h|^2\int_{B_t}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x\\
&+\vartheta|h|^2\int_{B_t}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x,
\end{aligned}
$$

where $\vartheta\in(0,1)$ is a constant that will be chosen later.

Using again assumption (F4) and Hölder's inequality, we estimate III as follows

$$
\begin{aligned}
III &\le \frac{c}{t-s}|h|\int_{B_t}\rho^{p-1}(|k(x+sh)|+|k(x)|)(1+|Du(x)|)^{p-1}|\tau_{s,h}u|\,\mathrm{d}x\\
&\le \frac{c}{t-s}|h|\Big(\int_{B_t}\rho^p(|k(x+sh)|+|k(x)|)^{\frac{p}{p-1}}(1+|Du(x)|)^p\,dx\Big)^{\frac{p-1}{p}}\\
&\quad\times\Big(\int_{B_t}|\tau_{s,h}u|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\le \frac{c}{t-s}|h|^2\Big(\int_{B_t}\rho^p(|k(x+sh)|+|k(x)|)^{\frac{p}{p-1}}(1+|Du(x)|)^p\,dx\Big)^{\frac{p-1}{p}}\\
&\quad\times\Big(\int_{B_{\lambda r}}|Du|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\le \frac{c}{t-s}|h|^2\Big(\int_{B_R}|k(x)|^{\frac{p(p+2)}{2(p-1)}}\Big)^{\frac{2(p-1)}{p(p+2)}}\Big(\int_{B_t}\rho^2(1+|Du(x)|)^{p+2}\,dx\Big)^{\frac{p-1}{p+2}}\\
&\quad\times\Big(\int_{B_{\lambda r}}|Du|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\\
&\le c|h|^2\Big(\int_{B_R}|k(x)|^{\frac{p(p+2)}{2(p-1)}}\Big)^{\frac{2}{p+2}}\Big(\int_{B_t}\rho^2(1+|Du(x)|)^{p+2}\,dx\Big)^{\frac{p}{p+2}}\\
&\quad+\frac{c}{(t-s)^p}|h|^2\int_{B_{\lambda r}}|Du|^p\,\mathrm{d}x.
\end{aligned}
$$

Similarly to the estimation of II, we use the interpolation inequality (2.6) and Young's inequality, thus getting

$$
\begin{aligned}
(3.9)\qquad III \le{}& \frac{c}{(t-s)^p}|h|^2\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x\\
&+c|h|^2\|u\|^{\frac{2p}{p+2}}_{L^\infty(B_t)}\Big(\int_{B_R}|k(x)|^{\frac{p(p+2)}{2(p-1)}}\Big)^{\frac{2}{p+2}}\Big(\int_{B_t}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x\Big)^{\frac{p}{p+2}}
\end{aligned}
$$

$$+ c|h|^2 \frac{c\|u\|_{L^\infty(B_t)}^{\frac{2p}{p+2}}}{(t-s)^{\frac{2p}{p+2}}} \Big(\int_{B_R} |k(x)|^{\frac{p(p+2)}{2(p-1)}} \Big)^{\frac{2}{p+2}} \Big(\int_{B_t} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x \Big)^{\frac{p}{p+2}}$$

$$\leq \frac{c}{(t-s)^p} |h|^2 \int_{B_{\lambda r}} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x + c|h|^2 \frac{\|u\|_{L^\infty(B_t)}^p}{(t-s)^p} \int_{B_R} |k(x)|^{\frac{p(p+2)}{2(p-1)}}$$

$$+ \vartheta |h|^2 \int_{B_t} (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x.$$

Inserting (3.7), (3.8) and (3.9) in (3.6), we obtain

$$\nu \int_{B_t} \rho^p (1+|Du(x+sh)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \,\mathrm{d}x$$

$$\leq \frac{\nu}{4} \int_{B_t} \rho^p (1+|Du(x)|^2 + |Du(x+sh)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \,\mathrm{d}x$$

$$+ c \int_{B_t \setminus B_s} (1+|Du(x+sh)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \,\mathrm{d}x$$

$$+ \frac{c}{(t-s)^p} |h|^2 \int_{B_{\lambda r}} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x + c|h|^2 \frac{\|u\|_{L^\infty(B_t)}^p}{(t-s)^p} \int_{B_R} (1+|k(x)|)^{p+2} \,\mathrm{d}x$$

$$+ 2\vartheta |h|^2 \int_{B_t} (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x.$$

