



Calculus of Variations — *Some recent results on the convergence of damage to fracture*, by SERGIO CONTI, MATTEO FOCARDI and FLAVIANA IURLANO, communicated on 13 November 2015.¹

ABSTRACT. — We discuss a recent approximation result for Barenblatt’s cohesive fracture energies in the case of antiplane shear. The regularized functionals are damage energies of Ambrosio–Tortorelli type and the approximation is obtained in the sense of Γ -convergence. The extension to the general case of linearized elasticity in dimension n is still an open problem. To prepare for this extension we study the structure of a subclass of functions of bounded deformation.

KEY WORDS: Cohesive fracture, phase field models, Γ -convergence, damage problems

MATHEMATICS SUBJECT CLASSIFICATION (primary; secondary): 49J45; 26B30, 74R10, 35A35

1. INTRODUCTION

The classical variational model in Fracture Mechanics takes the form

$$(1.1) \quad \int_{\Omega} h(|\nabla u|) dx + \int_{J_u} g(|[u]|) d\mathcal{H}^{n-1} + \kappa |D^c u|(\Omega)$$

in the case of antiplane shear, for a scalar displacement $u : \Omega \rightarrow \mathbb{R}$ in the space of functions with bounded variation (see for example [11]). In formula (1.1) the three terms represent the stored energy, the energy spent to open the crack, and the energy due to micro-cracking respectively.

The most renowned example of (1.1) is Griffith’s brittle fracture energy, where h is quadratic, g is constant, and $\kappa = +\infty$, so that $D^c u$ necessarily vanishes and u actually is a special function of bounded variation. This functional is known as Mumford–Shah functional in the context of image segmentation.

Another important case of (1.1) is Barenblatt’s cohesive fracture energy, where h is chosen quadratic near the origin and linear at $+\infty$, g is concave, linear near the origin, and grows from $g(0) = 0$ to some finite value $g(+\infty)$ representing the energetic cost of total fracture, while κ is finite and non-zero. A particular subcase is given by Dugdale’s energy, where the surface density is precisely $g(s) := \min\{s, 1\}$, $s \in [0, +\infty)$.

¹This paper reports the content of a talk given by Flavia Iurlano at “XXV Convegno Nazionale di Calcolo delle Variazioni”—Levico, 2–6 February 2015.

Both for numerical and physical applications, a very important issue is the approximation of models as (1.1) with more regular models within the framework of Γ -convergence. However an approximation of (1.1) with functionals of the form

$$(1.2) \quad \int_{\Omega} \phi_{\varepsilon}(\nabla u) \, dx, \quad u \in W^{1,1}(\Omega)$$

cannot be performed, since otherwise also the lower semicontinuous envelope of (1.2) would Γ -converge to the same limit, implying in turn the convexity of (1.1). Since Γ -convergence preserves convexity, the last assertion follows from the fact that the lower semicontinuous envelope of (1.2) can be written as

$$\int_{\Omega} \phi_{\varepsilon}^{**}(\nabla u) \, dx, \quad u \in W^{1,1}(\Omega),$$

where ϕ_{ε}^{**} is the convex envelope of ϕ_{ε} .

A standard approach to provide an approximation of (1.1) is to introduce an auxiliary variable, playing the role of regularization of the singularities of the displacement u . In the first work of this sort, Ambrosio and Tortorelli [8, 7] show that the elliptic functional

$$(1.3) \quad \int_{\Omega} \left((v^2 + o(\varepsilon)) |\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon} + \varepsilon |\nabla v|^2 \right) dx$$

Γ -converges in $L^1(\Omega) \times L^1(\Omega)$ to the Mumford–Shah functional. Though inspired by image segmentation problems, this result turns out to be useful also in the mechanical framework, since the approximating functionals can be regarded as energies for damaged materials with damage variable v (see [28, 33] and references therein). Indeed the behavior of v as $\varepsilon \rightarrow 0$ can be interpreted as concentration of the damage along suitable hypersurfaces, which will become cracks in the limit.

Again in the context of image reconstruction, Alicandro, Braides, and Shah propose in [2] (see also [3]) an approximation for functionals with more general dependence on the opening of the jump $[u]$, including in particular Barenblatt's energy. A key point here is that the regularizations they adopt depend on $|\nabla u|$ through an asymptotically linear function.

