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**Partial Differential Equations** — *Existence of entire solutions to a fractional Liouville equation in* $\mathbb{R}^n$, by Ali Hyder, communicated on 13 November 2015.[1]

Abstract. — We study the existence of solutions to the problem

$$(-\Delta)^{\frac{n}{2}}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \qquad V := \int_{\mathbb{R}^n} e^{nu}\,dx < \infty,$$

where $Q = (n-1)!$ or $Q = -(n-1)!$. Extending the works of Wei-Ye and Hyder-Martinazzi to arbitrary odd dimension $n \geq 3$ we show that to a certain extent the asymptotic behavior of $u$ and the constant $V$ can be prescribed simultaneously. Furthermore if $Q = -(n-1)!$ then $V$ can be chosen to be any positive number. This is in contrast to the case $n = 3$, $Q = 2$, where Jin–Maalaoui–Martinazzi–Xiong showed that necessarily $V \leq |S^3|$, and to the case $n = 4$, $Q = 6$, where C-S. Lin showed that $V \leq |S^4|$.

Key words: Q-curvature, fractional Laplacian, Liouville equation, variational methods

Mathematics Subject Classification: 35J35, 35R11, 53A30, 35G20

## 1. Introduction to the problem

In this paper we consider the equation

$$(1) \qquad (-\Delta)^{\frac{n}{2}}u = (n-1)!e^{nu} \quad \text{in } \mathbb{R}^n,$$

where $n \geq 1$ and

$$(2) \qquad V := \int_{\mathbb{R}^n} e^{nu}\,dx < \infty.$$

The operator $(-\Delta)^{\frac{n}{2}}$ can be defined as follows. For $s > 0$ we set

$$\widehat{(-\Delta)^s\varphi}(\xi) := |\xi|^{2s}\hat{\varphi}(\xi), \quad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^n),$$

where

$$\mathscr{S}(\mathbb{R}^n) := \left\{u \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x|^N |D^\alpha u(x)| < \infty \text{ for all } N \in \mathbb{N} \text{ and } \alpha \in \mathbb{N}^n\right\}$$

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is the Schwartz space. We recall the space (see [15])

$$L_s(\mathbb{R}^n) := \left\{ v \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|v(x)|}{1+|x|^{n+2s}} dx < \infty \right\}. \tag{3}$$

Then we have the following definition:

Definition 1.1. Let $f \in L^1(\mathbb{R}^n)$. A function $u \in L_{\frac{n}{2}}(\mathbb{R}^n)$ is said to be a solution of

$$(-\Delta)^{\frac{n}{2}} u = f \quad \text{in } \mathbb{R}^n,$$

if

$$\int_{\mathbb{R}^n} u(-\Delta)^{\frac{n}{2}} \varphi \, dx = \int_{\mathbb{R}^n} f \varphi \, dx, \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{4}$$

where the integral on the left-hand side of (4) is well-defined thanks to Proposition A.2 in the appendix.

Geometrically if $u$ is a smooth solution of (1)–(2) then the conformal metric $g_u := e^{2u}|dx|^2$ on $\mathbb{R}^n$ ($|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) has constant $Q$-curvature equal to $(n-1)!$. Moreover the volume and the total $Q$-curvature of the metric $g_u$ are $V$ and $(n-1)!V$ respectively.

It is well-known that

$$u_{\lambda, x_0}(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \tag{5}$$

is a solution of (1)–(2) with $V = |S^n|$ for every $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. Any such $u_{\lambda, x_0}$ is called spherical solution, because it can be obtained by pulling back the round metric of $S^n$ onto $\mathbb{R}^n$ via stereographic projection. When $n = 2$ W. Chen – C. Li [5] showed that these are the only solutions to (1)–(2). On the other hand in higher even dimension non-spherical solutions do exist as shown by A. Chang – W. Chen [3]:

Theorem A ([3]). *Let $n \geq 4$ be any even number. Then for every $V \in (0, |S^n|)$ there exists at least one solution to* (1)–(2).

Moreover J. Wei and D. Ye [17] in dimension 4 and A. Hyder and L. Martinazzi [8] for arbitrary even dimension $n \geq 4$ proved the existence of solution to (1)–(2) with prescribed volume and asymptotic behavior in the following sense:

Theorem B ([17], [8]). *Let $n \geq 4$ be even. For a given $V \in (0, |S^n|)$, and a given polynomial $P$ such that degree$(P) \leq n-2$ and*

$$x \cdot \nabla P(x) \to \infty \quad \textit{as } |x| \to \infty, \tag{6}$$

*there exists a solution $u$ to* (1)–(2) *having the asymptotic behavior*

$$u(x) = -P(x) - \alpha \log|x| + C + o(1), \tag{7}$$

*where* $\alpha := \frac{2V}{|S^n|}$, *and* $o(1) \to 0$ *as* $|x| \to \infty$.