Reabsorbing the first integral in the right hand side of the previous estimate by the left hand side and using the properties of ρ we get

$$\int_{B_s} (1+|Du(x+sh)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \,\mathrm{d}x$$

$$\leq c(\nu, \ell) \int_{B_t \setminus B_s} (1+|Du(x+sh)|^2 + |Du(x)|^2)^{\frac{p-2}{2}} |D(\tau_{s,h}u)|^2 \,\mathrm{d}x$$

$$+ \frac{c}{(t-s)^2} |h|^2 \int_{B_{\lambda r}} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x + c|h|^2 \frac{\|u\|_{L^\infty(B_t)}^p}{(t-s)^p} \int_{B_R} (1+|k(x)|)^{p+2} \,\mathrm{d}x$$

$$+ 2\vartheta |h|^2 \int_{B_t} (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x.$$

Since previous inequality is valid for all radii $r < s < t < \lambda r$, filling the hole, by the iteration Lemma 2.3, we deduce that

$$\int_{B_r}(1+|Du(x+sh)|^2+|Du(x)|^2)^{\frac{p-2}{2}}|D(\tau_{s,h}u)|^2\,\mathrm{d}x$$

$$\leq \frac{c}{r^p(\lambda-1)^p}|h|^2\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x+c|h|^2\frac{\|u\|^p_{L^\infty(B_{\lambda r})}}{r^p(\lambda-1)^p}\int_{B_R}(1+|k(x)|)^{p+2}\,\mathrm{d}x$$

$$+2\vartheta|h|^2\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x$$

and so, by virtue of Lemma 2.2,

$$\int_{B_r}|\tau_{s,h}(V(Du)|^2\,\mathrm{d}x \tag{3.10}$$

$$\leq \frac{c}{r^p(\lambda-1)^p}|h|^2\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x$$

$$+c|h|^2\frac{\|u\|^p_{L^\infty(B_{\lambda r})}}{r^p(\lambda-1)^p}\int_{B_R}(1+|k(x)|)^{p+2}\,\mathrm{d}x$$

$$+2\vartheta|h|^2\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x.$$

By the use of Lemma 2.7, estimate (3.10) yields that

$$\int_{B_r}|D(V(Du)|^2\,\mathrm{d}x \tag{3.11}$$

$$\leq \frac{c}{r^p(\lambda-1)^p}\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x$$

$$+c\frac{\|u\|^p_{L^\infty(B_{\lambda r})}}{(\lambda-1)^pr^p}\int_{B_R}(1+|k(x)|)^{p+2}\,\mathrm{d}x$$

$$+\tilde{c}(p,n,\nu,\ell)\vartheta\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x.$$

The elementary inequality (2.4) yields that

$$\int_{B_r}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x\leq\tilde{c}(p,n,\nu,\ell)\vartheta\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x \tag{3.12}$$

$$+\frac{c}{r^p(\lambda-1)^p}\int_{B_{\lambda r}}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x$$

$$+c\frac{\|u\|^p_{L^\infty(B_{\lambda r})}}{(\lambda-1)^pr^p}\int_{B_R}(1+|k(x)|)^{p+2}\,\mathrm{d}x.$$

Since previous estimate is valid for every $\lambda \in (1,2)$, choosing $\vartheta = \frac{1}{2\bar{c}}$, we can use again the iteration Lemma 2.3 thus obtaining

$$\int_{B_{R/2}} (1+|Du|^2)^{\frac{p-2}{2}} |D^2u|^2 \,\mathrm{d}x \leq \frac{c}{R^p}\int_{B_R} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x + \frac{c}{R^p}\|u\|^p_{L^\infty(B_R)} \int_{B_R} (1+|k(x)|)^{p+2} \,\mathrm{d}x. \tag{3.13}$$

By virtue of Theorem 2.8, we conclude with

$$\begin{aligned}\int_{B_{R/2}} (1+|Du|^2)^{\frac{p-2}{2}} |D^2u|^2 \,\mathrm{d}x &\leq \frac{c}{R^p}\int_{B_R} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x \\ &\quad + \frac{c}{R^p}\Big(\fint_{B_{2R}} |u|^p \,\mathrm{d}x\Big)\Big(\int_{B_R} (1+|k(x)|)^{p+2} \,\mathrm{d}x\Big),\end{aligned} \tag{3.14}$$

for a constant $c = c(n,p,\nu,\ell,L)$.