Contrary to the Ambrosio–Tortorelli case, these approximating functionals are not appropriate to describe damage because of the linear growth in $|\nabla u|$ of the term representing the elastic energy. The only approximations with quadratic bulk densities available in the literature so far have been obtained for energies which are linear [30] or affine in the jump $[u]$ [6, 24, 27], and have in common that the profiles of u and v in the optimal-transition problem related to the function g can be decoupled. Let us stress also the fact that from a numerical point of view a quadratic functional is easier to handle than a functional with linear growth.

Motivated by these remarks, in [19] we present a new Γ -convergence result for Barenblatt's energies using damage energy functionals introduced by [33], hence in particular quadratic in the gradients. Our functionals, which will be introduced precisely in Section 3, are of the type (1.3) with the only difference that the elastic coefficient v^2 is now replaced by a function of the form $f_\varepsilon^2(v) := \min\{1, \varepsilon\varphi^2(v)\}$, where φ vanishes at 0 (total damage), is nondecreasing, and diverges for $v \rightarrow 1$ (undamaged material).

The limit bulk density h depends only on the asymptotic behavior of φ in 1: it starts quadratically at the origin and then proceeds linearly with a suitable slope, while the fracture energy density g can be determined by solving a one-dimensional vectorial optimal profile problem (see (3.4) and (3.5)). If in addition the function φ depends on the parameter ε , for specific choices we recover in the limit Dugdale's and Griffith's fracture models, and models with surface energy density having a power-law growth at small openings (see Remark 3.3).

The proof of these results is based on several theoretical tools, such as a relaxation procedure in BV for functionals which are finite on the subspace SBV^2 of functions in SBV whose approximate gradient is square integrable and whose jump set has finite \mathcal{H}^{n-1} -dimensional measure. Another important tool is an integral representation theorem on SBV^2 .

A very interesting, though involved, issue is the extension of the previous result to the general case of linearly elastic fracture mechanics in dimension n . In this case the displacement u is vector-valued, the gradient ∇u is replaced by the symmetric gradient $e(u)$, and the natural space is the set of functions with bounded deformation BD (see [35] and Section 2 below). Many fine properties and results holding in the BV framework have a counterpart for BD (see [4]), but in general BD functions remain less well understood. In particular the study of lower semicontinuity and relaxation in BD is still in its beginnings (see [10, 29, 34, 26, 12, 13, 21, 9]) and integral representation results have been established only in some special cases (see [25]). Compactness and approximation results with more regular functions have been obtained in [10, 14, 15, 23, 31].

The aforementioned relaxation and integral representation results do not cover the SBD^2 case (the definition of SBD^2 is given in analogy to SBV^2 , see Section 2 below) and therefore they are not sufficient to extend the proof of the Γ -convergence in [19] to the vector-valued case. To this aim, it is convenient to explore more accurately the fine properties of SBD^2 and its relation to BV . In [20] we contribute to this topic studying three different problems, which will be detailed in Section 4 below.

On the positive side, we show that a function u in SBD^2 whose distributional strain consists only of a jump part belongs to $GSBV$ (see Theorem 4.1), and that SBD^2 functions are approximately continuous \mathcal{H}^{n-1} -a.e. away from the jump set (see Theorem 4.4). On the negative side, we construct a function which is BD but not in SBD^2 , nor in $GSBV$, and has distributional symmetric gradient consisting only of a jump part, and one which has a distributional symmetric gradient consisting of only a Cantor part (see Theorems 4.2 and 4.3). Finally, in the light of the previous results, in Subsection 4.2 we discuss the possibility that SBD^2 functions are actually of (generalized) bounded variation.

2. NOTATION

Let $\Omega \subset \mathbb{R}^n$ be open. For the definitions, the notation, and the main properties of the spaces BV , SBV , GBV , and $GSBV$ we refer to the book [5]. We recall that $SBV^2(\Omega)$ is defined by

$$SBV^2(\Omega) := \{u \in SBV(\Omega) : \nabla u \in L^2(\Omega, \mathbb{R}^n) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\}.$$

The space $BD(\Omega)$ of functions of bounded deformation (see [35]) is defined as follows

$$BD(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^n) : Eu \in \mathcal{M}_b(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})\},$$

where $Eu := (Du + Du^T)/2$ and \mathcal{M}_b denotes the set of bounded Radon measures.