When $n$ is odd things are more complex as the operator $(-\Delta)^{\frac{n}{2}}$ is nonlocal. In a recent work T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong [9] have proven:

THEOREM C ([9]). *For every $V \in (0, |S^3|)$ there exists at least one smooth solution to* (1)–(2) *with $n = 3$.*

Extending the results of [3], [17], [8] and [9] to arbitrary odd dimension $n \geq 3$ we prove the following theorem about the existence of solutions to (1)–(2) with prescribed asymptotic behavior:

THEOREM 1.1. *Let $n \geq 3$ be an odd integer. For any given $V \in (0, |S^n|)$ and any given polynomial $P$ of degree at most $n - 1$ such that*

$$P(x) \to \infty \quad \text{as } |x| \to \infty, \tag{8}$$

*there exists $u \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ solution to* (1)–(2) *having the asymptotic behavior given in* (7) *with* $\alpha = \frac{2V}{|S^n|}$.

Notice that, contrary to the result of Theorem C, in Theorem 1.1 we can now prescribe both the asymptotic behaviour and the volume, similar to Theorem B, but in fact in more generality, since the condition (6) has been replaced by the weaker condition (8). Actually with minor modifications one can prove that the condition (8) also suffices in even dimension. On the other hand we do not expect this assumption to be optimal, compare to Theorem D below.

We also remark that the condition $0 < V < |S^n|$ is necessary for the existence of non-spherical solution to (1)–(2) in dimension 3 and 4 as shown in [9] and [11] respectively, but in higher dimension solutions could exist for large $V$, as shown for instance in dimension 6 by L. Martinazzi [14].

Also the condition $n \geq 3$ in Theorem 1.1 is necessary, since for $n = 1$ any solution of (1)–(2) is spherical, i.e. as in (5), see F. Da Lio, L. Martinazzi and T. Rivière [6].

Now we move from the problem of existence to the problem of studying the most general asymptotic behavior of solutions to (1)–(2).

For $n$ even we have this result due to C. S. Lin for $n = 4$ and L. Martinazzi when $n \geq 6$:

THEOREM D ([11], [13]). *Any solution $u$ of* (1)–(2) *with $n$ even has the asymptotic behavior*

$$u(x) = -P(x) - \alpha \log|x| + o(\log|x|) \tag{9}$$

*where* $\alpha = \frac{2V}{|S^n|}$, $\frac{o(\log|x|)}{\log|x|} \to 0$ *as* $|x| \to \infty$ *and P is a polynomial bounded from below and of degree at most* $n-2$.

Under certain regularity assumptions (precisely, $\Delta u \in L_{\frac{1}{2}}(\mathbb{R}^3)$, where the space $L_{\frac{1}{2}}(\mathbb{R}^3)$ is defined in (3)) T. Jin, A. Maalaoui, L. Martinazzi, J. Xiong [9] extended the above result to dimension 3, and among other things they proved:

THEOREM E ([9]). *Let* $u \in W^{2,1}_{loc}(\mathbb{R}^3)$ *be such that* $\Delta u \in L_{\frac{1}{2}}(\mathbb{R}^3)$ *and u satisfies* (2). *If u solves* (1) *in the sense that*

$$\int_{\mathbb{R}^3} (-\Delta)u(-\Delta)^{\frac{1}{2}}\varphi\, dx = 2\int_{\mathbb{R}^3} e^{3u}\varphi\, dx, \quad \textit{for every } \varphi \in \mathscr{S}(\mathbb{R}^3),$$

*then u has the asymptotic behavior given by* (9) *with* $\alpha = \frac{2V}{|S^3|}$ *and P is a polynomial bounded from below and of degree* 0 *or* 2.

In our upcoming paper [7] extending Theorem E we study the asymptotic behavior of solutions to (1)–(2) in arbitrary odd dimension under much weaker regularity assumptions:

THEOREM F ([7]). *Let* $n \geq 3$ *be any odd integer. Let* $u \in L_{\frac{n}{2}}(\mathbb{R}^n)$ *be a solution to* (1)–(2) *in the sense of Definition* 1.1. *Then u has the asymptotic behavior given by* (9) *with* $\alpha = \frac{2V}{|S^n|}$ *and P a polynomial bounded from below and of degree at most* $n-1$.

Now we shall discuss the case when the $Q$-curvature is negative. We consider the equation

$$(-\Delta)^{\frac{n}{2}}u = -(n-1)!e^{nu} \quad \text{in } \mathbb{R}^n. \tag{10}$$

Geometrically a smooth solution of (10) corresponds to a conformally flat metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^n$ which has constant $Q$-curvature $-(n-1)!$.