Step 2. The approximation
Fix a compact set $\Omega' \Subset \Omega$, and for a smooth kernel $\phi \in C^\infty_c(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)} \phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$ and put

$$k_\varepsilon := k * \phi_\varepsilon$$

and

$$f_\varepsilon(x,\xi) := f(x,\xi) * \phi_\varepsilon = \int_{B_1} \phi(\omega) f(x+\varepsilon\omega,\xi)\,\mathrm{d}\omega \tag{3.15}$$

on Ω' for each positive $\varepsilon < \mathrm{dist}(\Omega',\Omega)$. Note that

$$D_\xi f_\varepsilon(x,\xi) := D_\xi f(x,\xi) * \phi_\varepsilon = \int_{B_1} \phi(\omega) D_\xi f(x+\varepsilon\omega,\xi)\,\mathrm{d}\omega.$$

One can easily check that the assumptions (F1)–(F3) imply

$$\frac{1}{L}|\xi|^p \leq f_\varepsilon(x,\xi) \leq L(1+|\xi|^p) \tag{A1}$$

$$|D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(x,\eta)| \leq \ell|\xi-\eta|(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} \tag{A2}$$

$$\langle D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(x,\eta), \xi-\eta\rangle \geq \nu(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\eta-\xi|^2. \tag{A3}$$

Moreover, by virtue of assumption (F4), we have that

$$(A4) \qquad |D_\xi f_\varepsilon(x,\xi) - D_\xi f_\varepsilon(y,\xi)| \le (|k_\varepsilon(x)| + |k_\varepsilon(y)|)|x-y|(1+|\xi|^{p-1}).$$

for almost every $x, y \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^n$. For further needs we record that, since $k \in L^{p+2}_{\mathrm{loc}}(\Omega)$,

$$k_\varepsilon \to k \quad \text{strongly in } \in L^{p+2}_{\mathrm{loc}}(\Omega') \tag{3.16}$$

and, since $D_\xi f_\varepsilon(x, Du) \in L^{\frac{p}{p-1}}_{\mathrm{loc}}(\Omega')$, that

$$D_\xi f_\varepsilon(x,Du) \to D_\xi f(x,Du) \quad \text{strongly in } \in L^{\frac{p}{p-1}}_{\mathrm{loc}}(\Omega'). \tag{3.17}$$

Let u be a local minimizer of the functional (1.1) and let fix a ball $B_R \Subset \Omega'$. Let us denote by $u_\varepsilon \in W^{1,p}(B_R)$ the unique minimizer of the functional

$$\mathscr{F}_\varepsilon(v, B_R) := \int_{B_R} f_\varepsilon(x, Dv)\,\mathrm{d}x$$

under the boundary condition

$$v = u \quad \text{on } \partial B_R.$$

Using $\varphi = u_\varepsilon - u$ as test function in the Euler Lagrange equation of the functionals $\mathscr{F}_\varepsilon(v, B_R)$ and $\mathscr{F}(v, \Omega)$, we have

$$\int_{B_R} \langle D_\xi f_\varepsilon(x, Du_\varepsilon), Du - Du_\varepsilon\rangle\,\mathrm{d}x = \int_{B_R} \langle D_\xi f(x,Du), Du - Du_\varepsilon\rangle\,\mathrm{d}x = 0. \tag{3.18}$$

Inequality (A3) yields

$$\begin{aligned} (3.19) \qquad \nu \int_{B_R} &(1+|Du|^2+|Du_\varepsilon|^2)^{\frac{p-2}{2}}|Du-Du_\varepsilon|^2\,\mathrm{d}x \\ &\le \int_{B_R} \langle D_\xi f_\varepsilon(x,Du_\varepsilon) - D_\xi f_\varepsilon(x,Du), Du-Du_\varepsilon\rangle\,\mathrm{d}x \\ &= \int_{B_R} \langle D_\xi f(x,Du) - D_\xi f_\varepsilon(x,Du), Du-Du_\varepsilon\rangle\,\mathrm{d}x \\ &\le \Big(\int_{B_R} |D_\xi f(x,Du) - D_\xi f_\varepsilon(x,Du)|^{\frac{p}{p-1}}\,\mathrm{d}x\Big)^{\frac{p-1}{p}} \Big(\int_{B_R} |Du-Du_\varepsilon|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}, \end{aligned}$$