Given $u \in BD(\Omega)$, we say that $x \in \Omega$ is a point of approximate continuity of u if there is $\tilde{u}(x) \in \mathbb{R}^n$ such that

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^n(B_r)} \int_{B_r(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

We denote by S_u the set of points $x \in \Omega$ which are not points of approximate continuity.

We say that $x \in \Omega$ is a jump point of u if there are two vectors $u^+(x) \neq u^-(x) \in \mathbb{R}^n$ and a vector $\nu_u(x) \in \mathbb{S}^{n-1}$ such that

$$\lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^n(B_r)} \int_{B_r^+(x)} |u(y) - u^+(x)| dy = \lim_{r \rightarrow 0} \frac{1}{\mathcal{L}^n(B_r)} \int_{B_r^-(x)} |u(y) - u^-(x)| dy = 0,$$

where $B_r^\pm(x) = B_r(x) \cap \{\pm(y-x) \cdot \nu_u(x) > 0\}$. Hence $J_u \subset S_u$ and if $u \in BV(\Omega, \mathbb{R}^n)$ it is well known that $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$, see, for example, [5].

One crucial property of $BD(\Omega)$ functions is that the strain Eu can be decomposed in a part absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n \llcorner \Omega$, a jump part and a third part, called Cantor part,

$$Eu = e(u)\mathcal{L}^n \llcorner \Omega + \frac{[u] \otimes \nu_u + \nu_u \otimes [u]}{2} \mathcal{H}^{n-1} \llcorner J_u + E^c u,$$

where $[u] := u^+ - u^-$ is the jump of u and J_u turns out to be $(n-1)$ -rectifiable, see [4] for details.

In addition, the approximate differentiability of BD functions has been established in [4, Theorem 7.4]. We denote by ∇u the approximate differential of $u \in BD(\Omega)$, so that $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$.

The set of special functions of bounded deformation is defined by

$$SBD(\Omega) := \{u \in BD(\Omega) : E^c u = 0\}$$

and its subspace SBD^p , for $p \in (1, \infty)$, by

$$(2.1) \quad SBD^p(\Omega) := \{u \in SBD(\Omega) : e(u) \in L^p(\Omega), \mathcal{H}^{n-1}(J_u) < \infty\}.$$

Concerning Γ -convergence, the definitions and main results can be found in [22].

3. DAMAGE CONVERGES TO COHESIVE FRACTURE

In this section we present our Γ -convergence result for Barenblatt's cohesive energy. To be precise, given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and an infinitesimal sequence $\varepsilon_k > 0$, we consider the sequence of energy functionals for damaged materials $F_k : L^1(\Omega) \times L^1(\Omega) \rightarrow [0, +\infty]$ defined by

$$(3.1) \quad F_k(u, v) := \begin{cases} \int_{\Omega} \left(f_k^2(v) |\nabla u|^2 + \frac{(1-v)^2}{4\varepsilon_k} + \varepsilon_k |\nabla v|^2 \right) dx & \text{if } (u, v) \in H^1(\Omega) \times H^1(\Omega) \\ & \text{and } 0 \leq v \leq 1 \text{ } \mathcal{L}^n\text{-a.e.}, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$(3.2) \quad f_k(z) := \min\{1, \varepsilon_k^{\frac{1}{2}} \varphi(z)\}, \quad f_k(1) = 1,$$

and

$\varphi \in C^0([0, 1], [0, +\infty))$ is a nondecreasing function satisfying $\varphi^{-1}(0) = \{0\}$

with

$$(3.3) \quad \lim_{z \rightarrow 1^-} (1-z)\varphi(z) = \ell, \quad \ell \in (0, +\infty).$$

In particular, the function $[0, 1) \mapsto (1-z)\varphi(z)$ can be continuously extended to $z = 1$ with value ℓ . The function $\varphi(z) := \frac{z}{1-z}$ is a prototype.