In even dimension $n \geq 4$ L. Martinazzi [12] has shown that any solution $u$ to (10)–(2) has the asymptotic behavior given by (9) with $\alpha = -\frac{2V}{|S^n|}$, while for any $V > 0$ and any given polynomial $P$ of degree at most $n-2$ satisfying (6), A. Hyder – L. Martinazzi [8] have proven the existence of solutions to (10)–(2) having the asymptotic behavior given by (7) with $\alpha = -\frac{2V}{|S^n|}$. As in the positive case, we shall extend this existence result to arbitrary odd dimension $n \geq 3$, again replacing condition (6) with the weaker condition (8).

THEOREM 1.2. *Let* $n \geq 3$ *be an odd integer. For any given* $V > 0$ *and any given polynomial P of degree at most* $n-1$ *satisfying* (8) *there exists* $u \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ *solution to* (10)–(2) *having the asymptotic behavior given in* (7) *with* $\alpha = -\frac{2V}{|S^n|}$.

Finally we remark that in dimension 1 and 2 (10)–(2) has no solution (compare to [12] and [6]).

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## 2. Proof of Theorem 1.1 and Theorem 1.2

The proof of Theorem 1.1 and 1.2 rests on the following theorem:

THEOREM 2.1. *Let* $w_0(x) = \log \frac{2}{1+|x|^2}$ *and let* $\pi : S^n \backslash \{N\} \to \mathbb{R}^n$ *be the stereographic projection and* $N = (0, \dots, 0, 1) \in S^n$ *be the North pole. Take any number* $\alpha \in (-\infty, 0) \cup (0, 2)$ *and consider two functions* $K, \varphi \in C^\infty(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n} \varphi \, dx = \gamma_n := \frac{(n-1)!}{2} |S^n|, \tag{11}$$

$\alpha K > 0$ *everywhere in* $\mathbb{R}^n$ *and whenever* $\alpha < 0$ *then* $|K| > \delta e^{-\delta |x|^p}$ *for some* $\delta > 0$, $0 < p < n$. *If both of* $Ke^{-nw_0}$ *and* $\varphi e^{-nw_0}$ *can be extended as* $C^{2n+1}$ *function on* $S^n$ *via the stereographic projection* $\pi$ *then the problem*

$$(-\Delta)^{\frac{n}{2}} w = K e^{n(w + c_w)} - \alpha \varphi \quad \text{in } \mathbb{R}^n, \quad c_w := -\frac{1}{n} \log\left(\frac{1}{\alpha \gamma_n} \int_{\mathbb{R}^n} K e^{nw} \, dx\right), \tag{12}$$

*has at least one solution* $w \in C^\infty(\mathbb{R}^n) \cap L_{\frac{n}{2}}(\mathbb{R}^n)$ *(in the sense of Definition* 1.1*) so that* $\lim_{|x| \to \infty} w(x) \in \mathbb{R}$.

Now the proof of Theorem 1.1 and Theorem 1.2 follows at once by taking

$$u := -P + \alpha u_0 + w + c_w,$$

where $u_0 \in C^\infty(\mathbb{R}^n)$ is given by Lemma 2.2 with $k = 2n + 3$, $w$ is the solution in Theorem 2.1 with $\varphi = (-\Delta)^{\frac{n}{2}} u_0$ which satisfies (11) thanks to Lemma 2.3, and $K := \operatorname{sign}(\alpha)(n-1)! e^{-nP + n\alpha u_0}$. Notice that $Ke^{-nw_0}$ can be extended smoothly on $S^n$ via the stereographic projection $\pi$ where as $\varphi e^{-nw_0}$ can be extended as a $C^{2n+1}$ function.

LEMMA 2.2. *For every positive integer* $k$ *there exists* $u_0 \in C^\infty(\mathbb{R}^n)$ *such that*

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } |x| \geq 1, \qquad |D^\alpha (-\Delta)^{\frac{n}{2}} u_0(x)| \leq \frac{C}{|x|^{2n+k+|\alpha|}} \quad \text{for } x \neq 0, \tag{13}$$

*for any multi-index* $\alpha \in \mathbb{N}^n$.

PROOF. Inductively we define

$$v_j(x) = \int_0^{x_1} v_{j-1}((t, \bar{x})) \, dt, \quad \text{for } x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}, \ j = 1, 2, \dots, k,$$

where

$$v_0(x) = \log \frac{1}{|x|}.$$

Let $\chi \in C^\infty(\mathbb{R}^n)$ be such that

$$\chi(x) = \begin{cases} 0 & for\ |x| \le \frac{1}{2} \\ 1 & for\ |x| \ge 1. \end{cases}$$