where we used the equality (3.18) and Hölder's inequality. Since $p \ge 2$, by well known means, from estimate (3.19) we deduce

$$\int_{B_R} |Du-Du_\varepsilon|^p\,\mathrm{d}x \le c \int_{B_R} |D_\xi f(x,Du) - D_\xi f_\varepsilon(x,Du)|^{\frac{p}{p-1}}\,\mathrm{d}x.$$

Taking the limit as $\varepsilon \to 0$ in previous inequality and recalling (3.17), we deduce that u_ε converges strongly to u in $W^{1,p}(B_R)$ and therefore a.e. in B_R for a not relabeled subsequence.

It is well known that $(1+|Du_\varepsilon|^2)^{\frac{p-2}{4}} D^2 u_\varepsilon \in L^2_{\mathrm{loc}}(B_R)$ and, since f_ε satisfies conditions (F1)–(F4), we are legitimate to apply estimate (3.14) to get

$$\begin{aligned}(3.20)\qquad &\int_{B_{r/4}} (1+|Du_\varepsilon|^2)^{\frac{p-2}{2}} |D^2 u_\varepsilon|^2 \,\mathrm{d}x \\ &\leq \frac{c}{r^p} \int_{B_{r/2}} (1+|Du_\varepsilon|^2)^{\frac{p}{2}} \,\mathrm{d}x \\ &\quad + \frac{c}{r^p} \Big(\fint_{B_r} |u_\varepsilon|^p \,\mathrm{d}x \Big) \Big(\int_{B_{r/2}} (1+|k_\varepsilon(x)|)^{p+2} \,\mathrm{d}x \Big),\end{aligned}$$

for every ball $B_r \Subset B_R$. The strong convergence of u_ε to u in $W^{1,p}(B_R)$ allows us to pass to the limit in (3.20) and by virtue of the Fatou's Lemma and by (3.16), we get

$$\begin{aligned}\int_{B_{r/4}} (1+|Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 \,\mathrm{d}x &\leq \frac{c}{r^p} \int_{B_{r/2}} (1+|Du|^2)^{\frac{p}{2}} \,\mathrm{d}x \\ &\quad + \frac{c}{r^p} \Big(\fint_{B_r} |u|^p \,\mathrm{d}x \Big) \Big(\int_{B_{r/2}} (1+|k(x)|)^{p+2} \,\mathrm{d}x \Big),\end{aligned}$$

i.e. the conclusion. □

References

[1] E. Acerbi - N. Fusco. *Regularity for minimizers of non-quadratic functionals. The case* $1 < p < 2$, J. Mat. Anal. Appl., 1 (1989), 115–135.

[2] E. Acerbi - N. Fusco. *A regularity theorem for minimizers of quasiconvex integrals*, Arch. Rat. Mech Anal., 99 (1987), 261–281.

[3] E. Acerbi - N. Fusco. *Partial regularity under anisotropic* (p,q)*-growth conditions*, J. of Diff. Equations, 107, (1994), 46–67.

[4] M. Bildhauer. *Convex variational problems. Linear, nearly linear and anisotropic growth conditions*, Lecture Notes in Math. 1818, Springer-Verlag, Berlin (2002).

[5] V. Bögelein - F. Duzaar - J. Habermann - C. Scheven. *Partial Hölder continuity for discontinuous elliptic problems with VMO-coefficients*, Proc. London Math. Soc. 103 (3) (2011), 1–34.

[6] M. Carozza - J. Kristensen - A. Passarelli di Napoli. *Higher differentiability of minimizers of convex variational integrals*, Annales Inst. H. Poincaré (C) Non Linear Analysis, 28 (3) (2011), 395–411.

[7] M. Carozza - J. Kristensen - A. Passarelli di Napoli. *Regularity of minimizers of autonomous convex variational integrals*, Ann. Scuola Normale di Pisa XIII, 4 (2014), 1065–1089.

[8] M. Carozza - J. Kristensen - A. Passarelli di Napoli. *On the validity of the Euler-Lagrange system*, Comm. on Pure and Applied Anal. 1 (2015), 51–62.