Let now $\Phi : L^1(\Omega) \rightarrow [0, +\infty]$ be defined by

$$\Phi(u) := \begin{cases} \int_{\Omega} h(|\nabla u|) dx + \int_{J_u} g(|[u]|) d\mathcal{H}^{n-1} + \ell |D^c u|(\Omega) & \text{if } u \in GBV(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

with $h, g : [0, +\infty) \rightarrow [0, +\infty)$ given by

$$(3.4) \quad h(s) := \begin{cases} s^2 & \text{if } s \leq \ell/2, \\ \ell s - \ell^2/4 & \text{if } s \geq \ell/2, \end{cases}$$

and

$$(3.5) \quad g(s) := \inf_{(\alpha, \beta) \in \mathcal{U}_s} \int_0^1 |1 - \beta| \sqrt{\varphi^2(\beta) |\alpha'|^2 + |\beta'|^2} dt,$$

where

$$(3.6) \quad \mathcal{U}_s := \{\alpha, \beta \in H^1((0, 1)) : 0 \leq \beta \leq 1, \alpha(0) = 0, \alpha(1) = s, \beta(0) = \beta(1) = 1\}.$$

In the one-dimensional setting the functional Φ turns out to be finite on $BV(\Omega)$. It is also easy to check that h and g satisfy all the properties of densities in Barenblatt's cohesive fracture model as discussed in the introduction.

Our main result is the following.

THEOREM 3.1. *Under the assumptions (3.1–3.6), the functionals F_k Γ -converge in $L^1(\Omega) \times L^1(\Omega)$ to the functional F defined by*

$$F(u, v) := \begin{cases} \Phi(u) & \text{if } v = 1 \text{ } \mathcal{L}^n\text{-a.e. in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

REMARK 3.2. Up to a slight modification in the definition of the functionals (3.1) a compactness result can be proved. Hence the convergence of minima and minimizers of problems connected to (3.1) to the correspondent quantities in the limit is ensured by a general result of Γ -convergence.

REMARK 3.3. By slightly changing the approximating energies F_k 's, we can build upon the results above to obtain different models in the limit. To be precise we are able to reach Griffith's and Dugdale's energies, and a cohesive energy with power-law growth at small openings. Indeed, if we replace the function φ in (3.2) by a function which pointwise diverges as $\ell_k/(1-z)$, with $\ell_k \uparrow \infty$, then one can show that Griffith's energy is recovered.

We now consider elastic coefficients of the form

$$f_k(z) := \min\{1, \varepsilon_k^{\frac{1}{2}} \max\{\varphi(z), a_k z\}\}.$$

If $a_k \rightarrow \infty$ and $a_k \varepsilon_k^{\frac{1}{2}} \rightarrow 0$, then Dugdale's energy is obtained, i.e., $g(s) = \min\{1, \ell s\}$.

Finally we consider a situation in which φ diverges with exponent $p > 1$ close to $z = 1$, so that (3.3) is replaced by

$$\lim_{z \rightarrow 1} (1-z)^p \varphi(z) = \gamma.$$

Taking

$$f_k(z) := \min\left\{1, \varepsilon_k^{\frac{1}{2}} \min\left\{\frac{kz}{1-z}, \varphi(z)\right\}\right\},$$

one finds that the functionals Γ -converge to a problem of the form of (1.1). The fracture energy g turns out to be proportional to the opening $s^{2/(p+1)}$ at small s . Moreover the coefficient of the Cantor part is infinite, so that the effective domain of the limit problem is contained in the space $GSBV$.

4. A COMPARISON BETWEEN SBD^p AND BV

4.1. The main results. In this section we present some fine results on SBD^p functions, whose definition has been given in Section 2.

The first theorem says that piecewise affine functions induced by Caccioppoli partitions have components in $GSBV$. For the definition and the main properties of Caccioppoli partitions we refer to [5, Section 4.4]. We call a function $u : \Omega \rightarrow \mathbb{R}^m$ *Caccioppoli-affine* if there exist matrices $A_k \in \mathbb{R}^{m \times n}$ and vectors $b_k \in \mathbb{R}^m$ such that

$$u(x) = \sum_k (A_k x + b_k) \chi_{E_k}(x),$$

where $(E_k)_k$ is a Caccioppoli partition of Ω .