We claim that $u_0 = \frac{\partial^k}{\partial x_1^k}(\chi v_k)$ satisfies (13). It is easy to see that $u_0(x) = \log \frac{1}{|x|}$ for $|x| \ge 1$. By Lemma A.1 $\frac{1}{\gamma_n}(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial x_1^k} v_k$ is a fundamental solution of $(-\Delta)^{\frac{1}{2}}$ on $\mathbb{R}^n$ and hence for $x \ne 0$, $(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial x_1^k} v_k(x) = 0$. For $|x| > 2$ using integration by parts we compute

$$\begin{aligned}(-\Delta)^{\frac{n}{2}} u_0(x) &= (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial x_1^k}(\chi v_k - v_k)(x) \\ &= C_n \int_{|y|<1} \frac{(-\Delta)^{\frac{n-1}{2}} \frac{\partial^k}{\partial y_1^k}(\chi v_k - v_k)(y)}{|x-y|^{n+1}} dy \\ &= C_n \int_{|y|<1} (\chi v_k - v_k)(y) \frac{\partial^k}{\partial y_1^k} (-\Delta)^{\frac{n-1}{2}} \Big( \frac{1}{|x-y|^{n+1}} \Big) dy,\end{aligned}$$

and

$$D^\alpha (-\Delta)^{\frac{n}{2}} u_0(x) = C_n \int_{|y|<1} (\chi(y) v_k(y) - v_k(y)) D_x^\alpha \frac{\partial^k}{\partial y_1^k} (-\Delta)^{\frac{n-1}{2}} \Big( \frac{1}{|x-y|^{n+1}} \Big) dy.$$

Hence

$$|D^\alpha (-\Delta)^{\frac{n}{2}} u_0(x)| \le C \frac{\|v_k\|_{L^1(B_1)}}{|x|^{2n+k+|\alpha|}}. \qquad \square$$

LEMMA 2.3. *Let* $u_0 \in C^\infty(\mathbb{R}^n)$ *be as given by Lemma 2.2 for a given* $k \in \mathbb{N}$. *Then* $(-\Delta)^{\frac{n}{2}} u_0$ *satisfies* (11)*, that is,*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{2}} u_0(x)\, dx = \gamma_n.$$

PROOF. Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

We set $\eta_k(x) = \eta\left(\frac{x}{k}\right)$. Then noticing that $(-\Delta)^{\frac{n}{2}}u_0 \in L^1(\mathbb{R}^n)$ one has

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{2}}u_0(x)\,dx &= \lim_{k\to\infty}\int_{\mathbb{R}^n}(-\Delta)^{\frac{n}{2}}u_0(x)\eta_k(x)\,dx\\ &= \lim_{k\to\infty}\int_{B_1}\Big(u_0(x) - \log\frac{1}{|x|}\Big)(-\Delta)^{\frac{n}{2}}\eta_k(x)\,dx + \gamma_n\\ &= \gamma_n,\end{aligned}$$

where in the second equality we used the fact that $\frac{1}{\gamma_n}\log\frac{1}{|x|}$ is a fundamental solution of $(-\Delta)^{\frac{n}{2}}$ and the third equality follows from the locally uniform convergence of $(-\Delta)^{\frac{n}{2}}\eta_k \to 0$. □

It remains to prove Theorem 2.1. In order to do that we recall the definition of $H^n(S^n)$.

Definition 2.1. Let $n \geq 3$ be an odd integer. Let $\{Y_l^m \in C^\infty(S^n) : 1 \leq m \leq N_l, l = 0, 1, 2, \ldots\}$ be a orthonormal basis of $L^2(S^n)$ where $Y_l^m$ is an eigenfunction of the Laplace–Beltrami operator $-\Delta_{g_0}$ ($g_0$ denotes the round metric on $S^n$) corresponding to the eigenvalue $\lambda_l = l(l+n-2)$ and $N_l$ is the multiplicity of $\lambda_l$ (see [16, p. 68]). The space $H^n(S^n)$ is defined by

$$H^n(S^n) = \{u \in L^2(S^n) : \|u\|_{\dot{H}^n(S^n)} < \infty\},$$

where for any

$$u = \sum_{l=0}^{\infty}\sum_{m=1}^{N_l} u_l^m Y_l^m$$

we set

$$\|u\|^2_{\dot{H}^n(S^n)} := \sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right)\prod_{k=0}^{\frac{n-3}{2}}(\lambda_l + k(n-k-1))^2(u_l^m)^2.$$

Notice that the norm $\|u\|^2_{\dot{H}^n(S^n)}$ is equivalent to the simpler norm $\|u\|^2 := \sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\lambda_l^n(u_l^m)^2$, but has the advantage of taking the form

$$\|u\|_{\dot{H}^n(S^n)} = \|P^n_{g_0}u\|_{L^2(S^n)},$$

where for $n$ odd the Paneitz operator $P^n_{g_0}$ can be defined on $H^n(S^n)$ by (see for instance [4] and the refferences there in)

$$P^n_{g_0}u = \sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}}\prod_{k=0}^{\frac{n-3}{2}}(\lambda_l + k(n-k-1))u_l^m Y_l^m.$$