[9] A. Baison - A. Clop - R. Giova - J. Orobitg - A. Passarelli di Napoli. *Regularity results for equations with coefficients in fractional order Sobolev space*, in preparation.

[10] F. Duzaar - A. Gastel - J. F. Grotowski. *Partial regularity for almost minimizers of quasiconvex integrals*, SIAM J. Math. Anal. 32 (2000), 665–687.

[11] E. De Giorgi. *Sulla differenziabilitá e l'analiticitá delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino, P. I., III. Ser. 3 (1957), 25–43.

[12] E. De Giorgi. *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. It. 1 (4) (1968), 135–137.

[13] B. De Maria - A. Passarelli di Napoli. *Partial regularity for non autonomous functionals with non standard growth conditions*, Calc. Var. Partial Differential Equations 38 (2010), no. 3–4, 417–439.

[14] F. Duzaar - M. Krontz. *Regularity of ω-minimizers of quasiconvex integrals*, Diff. Geom. Appl. 17 (2002), 139–152.

[15] M. Foss - G. R. Mingione. *Partial continuity for elliptic problems*, Ann. I. H. Poincaré AN 25 (2008), 471–503.

[16] N. Fusco - J. E. Hutchinson. *$C^{1,\alpha}$ partial of function minimizing quasiconvex integrals*, Manu. Math. 54 (1985), 121–143.

[17] F. Giannetti - A. Passarelli di Napoli. *Hölder continuity of degenerate p-harmonic functions*, Ann. Acad. Sci. Fennicae Math. 39 (2014), 567–577.

[18] F. Giannetti - A. Passarelli di Napoli. *Higher differentiability of minimizers of variational integrals with variable exponents*, Math. Zeitschrift (2015) DOI:10.1007/s00209-015-1453-4.

[19] M. Giaquinta. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies 105 (1983), Princeton Univ. Press.

[20] M. Giaquinta - G. Modica. *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manu. Math. 57 (1986), 55–99.

[21] R. Giova. *Higher differentiability for n-harmonic systems with Sobolev coefficients*, J. Diff. Equations (to appear).

[22] R. Giova - A. Passarelli di Napoli. *Higher differentiability for a priori bounded minimizers of variational integrals with discontinuous coefficients*, in preparation.

[23] E. Giusti. *Direct methods in the calculus of variations*, World Scientific, River Edge, NJ, (2003).

[24] P. Hajlasz. *Sobolev Spaces on an Arbitrary Metric Space*, Potential Anal. 5 (1996), 403–415.

[25] T. Iwaniec - C. Sbordone. *Riesz transform and elliptic PDE's with VMO coefficients*, J. Anal. Math. 74 (1) (1998), 183–212.

[26] J. Kristensen - G. R. Mingione. *The singular set of minima of integral functionals*, Arch. Ration. Mech. Anal. 180 (2006), 331–398.

[27] P. Marcellini. *Un example de solution discontinue d'un probleme variationel dans le cas scalaire*, Preprint Ist. Mat. Dini Univ. di Firenze (1987).

[28] P. Marcellini. *Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions*, Journal of Diff. Equations, 90 (1991), pp. 1–30.

[29] P. Marcellini. *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa, Ser. 4, 23, 1 (1996), pp. 1–25.

[30] A. Passarelli di Napoli. *Higher differentiability of minimizers of variational integrals with Sobolev coefficients*, Adv. Cal. Var. 7 (1) (2014), 59–89.

[31] A. PASSARELLI DI NAPOLI. *Higher differentiability of solutions of elliptic systems with Sobolev coefficients: the case* $p = n = 2$, Pot. Anal. 41 (3) (2014), 715–735.

[32] A. PASSARELLI DI NAPOLI - F. SIEPE. *A regularity result for a class or anisotropic systems*, Rend. Ist. Mat. di Trieste, 28 (1997), pp. 13–31.

[33] V̌. SVERÁK - X. YAN. *A singular minimizer of a smooth strongly convex functional in three dimensions*, Calc. Var. 10 (2000), 213–221.

[34] V. ŠVERÁK - X. YAN. *Non Lipschitz minimizers of smooth strongly convex variational integrals*, Proc. Nat. Acad. Sc. USA 99 24 (2002), 15269–15276.

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