THEOREM 4.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz set, $u : \Omega \rightarrow \mathbb{R}^m$ be Caccioppoli-affine. Then $u \in (GSBV(\Omega))^m \subset GSBV(\Omega; \mathbb{R}^m)$ with*

$$\nabla u = A_k \quad \mathcal{L}^n\text{-a.e. on } E_k, \quad \text{and} \quad \mathcal{H}^{n-1}(J_u \setminus J_\emptyset) = 0,$$

where $J_\emptyset = \bigcup_k \partial^* E_k \cap \Omega$ and $\partial^* E_k$ denotes the essential boundary of the set E_k (see [5, Definition 3.60]). In particular, if $m = n$ and $u \in SBD(\Omega)$ with $e(u) = 0$ \mathcal{L}^n -a.e. on Ω and $\mathcal{H}^{n-1}(J_u) < \infty$ then $u \in (GSBV(\Omega))^n \subset GSBV(\Omega; \mathbb{R}^n)$.

The second result we present goes in the opposite direction, showing that functions with vanishing symmetrized strain and jump set of infinite measure are not necessarily in the space $GSBV$. The proof follows an idea of [18, Theorem 1].

THEOREM 4.2. *For any nonempty open set $\Omega \subseteq \mathbb{R}^n$ there is $u \in SBD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that $e(u) = 0$ \mathcal{L}^n -a.e., $\mathcal{H}^{n-1}(J_u) = \infty$, and $\nabla u \notin L^1(\Omega; \mathbb{R}^{n \times n})$. In particular $u \notin GBV(\Omega; \mathbb{R}^n)$.*

In the same spirit a function u in $BD \setminus GBV$ for which $Eu = E^c u$ can be constructed.

THEOREM 4.3. *For any nonempty open set $\Omega \subseteq \mathbb{R}^n$ there is $u \in BD(\Omega) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that $Eu = E^c u$ and $\nabla u \notin L^1(\Omega; \mathbb{R}^{n \times n})$. In particular $u \notin GBV(\Omega; \mathbb{R}^n)$.*

In conclusion of the section we present the following theorem concerning the size of the set $S_u \setminus J_u$ for a function u belonging to $SBD^p(\Omega)$. It is based on a Korn–Poincaré inequality for SBD^p functions (see [16]).

THEOREM 4.4. *If $u \in SBD^p(\Omega)$ for some $p > 1$, with $\Omega \subset \mathbb{R}^n$ open, then $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.*

4.2. Discussion. As already mentioned, the space $SBD^p(\Omega)$ defined in (2.1) naturally appears in the variational formulation of some mathematical problems of Mechanics of Materials, and the study of its fine properties is at the base of a thorough understanding of those problems.

It is reasonable to expect that an SBD^p function is significantly more regular than a general BD function, due to the higher integrability of its strain and to the finiteness \mathcal{H}^{n-1} -measure of its jump set. The results known so far and those described above seem to suggest that SBD^p functions might be in fact of generalized bounded variation (and in this case also special, due to Alberti's rank one theorem [1]).

Let us now motivate the previous assertion discussing the possible optimality of the aforementioned inclusion. First let us highlight that if $u \in BD(\Omega) \setminus SBD^p(\Omega)$ then u is not necessarily in $GBV(\Omega; \mathbb{R}^n)$. Indeed, we recall that a generic function $u \in SBD(\Omega)$ is not necessarily of generalized bounded variation, even if $Eu = e(u)\mathcal{L}^n$ (see [4, Example 7.7], which is based on [32]). In this last case obviously a higher integrability of the strain would give the desired property.

However the condition $e(u) = 0$ \mathcal{L}^n -a.e. alone is not sufficient to guarantee that $u \in SBD(\Omega)$ is of generalized bounded variation, as Theorem 4.2 shows, but a bound on the size of the jump set has also to be assumed. Analogously Theorem 4.3 says that also the property $E^c u = 0$ is mandatory for our purposes.

On the contrary, one cannot expect an inclusion better than $SBD^p(\Omega) \subset GSBV(\Omega; \mathbb{R}^n)$, to be precise one cannot hope for a higher integrability of the gradient, a counterexample is given in [20].

In conclusion of this discussion, we mention the result proved in [17, Theorem A.1]. The authors show that any $u \in SBD(\Omega)$ with $e(u) = 0$ \mathcal{L}^n -a.e. and $\mathcal{H}^{n-1}(J_u) < \infty$ is in fact a Caccioppoli-affine function (see Section 4.1 for the definition). Theorem 4.1 thus yields that it also belongs to $GSBV(\Omega; \mathbb{R}^n)$.

ACKNOWLEDGMENTS. This work was partially supported by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 1060 "The mathematics of emergent effects", project A5.

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Received 29 April 2015,
and in revised form 17 June 2015.

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