Since the operator $P^n_{g_0}$ is positive we can define its square root, namely

$$(P^n_{g_0})^{\frac{1}{2}}u := \sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\Big(\lambda_l + \Big(\frac{n-1}{2}\Big)^2\Big)^{\frac{1}{4}}\prod_{k=0}^{\frac{n-3}{2}}(\lambda_l + k(n-k-1))^{\frac{1}{2}}u_l^m Y_l^m, \quad u \in H^{\frac{n}{2}}(S^n),$$

where the space $H^{\frac{n}{2}}(S^n)$ is defined by

$$H^{\frac{n}{2}}(S^n) := \Bigg\{u \in L^2(S^n) : \sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\Big(\lambda_l + \Big(\frac{n-1}{2}\Big)^2\Big)^{\frac{1}{2}} \times \prod_{k=0}^{\frac{n-3}{2}}(\lambda_l + k(n-k-1))(u_l^m)^2 < \infty\Bigg\},$$

endowed with the norm

$$\begin{aligned}\|u\|^2_{H^{\frac{n}{2}}(S^n)} &:= \|u\|^2_{L^2(S^n)} + \|u\|^2_{\dot{H}^{\frac{n}{2}}(S^n)}\\ &:= \|u\|^2_{L^2(S^n)} + \|(P^n_{g_0})^{\frac{1}{2}}u\|^2_{L^2(S^n)}.\end{aligned}$$

Definition 2.2. Let $f \in H^{-\frac{n}{2}}(S^n)$ the dual of $H^{\frac{n}{2}}(S^n)$. A function $u \in H^{\frac{n}{2}}(S^n)$ is said to be a weak solution of

$$P^n_{g_0}u = f,$$

if

$$\int_{S^n}(P^n_{g_0})^{\frac{1}{2}}u(P^n_{g_0})^{\frac{1}{2}}\varphi\, dV_0 = \langle f, \varphi\rangle, \quad \text{for every } \varphi \in H^{\frac{n}{2}}(S^n). \tag{14}$$

The following estimate of Beckner is crucial in the proof of Theorem 2.1.

Theorem 2.4 ([1]). *For every $u \in H^{\frac{n}{2}}(S^n)$ one has*

$$\log\Big(\frac{1}{|S^n|}\int_{S^n}e^{u-\bar{u}}\,dV_0\Big) \le \frac{1}{2|S^n|n!}\int_{S^n}|(P^n_{g_0})^{\frac{1}{2}}u|^2\,dV_0, \quad \bar{u} := \frac{1}{|S^n|}\int_{S^n}u\,dV_0.$$

Proof of Theorem 2.1. Let $\tilde{K} = K\circ\pi$, $\varphi_1 = \varphi e^{-nw_0}$ and $\tilde{\varphi}_1 = \varphi_1\circ\pi$. Define the functional $J$ on $H^{\frac{n}{2}}(S^n)$ by

$$J(w) := \int_{S^n}\Big(\frac{1}{2}|(P^n_{g_0})^{\frac{1}{2}}w|^2 + \alpha\widetilde{\varphi}_1 w\Big)dV_0 - \frac{\alpha\gamma_n}{n}\log\Big(\int_{S^n}|\tilde{K}|e^{nw}e^{-nw_0\circ\pi}\,dV_0\Big).$$

Using Theorem 2.4 we bound

$$
\begin{aligned}
(15)\qquad &\log\Big(\int_{S^n}|\tilde{K}|e^{nw}e^{-nw_0\circ\pi}\,dV_0\Big)\\
&=\Big(\log\Big(\frac{1}{|S^n|}\int_{S^n}e^{nw-n\overline{w}}|\tilde{K}|e^{-nw_0\circ\pi}\,dV_0\Big)+n\overline{w}+C\Big)\\
&\le\log\Big(\frac{1}{|S^n|}\int_{S^n}e^{nw-n\overline{w}}\,dV_0\Big)+\log(\|\tilde{K}e^{-nw_0\circ\pi}\|_{L^\infty})+n\overline{w}+C\\
&\le\frac{n^2}{2|S^n|n!}\int_{S^n}|(P^n_{g_0})^{\frac{1}{2}}w|^2\,dV_0+n\overline{w}+C.
\end{aligned}
$$

Since for any $c\in\mathbb{R}$ $J(w+c)=J(c)$ we can assume $\overline{w}=0$. Then from (15) we have

$$
J(w)\ge\min\Big\{\frac{1}{2},\underbrace{\Big(\frac{1}{2}-\frac{\alpha\gamma_n}{n}\frac{n^2}{2|S^n|n!}\Big)}_{=(2-\alpha)/4}\Big\}\|w\|^2_{\dot{H}^{\frac{n}{2}}}-\varepsilon\|w\|^2_{\dot{H}^{\frac{n}{2}}}-\frac{1}{\varepsilon}\|\tilde{\varphi}_1\|^2_{L^2}-C,
$$

where $0<\varepsilon<\frac{1}{2}$ is sufficiently small so that $\frac{2-\alpha}{4}-\varepsilon>0$ and for $\alpha<0$ using $|K|>\delta e^{-\delta|x|^p}$ one has

$$
\log\Big(\int_{S^n}|\tilde{K}|e^{nw}e^{-nw_0\circ\pi}\,dV_0\Big)\ge\frac{1}{|S^n|}\int_{S^n}\log(|\tilde{K}|e^{-nw_0\circ\pi})\,dV_0+n\overline{w}+\log|S^n|\ge -C.
$$

Thus a minimizing sequence $\{w_k\}$ of $J$ with $\overline{w}_k=0$ is bounded in $\dot{H}^{\frac{n}{2}}(S^n)$. With the help of Poincaré's inequality

$$
\|w-\overline{w}\|_{L^2(S^n)}\le\|(P^n_{g_0})^{\frac{1}{2}}w\|_{L^2(S^n)},\quad\text{for every } w\in H^{\frac{n}{2}}(S^n),
$$

which easily follows from the definition of $\|(P^n_{g_0})^{\frac{1}{2}}w\|_{L^2(S^n)}$, we conclude that the sequence $\{w_k\}$ is bounded in $H^{\frac{n}{2}}(S^n)$. Then up to a subsequence $w_k$ converges weakly to $u$ for some $u\in H^{\frac{n}{2}}(S^n)$. From the compactness of the map $v\mapsto e^v$ from $H^{\frac{n}{2}}(S^n)$ to $L^p(S^n)$ for any $p\in[1,\infty)$ (for a simple proof see Proposition 7 in [9] which holds in higher dimension as well) we have (up to a subsequence)

$$
\lim_{k\to\infty}\log\Big(\int_{S^n}|\tilde{K}|e^{nw_k}e^{-nw_0\circ\pi}\,dV_0\Big)=\log\Big(\int_{S^n}|\tilde{K}|e^{nu}e^{-nw_0\circ\pi}\,dV_0\Big).
$$

Moreover from the weak convergence of $w_k$ to $u$ we have

$$
\lim_{k\to\infty}\int_{S^n}\tilde{\varphi}_1w_k\,dV_0=\int_{S^n}\tilde{\varphi}_1u\,dV_0\quad\text{and}\quad\|u\|_{H^{\frac{n}{2}}(S^n)}\le\liminf_{k\to\infty}\|w_k\|_{H^{\frac{n}{2}}(S^n)},
$$

and from the compact embedding $H^{\frac{n}{2}}(S^n)\hookrightarrow L^2(S^n)$ we get

$$
\lim_{k\to\infty}\|w_k\|_{L^2(S^n)}=\|u\|_{L^2(S^n)}.
$$

Thus $\|(P^n_{g_0})^{\frac{1}{2}}u\|_{L^2(S^n)} \le \liminf_{k\to\infty} \|(P^n_{g_0})^{\frac{1}{2}}w_k\|_{L^2(S^n)}$ which implies that $u$ is a minimizer of $J$ and hence $u$ is a weak solution of (in the sense of Definition 2.2)

$$P^n_{g_0}u + \alpha\widetilde{\varphi}_1 = \frac{\alpha\gamma_n}{\int_{S^n}\tilde{K}e^{nu}e^{-nw_0}\,dV_0}\tilde{K}e^{-nw_0\circ\pi}e^{nu} =: C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu}.$$

Since $\widetilde{\varphi}_1 \in C^{2n+1}(S^n)$ and $\tilde{K}e^{-nw_0\circ\pi} \in C^\infty(S^n)$ we have

$$P^n_{g_0}u = C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu} - \alpha\widetilde{\varphi}_1 \in L^2(S^n),$$

and by Lemma 2.5 below $u \in H^n(S^n)$ and a repeated use of Lemma 2.6 gives $u \in C^{2n+1}(S^n)$.

We set $w := u\circ\pi^{-1}$ and $w_k := u_k\circ\pi^{-1}$ where $u_k \in C^\infty(S^n)$ be such that $u_k \xrightarrow{C^{2n+1}(S^n)} u$. It is easy to see that $(-\Delta)^{\frac{n}{2}}w_k \xrightarrow{C^0(\mathbb{R}^n)} (-\Delta)^{\frac{n}{2}}w$ and $P^n_{g_0}u_k \xrightarrow{C^0(S^n)} P^n_{g_0}u$ which easily follows from

$$P^n_{g_0}u_k \xrightarrow{H^{n+1}(S^n)} P^n_{g_0}u \quad \text{and} \quad H^{n+1}(S^n) \hookrightarrow C^0(S^n).$$

Now using the following identity of T. Branson (see [2])

$$(-\Delta)^{\frac{n}{2}}(v\circ\pi^{-1}) = e^{nw_0}(P^n_{g_0}v)\circ\pi^{-1} \quad \text{for every } v \in C^\infty(S^n),$$

we get

$$\begin{aligned}(-\Delta)^{\frac{n}{2}}w = (-\Delta)^{\frac{n}{2}}(u\circ\pi^{-1}) &= e^{nw_0}(C_0\tilde{K}e^{-nw_0\circ\pi}e^{nu} - \alpha\widetilde{\varphi}_1)\circ\pi^{-1} \\ &= C_0Ke^{nw} - \alpha\varphi = Ke^{n(w+c_w)} - \alpha\varphi.\end{aligned}$$

Since $(-\Delta)^{\frac{n}{2}}w \in L_{\frac{1}{2}}(\mathbb{R}^n) \cap C^{2n+1}(\mathbb{R}^n)$ we have

$$(-\Delta)^{\frac{n+1}{2}}w = (-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{n}{2}}w \in C^{2n}(\mathbb{R}^n),$$

and by bootstrap argument we conclude that $w \in C^\infty(\mathbb{R}^n)$. □

The following lemma is probably known. Since we could not find a precise refference for this, we give a proof.

Lemma 2.5. *Let $f \in L^2(S^n)$. Let $u \in H^{\frac{n}{2}}(S^n)$ be a weak solution (in the sense of Definition 2.2) of*

$$P^n_{g_0}u = f \quad \text{on } S^n.$$

*Then $u \in H^n(S^n)$.*

Proof. Let

$$u = \sum_{l=0}^{\infty}\sum_{m=1}^{N_l} u_l^m Y_l^m \quad \text{and} \quad f = \sum_{l=0}^{\infty}\sum_{m=1}^{N_l} f_l^m Y_l^m.$$

Taking the test function $\varphi = Y_l^m$ in (14) we get

$$f_l^m = \int_{S^n} (P_{g_0}^n)^{\frac{1}{2}} u (P_{g_0}^n)^{\frac{1}{2}} \varphi \, dV_0 = \left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1)) u_l^m.$$

Hence

$$\begin{aligned} \|P_{g_0}^n u\|_{L^2(S^n)} &= \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} \left(\lambda_l + \left(\frac{n-1}{2}\right)^2\right) \prod_{k=0}^{\frac{n-3}{2}} (\lambda_l + k(n-k-1))^2 (u_l^m)^2 \\ &= \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} (f_l^m)^2 < \infty, \end{aligned}$$

and we conclude the proof. □

LEMMA 2.6. *Let $u \in H^s(S^n)$ and $f \in H^{s-n+t}(S^n)$ for some $s \geq n$ and $t \geq 0$. If $u$ solves*

$$P_{g_0}^n u = f \quad \text{on } S^n,$$

*then $u \in H^{s+t}(S^n)$.*

PROOF. Let

$$u = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} u_l^m Y_l^m,$$

and

$$(-\Delta_{g_0})^{\frac{s-n}{2}} f =: h = \sum_{i=0}^{\infty} \sum_{j=1}^{N_i} h_i^j Y_i^j,$$

where for any $r > 0$

$$(-\Delta_{g_0})^r v = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} v_l^m \lambda_l^r Y_l^m \quad \text{for } v = \sum_{l=0}^{\infty} \sum_{m=1}^{N_l} v_l^m Y_l^m \in H^{2r}(S^n).$$

Then

$$(-\Delta_{g_0})^{\frac{s-n}{2}} P_{g_0}^n u = h \quad \text{on } S^n. \tag{16}$$

Multiplying both sides of (16) by $Y_j^i$ and integrating on $S^n$ one has

$$\left(\lambda_j + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}} (\lambda_j + k(n-k-1)) \lambda_j^{\frac{s-n}{2}} u_j^i = h_j^i.$$

Since $h \in H^t(S^n)$ we have

$$\sum_{l=0}^{\infty}\sum_{m=1}^{N_l}\Big(\lambda_l+\Big(\frac{n-1}{2}\Big)^2\Big)\prod_{k=0}^{\frac{n-3}{2}}(\lambda_l+k(n-k-1))^2\lambda_l^{s-n}\lambda_l^t(u_l^m)^2<\infty,$$

and hence $u \in H^{s+t}(S^n)$. □

## A. Appendix

Lemma A.1 (Fundamental solution). *For $n \geq 3$ odd integer the function*

$$\Phi(x) := \frac{\big(\frac{n-3}{2}\big)!}{2\pi^{\frac{n+1}{2}}}\frac{1}{|x|^{n-1}} = \frac{1}{\gamma_n}(-\Delta)^{\frac{n-1}{2}}\log\frac{1}{|x|}$$

*is a fundamental solution of $(-\Delta)^{\frac{1}{2}}$ in $\mathbb{R}^n$ in the sense that for all $f \in L^1(\mathbb{R}^n)$ we have $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^n)$ and for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n}(-\Delta)^{\frac{1}{2}}(\Phi * f)\varphi\,dx := \int_{\mathbb{R}^n}(\Phi * f)(-\Delta)^{\frac{1}{2}}\varphi\,dx = \int_{\mathbb{R}^n} f\varphi\,dx. \tag{17}$$

Proof. To show $\Phi * f \in L_{\frac{1}{2}}(\mathbb{R}^n)$ we bound

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|\Phi * f(x)|}{1+|x|^{n+1}}dx &\leq C\int_{\mathbb{R}^n}\frac{1}{1+|x|^{n+1}}\Big(\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-1}}|f(y)|\,dy\Big)dx\\ &= C\int_{\mathbb{R}^n}|f(y)|\Big(\int_{\mathbb{R}^n}\frac{1}{1+|x|^{n+1}}\frac{1}{|x-y|^{n-1}}dx\Big)dy\\ &\leq C\int_{\mathbb{R}^n}|f(y)|\Big(\int_{B_1}\frac{dx}{|x|^{n-1}}+\int_{\mathbb{R}^n}\frac{dx}{1+|x|^{n+1}}\Big)dy\\ &\leq C\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned} \tag{18}$$

If $f \in C_c^\infty(\mathbb{R}^n)$ then (17) is true by Theorem 5.9 in [10]. For the general case $f \in L^1(\mathbb{R}^n)$ choose $f_k \in C_c^\infty(\mathbb{R}^n)$ such that $f_k \to f$ in $L^1(\mathbb{R}^n)$. Then using (18) with $f \equiv f_k - f$ one has

$$\begin{aligned}\int_{\mathbb{R}^n}|\Phi * (f_k - f)|\,|(-\Delta)^{\frac{1}{2}}\varphi|\,dx &\leq C\int_{\mathbb{R}^n}\frac{|\Phi * (f_k-f)(x)|}{1+|x|^{n+1}}dx\\ &\leq C\|f_k - f\|_{L^1(\mathbb{R}^n)} \to 0,\end{aligned}$$

that is

$$\int_{\mathbb{R}^n}(\Phi * f_k)(-\Delta)^{\frac{1}{2}}\varphi\,dx \to \int_{\mathbb{R}^n}(\Phi * f)(-\Delta)^{\frac{1}{2}}\varphi\,dx.$$

Now the proof follows from

$$\int_{\mathbb{R}^n} (\Phi * f_k)(-\Delta)^{\frac{1}{2}}\varphi\, dx = \int_{\mathbb{R}^n} f_k\varphi\, dx \to \int_{\mathbb{R}^n} f\varphi\, dx. \qquad \square$$

Proof of the following Proposition can be found in [7].

Proposition A.2. *For any $s > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$|(-\Delta)^s\varphi(x)| \le \frac{C}{|x|^{n+2s}}.$$

## References


[1] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) 138 (1993), 213–242.

[2] T. Branson, *Sharp inequality, the functional determinant and the complementary series*, Trans. Amer. Math. Soc. 347 (1995), 3671–3742.

[3] S-Y. A. Chang - W. Chen, *A note on a class of higher order conformally covariant equations*, Discrete Contin. Dynam. Systems 63 (2001), 275–281.

[4] S-Y. A. Chang - P. C. Yang, *On uniqueness of solutions of n-th order differential equations in conformal geometry*, Math. Res. Lett. 4 (1997), 91–102.

[5] W. Chen - C. Li, *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. 63 (3) (1991), 615–622.

[6] F. Da Lio - L. Martinazzi - T. Rivière, *Blow-up Analysis of a nonlocal Liouville-type equation*, Analysis & PDE 8 (2015) no. 7, 1757–1805.

[7] A. Hyder, *Structure of conformal metrics on $\mathbb{R}^n$ with constant Q-curvature*, arXiv: 1504.07095 (2015).

[8] A. Hyder - L. Martinazzi, *Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature, prescribed volume and asymptotic behavior*, Discrete Contin. Dynam. Systems A 35 (2015), no. 1, 283–299.

[9] T. Jin - A. Maalaoui - L. Martinazzi - J. Xiong, *Existence and asymptotics for solutions of a non-local Q-curvature equation in dimension three*, Calc. Var. Partial Differential Equations 52 (2015) no. 3–4, 469–488.

[10] E. H. Lieb - M. Loss, *Analysis*, Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. ISBN:0-8218-2783-9.

[11] C. S. Lin, *A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$*, Comment. Math. Helv. 73 (1998), no. 2, 206–231.

[12] L. Martinazzi, *Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature*, Rend. Lincei. Mat. Appl. 19 (2008), 279–292.

[13] L. Martinazzi, *Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$*, Math. Z. 263 (2009), no. 2, 307–329.

[14] L. Martinazzi, *Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature and large volume*, Ann. Inst. Henri Poincaré (C) 30 (2013), 969–982.

[15] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 60 (2007), no. 1, 67–112.

[16] E. M. Stein, *Singular integrals and differentiable properties of functions*, Princeton Math. Ser. 30 Princeton University Press, Princeton, NJ, 1970, xiv+290 pp.

[17] J. Wei - D. Ye, *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$, Calc. Var. Partial Differential Equations 32 (2008), no. 3, 373–386.

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